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# Backstepping/Nonlinear $\mathscr{H}_\infty$ Control for Path Tracking of a QuadRotor Unmanned Aerial Vehicle

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***Abstract*— This paper presents a nonlinear robust control strategy to solve the path tracking problem for a quadrotor unmanned aerial vehicle. The helicopter motion equations is obtained by the Lagrange-Euler formalism. The control structure is performed through a nonlinear $\mathscr{H}_\infty$ controller to stabilize the rotational movements and a control law based on backstepping approach to track the reference trajectory. Finally, simulations results in presence of aerodynamic moments disturbances and parametric uncertainty is carried out to corroborate the effectiveness and the robustness of the strategy proposed.**

***Index Terms*— Nonlinear $\mathscr{H}_\infty$ control, backstepping approach, robust control, autonomous aerial vehicle.**

## I. INTRODUCTION

Unmanned Aerial Vehicles (UAVs) constitute a field that has motivated the control community during the last years. UAVs are used in military and civil scopes, focusing on tasks such as searching, rescue, vigilance, inspection, and so on.

These vehicles are underactuated mechanical systems, which complicates the control design stage. Techniques developed for fully actuated robots can not be directly applied to the underactuated nonlinear mechanical systems [1]. Therefore, nonlinear modelling techniques and modern nonlinear control theory are usually employed to achieve autonomous flight with high performance [2].

This paper deals with a quadrotor UAV, in which the VTOL (Vertical Take-Off and Landing) is one of the concepts usually used to develop control laws. This kind of helicopter tries to reach a stable hovering and flight using the forces equilibrium produced by the four rotors [2]. One of the advantages of the quadrotor configuration is its load capacity. Moreover, this helicopter is highly maneuverable, which allows take-off and landing as well as flight in hard environment. As a drawback, this type of UAV presents a weight and energy consumption augmentation due to the extra motors.

Many efforts have been made to control the quadrotor helicopter and many strategies have been developed to solve the path tracking problem for this type of system. In [3], a nonlinear model was proposed, presenting the helicopter kinematics and dynamics based on a Newton-Euler formalism. The aerodynamic forces and moments acting on this model were considered. The path tracking problem was solved using exact linearization techniques and noninteracting control via dynamic feedback. In [4] the same methodology was used to obtain its motion equations, but considering also the rotor dynamics. The system equations were written in state space for the controller design. The model was split up in two subsystems: the angular rotations and the linear translations. A backstepping and sliding-mode technique was used to control the helicopter. In [5], the same authors described the helicopter dynamics by the Lagrange-Euler formalism. Again, two control techniques were compared, a PID and a Linear Quadratic Regulator, where a linearized model was considered to design the PID controller. The development of the LQR was based on a time variant model.

Several control strategies have been tested on the quadrotor helicopter, but most of them do not consider external disturbances and parametric uncertainty of the model.

In some publications the quadrotor helicopter has been controlled using a linear $\mathscr{H}_\infty$ controller based on linearized models. In [6], a simplified nonlinear model of the UAV movements was presented. The path tracking problem was divided into two parts, the first one to achieve the angular rates and vertical velocity stabilization by a 2DOF $\mathscr{H}_\infty$ controller using the loop shaping technique. The same technique was used to control the longitudinal and lateral velocities, the yaw angle and the height in the outer loop. A predictive control was designed to solve the path tracking problem.

In this paper a nonlinear robust control strategy to solve the path tracking problem of the quadrotor helicopter is proposed. A nonlinear $\mathscr{H}_\infty$ controller is synthesized to stabilize the rotational movements, whereas a control law based on backstepping is used to track the reference trajectory ([4]).

The goal of the nonlinear $\mathscr{H}_\infty$ control theory, introduced by van der Schaft in his prominent article [7], is to achieve a bounded ratio between the energy of the so-called error signals and the energy of the disturbance signals. In general, the nonlinear approach of this theory considers two Hamilton-Jacobi-Bellman-Isaacs partial derivative equations (HJBI PDEs), which replace the Riccati equations in the case of the linear $\mathscr{H}_\infty$ control formulation. The main problem in the nonlinear case is the absence of a general method to solve these HJBI PDEs.

In [8] a strategy to control mechanical systems considering the tracking error dynamic equation was proposed. In such strategies a nonlinear $\mathscr{H}_\infty$ control, formulated via game theory, was applied. This strategy provides, by an analytical solution, a constant gain similar to the results obtained with the feedback linearization procedures.

The remainder of the paper is organized as follows: in Section II, a description of the quadrotor helicopter modeling is given. The nonlinear $\mathscr{H}_\infty$ controller for the rotational subsystem is developed in Section III. In Section IV, the backstepping control for the translational movements is pre-

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sented. Some simulation results are presented in Section V. Finally, the major conclusions of the work are drawn in Section VI.

## II. SYSTEM MODELLING

### A. Description

The autonomous aerial vehicle used in this paper is a miniature four rotor helicopter. The movement of the UAV results from changes in the velocities of the rotors. In order to achieve forward motion the velocity of the rear rotor must be increased and, simultaneously, the front rotor velocity must be decreased. The lateral displacement is performed with the same procedure but using the right and left motors. Yaw movement is obtained from the difference in the counter-torque between each pair of propellers, i.e., accelerating the two clockwise turning rotors while decelerating the counter-clockwise turning rotors, and vice-versa.

The dynamic model of the system is obtained under the assumption that the vehicle is a rigid body in the space subject to one main force (thrust) and three torques. However, this type of vehicle is a flight system of lightweight structure and, therefore, gyroscopes effects resulting from the rotation of the rigid body and the four propellers must be included in the dynamic model [4].

Besides, a helicopter is an underactuated mechanical system with six degrees of freedom and only four control inputs. Due to the complexities presented, some assumptions are made for modeling purposes [9]. The moment effects caused by the rigid body on the translational dynamic are neglected, as well as the ground effect. The center of mass and the body fixed frame origin are assumed coincident. Moreover, the helicopter structure is assumed to be symmetric, which results in a diagonal inertia matrix.

### B. Helicopter Kinematics

The helicopter as a rigid body is characterized by a frame linked to him. Let $\mathscr{B} = \{B_1^b, B_2^b, B_3^b\}$ be the body fixed frame, where the $B_1^b$ axis is the helicopter normal flight direction, $B_2^b$ is orthogonal to $B_1^b$ and positive to starboard in the horizontal plane, whereas $B_3^b$ is oriented in ascendant sense and orthogonal to the plane $B_1^b O B_2^b$. The inertial frame $\mathscr{I} = \{E_x, E_y, E_z\}$ is considerad fixed with respect to the earth (see Fig. 1).

The vector $\xi = \{x, y, z\}$ represents the position of the helicopter mass center expressed in the inertial frame $\mathscr{I}$. The vehicle orientation is given by a rotation matrix $R_{\mathscr{I}} : \mathscr{B} \to \mathscr{I}$, where $R_{\mathscr{I}} \in SO(3)$ is an orthonormal rotation matrix [1]. The rotation matrix is obtained through three successive rotations around the axis of the body fixed frame. The first one is given by a rotation around the $E_x$ axis by roll angle, $(-\pi < \phi < \pi)$, followed by a rotation of pitch angle, $(-\pi/2 < \theta < \pi/2)$, around the $E_y$ axis from the new axis $B_2^b$. Finally, a rotation of the yaw angle, $(-\pi < \psi < \pi)$, is carried out around the $E_z$ axis from the new axis $B_3^b$ to carry the helicopter to the final position.

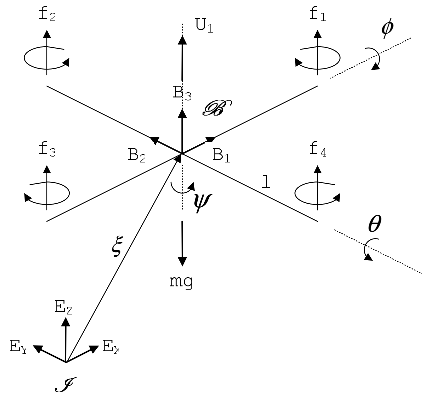


Fig. 1. Quadrotor helicopter scheme.

From these three rotations, the following rotation matrix from $\mathscr{B}$ to $\mathscr{I}$ is obtained:

$$R_{\mathscr{I}} = \begin{bmatrix} C\psi C\theta & \cos\psi S\theta S\phi - S\psi\cos\phi & \cos\psi S\theta\cos\phi + S\psi S\phi \\ S\psi C\theta & S\psi S\theta S\phi + \cos\psi\cos\phi & S\psi S\theta\cos\phi - \cos\psi S\phi \\ -S\theta & C\theta S\phi & C\theta\cos\phi \end{bmatrix} \quad (1)$$

where $C\cdot = \cos(\cdot)$ and $S\cdot = \sin(\cdot)$.

The kinematic equations of the rotational and translational movements are obtained by means of the rotation matrix. The translational kinematic can be written as:

$$v = R_{\mathscr{I}} \cdot V \quad (2)$$

where $v = [u_0 \quad v_0 \quad w_0]^T$ and $V = [u_L \quad v_L \quad w_L]^T$ are linear velocities expressed in the inertial frame and body fixed frame, respectively.

The rotational kinematic can be obtained from the relationship between the rotation matrix and its derivative with an skew-symmetric matrix ([10]) as follows:

$$\dot{R}_{\mathscr{I}} = R_{\mathscr{I}} \cdot S(\omega) \quad (3)$$

$$\dot{\eta} = W_\eta^{-1} \omega$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad (4)$$

where $\eta = (\phi, \theta, \psi)$, $\omega = (p, q, r)$ are the angular velocities in the body fixed frame.

### C. Lagrange-Euler Equations

The helicopter motion equations can be expressed by the Lagrange-Euler formalism based on the kinetic and potential energy concept:

$$\Gamma_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} \quad (5)$$

$$L = E_c - E_p$$

where $L$ is the Lagrangian, $E_c$ is the total kinetic energy, $E_p$ is the total potential energy, $q_i$ is the generalized coordinate and $\Gamma_i$ are the generalized forces/torques given by nonconservative forces/torques.

The generalized coordinates for a rigid body rotating in the three-dimensional space can be written as [11]:

$$q = [x \quad y \quad z \quad \phi \quad \theta \quad \psi]^T \in \Re^6$$

The Lagrangian expression of the helicopter is given by:

$$L(q, \dot{q}) = E_{c_{Trans}} + E_{c_{Rot}} - E_p \quad (6)$$

where $E_{c_{Trans}}$ is the translational energy and $E_{c_{Rot}}$ is the rotational energy.

Firstly, the translational energy term is developed requiring the knowledge of each generalized coordinate velocity. The linear velocity is given by (2), where $\dot{\xi} = v$ and the quadratic velocity is $\dot{\xi}^2(x,y,z) = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. Thus, the translational kinetic energy can be written as:

$$E_{c_{Trans}} = \frac{1}{2}\int \dot{\xi}^2(x,y,z)\,dm = \frac{m}{2}\dot{\xi}^2(x,y,z) = \frac{m}{2}\dot{\xi}^T\dot{\xi}$$

Let $E_{c_{Rot}}$ be the rotational kinetic energy in $\mathscr{B}$ expressed in $\mathscr{I}$, and let $dE_{c_{Rot}}$ be the kinetic energy of a particle with differential mass $dm$ in $\mathscr{B}$. Then:

$$dE_{c_{Rot}} = \frac{1}{2}\left({}^{\mathscr{I}}v_{\mathscr{B}}^2\right)dm = \frac{1}{2}\left({}^{\mathscr{I}}v_{\mathscr{B}x}^2 + {}^{\mathscr{I}}v_{\mathscr{B}y}^2 + {}^{\mathscr{I}}v_{\mathscr{B}z}^2\right)dm \tag{7}$$

Therefore, the rotational kinetic energy can be obtained solving (7). Furthermore, from the hypothesis assumed on the inertia matrix, the cross products can be neglected and consequently the inertia matrix becomes diagonal. Like this the rotational kinetic energy is given by:

$$E_{c_{Rot}} = \frac{1}{2}\int {}^{\mathscr{I}}v_{\mathscr{B}}^2 dm = \frac{1}{2}I_{xx}\left(\dot{\phi} - \dot{\psi}\sin\theta\right)^2 + \frac{1}{2}I_{yy}\left(\dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta\right)^2 + \frac{1}{2}I_{zz}\left(\dot{\theta}\sin\phi - \dot{\psi}\cos\phi\cos\theta\right)^2 \tag{8}$$

or in a compact form using (4):

$$E_{c_{Rot}} = \frac{1}{2}I_{xx}p^2 + \frac{1}{2}I_{yy}q^2 + \frac{1}{2}I_{zz}r^2 = \frac{1}{2}\omega^T J\omega \tag{9}$$

If the Jacobian from $\omega$ to $\dot{\eta}$ in (4) is named $W_\eta$ and the following matrix is defined:

$$\mathscr{J} = \mathscr{J}(\eta) = W_\eta^T J W_\eta \tag{10}$$

then the kinetic energy equation (9) can be rewritten as function of the generalized coordinate $\eta$ as follows:

$$E_{c_{Rot}} = \frac{1}{2}\dot{\eta}^T\mathscr{J}\dot{\eta} \tag{11}$$

The potential energy $E_p$ expressed in terms of the generalized coordinates is given by:

$$E_p = mgz \tag{12}$$

The complete movement equation is obtained from the Lagrangian expresion (6), as follows:

$$\begin{bmatrix} F_\xi \\ \tau_\eta \end{bmatrix} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} \tag{13}$$

where $\tau_\eta \in \Re^3$ represents the roll, pitch and yaw moments and $F_\xi = R_{\mathscr{I}}\hat{F} + A_T$ is the translational force applied to the helicopter due to the main control input $U_1$ in $z$ axis direction, with $R_{\mathscr{I}}\hat{F} = R_{\mathscr{I}_{E_3}}U_1 + A_T$ and $A_T$ the external disturbances.

Since the Lagrangian does not contain kinetic energy terms combining $\dot{\xi}$ and $\dot{\eta}$, the Lagrange-Euler equations can be divided into translational and rotational dynamics, being the Lagrange-Euler equations of the translational movement:

$$m\ddot{\xi} + mgE_3 = F_\xi \tag{14}$$

Then, (14) can be expressed by means of state vector $\xi$, yielding:

$$\begin{cases} \ddot{x} = \frac{1}{m}(\cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi)U_1 + \frac{A_x}{m} \\ \ddot{y} = \frac{1}{m}(\sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi)U_1 + \frac{A_y}{m} \\ \ddot{z} = -g + \frac{1}{m}(\cos\theta\cos\phi)U_1 + \frac{A_z}{m} \end{cases} \tag{15}$$

The Lagrange-Euler equations for the coordinate $\eta$,written in general form, are [11]:

$$M(\eta)\ddot{\eta} + C(\eta,\dot{\eta})\dot{\eta} = \tau_\eta \tag{16}$$

where $M(\eta) = \mathscr{J}(\eta)$.

Thus, the mathematical model that describes the helicopter rotational movement obtained from the Lagrange-Euler formalism is given by:

$$\ddot{\eta} = M(\eta)^{-1}(\tau_\eta - C(\eta,\dot{\eta})\dot{\eta}) \tag{17}$$

## III. NONLINEAR $\mathscr{H}_\infty$ CONTROLLER FOR STABILIZATION

In this section the rotational subsystem control law to achieve robustness in presence of sustained disturbances and parametric uncertainty is developed. A nonlinear $\mathscr{H}_\infty$ controller is able to execute this task. The controller design for mechanical system models using Lagrange-Euler equations is carried out by a direct method.

### A. Nonlinear $\mathscr{H}_\infty$ Control Theory

The dynamic equation of an $n$th order smooth nonlinear system which is affected by an unknown disturbance can be expressed as follows:

$$\dot{x} = f(x,t) + g(x,t)u + k(x,t)\omega\ , \tag{18}$$

where $u \in \Re^p$ is the vector of control inputs, $\omega \in \Re^q$ is the vector of external disturbances and $x \in \Re^n$ is the vector of states. Performance can be defined using the cost variable $z \in \Re^{(m+p)}$ given by the expression:

$$z = W\begin{bmatrix} h(x) \\ u \end{bmatrix}, \tag{19}$$

where $h(x) \in \Re^m$ represents the error vector to be controlled and $W \in \Re^{(m+p)\times(m+p)}$ is a weighting matrix. If states $x$ are assumed to be available for measurement, then the optimal $\mathscr{H}_\infty$ problem can be posed as follows [7]: *Find the smallest value $\gamma^* \geq 0$ such that for any $\gamma \geq \gamma^*$ exists a state feedback $u = u(x,t)$, such that the $L_2$ gain from $\omega$ to $z$ is less than or equal to $\gamma$, that is:*

$$\int_0^T \|z\|_2^2 dt \leq \gamma^2 \int_0^T \|\omega\|_2^2 dt\ . \tag{20}$$

The internal term of the integral expression on the left-hand side of inequality (20) can be written as:

$$\|z\|_2^2 = z^T z = \begin{bmatrix} h^T(x) & u^T \end{bmatrix} W^T W \begin{bmatrix} h(x) \\ u \end{bmatrix}$$

and the symmetric positive definite matrix $W^TW$ can be partitioned as follows:

$$W^TW = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \tag{21}$$

Matrices $Q$ and $R$ are symmetric positive definite and the fact that $W^TW > 0$ guarantees that $Q - SR^{-1}S^T > 0$.

Under these assumptions, an optimal control signal $u^*(x,t)$ may be computed for system (18) if there is a smooth solution $V(x,t)$, with $V(x_0,t) \equiv 0$ for $t \geq 0$, to the following HJBI equation [12]:

$$\frac{\partial V}{\partial t} + \frac{\partial^T V}{\partial x}f(x,t) + \frac{1}{2}\frac{\partial^T V}{\partial x}\left[\frac{1}{\gamma^2}k(x,t)k^T(x,t) - g(x,t)R^{-1}g^T(x,t)\right]\frac{\partial V}{\partial x} - \frac{\partial^T V}{\partial x}g(x,t)R^{-1}S^T h(x) + \frac{1}{2}h^T(x)\left(Q - SR^{-1}S^T\right)h(x) = 0 \tag{22}$$

for each $\gamma > \sqrt{\sigma_{\max}(R)} \geq 0$, where $\sigma_{\max}$ stands for the maximum singular value. In such a case, the optimal state feedback control law is derived as follows [13]:

$$u^* = -R^{-1}\left(S^T h(x) + g^T(x,t)\frac{\partial V(x,t)}{\partial x}\right)\ . \tag{23}$$

### B. Rotational Subsystem Nonlinear $\mathscr{H}_\infty$ Control

In order to develop the nonlinear $\mathscr{H}_\infty$ controller the rotational movements dynamic model (16), obtained from the Lagrange-Euler formalism, is used. $\tau_\eta$ adds the control torques and external disturbances, and is redefined as:

$$\tau_\eta = \tau_{\eta_a} + \tau_{\eta_d}$$

where $\tau_{\eta_a}$ is the applied torques vector and $\tau_{\eta_d}$ represents the total effect of system modeling errors and external disturbances.

As a first step to synthesize the control law, the tracking error vector is defined as follows:

$$\hat{x} = \begin{bmatrix} \dot{\tilde{\eta}} \\ \tilde{\eta} \\ \int \tilde{\eta} dt \end{bmatrix} = \begin{bmatrix} \dot{\eta} - \dot{\eta}^d \\ \eta - \eta^d \\ \int \left(\eta - \eta^d\right) dt \end{bmatrix} \tag{24}$$

where $\eta^d$ and $\dot{\eta}^d \in \Re^n$ are the desired trajectory and the corresponding velocity, respectively. Note that an integral term has been included in the error vector. This term will allow to find a null steady-state error when sustained disturbances are acting on the system [8].

The following control law is proposed for the rotational subsystem:

$$\tau_{\eta_a} = M(\eta)\ddot{\eta} + C(\eta,\dot{\eta})\dot{\eta} - T_1^{-1}\left(M(\eta)T\dot{\hat{x}} + C(\eta,\dot{\eta})T\hat{x}\right) + T_1^{-1}u \tag{25}$$

The proposed control law can be split up into three different parts: the first one consists of the first three terms of that equation, which are designed in order to compensate the system dynamics (16). The second part consists of terms including the error vector $\hat{x}$ and its derivative, $\dot{\hat{x}}$. Assuming $\tau_{\eta_d} \equiv 0$, these two terms of the control law enable perfect tracking, which means that they represent the *essential* control effort needed to perform the task. Finally, the third part includes a vector $u$, which represents the *additional* control effort needed for disturbance rejection.

It can also be pointed out that, despite the preceding control law might seem a not well posed system, it will be shown afterwards that the computed torque does not rely on joint accelerations, but on their references.

Matrix $T$ in (25) can be partitioned as follows:

$$T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}$$

with $T_1 = \rho I$, where $\rho$ is a positive scalar and $I$ is the $n$th-order identity matrix.

Substituting the expression of the control law from (25) into the Lagrange-Euler equation of the system (16) and defining $\omega = M(\eta)T_1 M^{-1}(\eta)\tau_{\eta_d}$, one gets:

$$M(\eta)T\dot{\hat{x}} + C(\eta,\dot{\eta})T\hat{x} = u + \omega \tag{26}$$

This expression represents the *dynamic equation of the system error*. Taking into account this nonlinear equation, the nonlinear $\mathscr{H}_\infty$ control problem can be posed as follows:

*"Find a control law $u(t)$ such that the ratio between the energy of the cost variable $z = W\left[h^T(\hat{x})\ u^T\right]^T$ and the energy of the disturbance signals $\omega$ is less than a given attenuation level $\gamma$".*

Taking into account the definition of the vector error, $\hat{x}$, and the definition of the cost variable, $z$, the following structures are considered for matrices $Q$ and $S$ in (21):

$$Q = \begin{bmatrix} Q_1 & Q_{12} & Q_{13} \\ Q_{12} & Q_2 & Q_{23} \\ Q_{13} & Q_{23} & Q_3 \end{bmatrix}, \quad S = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix}.$$

To apply the theoretical results presented in Section III-A, it is necessary to rewrite the nonlinear dynamic equation of the error (26) into the standard form of the nonlinear $\mathscr{H}_\infty$ problem (see (18)). This can be done by defining the following expressions:

$$\dot{\hat{x}} = f(\hat{x},t) + g(\hat{x},t)u + k(\hat{x},t)\omega, \tag{27}$$

$$f(\hat{x},t) = T_0^{-1}\begin{bmatrix} -M^{-1}C & O & O \\ T_1^{-1} & I - T_1^{-1}T_2 & -I + T_1^{-1}(T_2 - T_3) \\ O & I & -I \end{bmatrix} T_0,$$

$$g(\hat{x},t) = k(\hat{x},t) = T_0^{-1}\begin{bmatrix} M(\eta)^{-1} \\ O \\ O \end{bmatrix}$$

where $I$ is the identity matrix, $O$ the zero matrix, both of $n$-th order, and

$$T_0 = \begin{bmatrix} T_1 & T_2 & T_3 \\ O & I & I \\ O & O & I \end{bmatrix}. \tag{28}$$

As stated in Section III-A, the solution of the HJBI equation depends on the choice of the cost variable, $z$, and particularly on the selection of function $h(\hat{x})$ (see (19)). In this paper, this function is taken to be equal to the error vector, that is, $h(\hat{x}) = \hat{x}$. Once this function has been selected, computing the control law, $u$, will require finding the solution, $V(\hat{x},t)$, to the HJBI equation posed in the previous section (see (22)). Refer to [8] for more details.

Matrix $T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}$ can be computed by solving some Riccati algebraic equations (see [8]).

Once matrix $T$ is computed, substituting $V(\hat{x},t)$ in (23), control law $u^*$ corresponding to the $\mathscr{H}_\infty$ optimal index $\gamma$ is given by

$$u^* = -R^{-1}\left(S^T + T\right)\hat{x} \tag{29}$$

Finally, if the control law (29) is replaced into (25), and after some manipulations, the optimal control law can be written as:

$$\tau_{\eta_a}^* = M(\eta)\ddot{\eta}^d + C(\eta,\dot{\eta})\dot{\eta} - M(\eta)\left(K_D\dot{\tilde{\eta}} + K_P\tilde{\eta} - K_I\int\tilde{\eta}dt\right) \tag{30}$$

A particular case can be obtained when the components of weighting compound $W^TW$ verify [8]:

$$Q_1 = \omega_1^2 I, \quad Q_2 = \omega_2^2 I, \quad Q_3 = \omega_3^2 I, \quad R = \omega_u^2 I, \tag{31}$$

$$Q_{12} = Q_{13} = Q_{23} = 0, \quad S_1 = S_2 = S_3 = 0.$$

In this case, the following analytical expressions for the gain matrices have been obtained:

$$K_D = \frac{\sqrt{\omega_2^2 + 2\omega_1\omega_3}}{\omega_1}I + M(\eta)^{-1}\left(C(\eta,\dot{\eta}) + \frac{1}{\omega_u^2}I\right),$$

$$K_P = \frac{\omega_3}{\omega_1}I + \frac{\sqrt{\omega_2^2 + 2\omega_1\omega_3}}{\omega_1}M(\eta)^{-1}\left(C(\eta,\dot{\eta}) + \frac{1}{\omega_u^2}I\right),$$

$$K_I = \frac{\omega_3}{\omega_1}M(\eta)^{-1}\left(C(\eta,\dot{\eta}) + \frac{1}{\omega_u^2}I\right).$$

These expressions have an important property: they do not depend on the parameter $\gamma$. So, we obtain an algebraic expression for computing the general optimal solution for this particular case [8].

## IV. BACKSTEPPING CONTROL FOR PATH TRACKING

In this section a control law to solve the path tracking problem by translational movements is designed. This controller was presented in [4], where the rotational movements were also controlled by the same technique.

For the controller design we rewrite the system (15) in state space form $\dot{X} = f(X,U)$ introducing $X = (x_1 \dots x_6)$ as the state space vector of the system:

$$\begin{array}{ccc} x_1 = z\,, & x_2 = \dot{x}_1 = \dot{z}\,, & x_3 = x\,, \\ x_4 = \dot{x}_3 = \dot{x}\,, & x_5 = y\,, & x_6 = \dot{x}_5 = \dot{y} \end{array}$$

From (15) and the new state space vector the system can be written in the following form:

$$f(X,U) = \begin{bmatrix} x_1 \\ -g + (\cos\phi\cos\theta)\frac{U_1}{m} \\ x_4 \\ u_x\frac{U_1}{m} \\ x_6 \\ u_y\frac{U_1}{m} \end{bmatrix} \tag{32}$$

with:

$$\begin{array}{l} u_x = (\cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi) \\ u_y = (\sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi) \end{array} \tag{33}$$

### A. Translational Subsystem Backstepping Control

*1) Height Control:* The Backstepping technique can be used to design the control law in such way that the subsystem is forced to track the reference trajectory. The first step considers the tracking error [14]:

$$z_1 = x_{1_d} - x_1 \tag{34}$$

Using the Lyapunov theorem under consideration of $z_1$ positive definite and its time derivative negative semi-definite, the Lyapunov function of $z_1$, is given by:

$$V(z_1) = \frac{1}{2}z_1^2 \tag{35}$$

$$\dot{V}(z_1) = z_1(\dot{x}_{1_d} - x_2) \tag{36}$$

The stabilization of $z_1$ can be obtained by introducing a virtual control input $x_2$:

$$x_2 = \dot{x}_{1_d} + \alpha_1 z_1 \tag{37}$$

with $\alpha_1 > 0$.

The time derivative of the Lyapunov function can be written as:

$$\dot{V}(z_1) = -\alpha_1 z_1^2 \tag{38}$$

allowing to proceed with a change of variables:

$$z_2 = x_2 - \dot{x}_{1_d} - \alpha_1 z_1 \tag{39}$$

The second step considers the augmented Lyapunov function:

$$V(z_1,z_2) = \frac{1}{2}(z_1^2 + z_2^2) \tag{40}$$

and its temporal derivative is given by:

$$\begin{array}{rcl} \dot{V}(z_1,z_2) & = & z_2(-g + (\cos\phi\cos\theta)\frac{U_1}{m}) \\ & & -z_2(\ddot{x}_{1_d} - \alpha_1(z_2 + \alpha_1 z_1)) - z_1 z_2 - \alpha_1 z_1^2 \end{array} \tag{41}$$

The height control input $U_1$ obtains ($\ddot{x}_{1,2,3_d} = 0$), satisfying $\dot{V}(z_1,z_2) < 0$:

$$U_1 = \frac{m}{\cos\phi\cos\theta}(z_1 + g - \alpha_1(z_2 + \alpha_1 z_1) - \alpha_2 z_2) \tag{42}$$

The $\alpha_2 z_2$ term with $\alpha_2 > 0$ is added to stabilize $z_1$.

*2) Longitudinal and Lateral Movement Control:* From (15) can be seen that the movement through the $x$ and $y$ axis depends on the control input $U_1$. In fact, $U_1$ is the designed total thrust vector to obtain the desired linear movement. $u_x$ and $u_y$ can be considered the orientations of $U_1$ responsible for the movement through the $x$ and $y$ axis, respectively. Thus, the roll, $\phi$, and pitch, $\theta$, angles necessary to compute the control signals $u_x$ and $u_y$ can be extract from (33), satisfying $\dot{V}(z_1,z_2) < 0$.

$$u_x = \frac{m}{U_1}(z_3 - \alpha_3(z_4 + \alpha_3 z_3) - \alpha_4 z_4) \tag{43}$$

$$u_y = \frac{m}{U_1}(z_5 - \alpha_5(z_6 + \alpha_5 z_5) - \alpha_6 z_6) \tag{44}$$

## V. SIMULATION RESULTS

The proposed control strategy, using a Backstepping controller in cascade with a nonlinear $\mathscr{H}_\infty$ controller, was applied to the helicopter to corroborate the effectiveness to solve the path tracking problem. Simulations has been performed considering external disturbances and parametric uncertainties.

A vertical helix is used as reference trajectory and is defined as:

$$x_d = \tfrac{1}{2}\cos\left(\tfrac{t}{2}\right), \quad y_d = \tfrac{1}{2}\sin\left(\tfrac{t}{2}\right), \quad z_d = 1 + \tfrac{t}{10}, \quad \psi_d = \tfrac{\pi}{3}$$

The initial conditions of the helicopter are $(x,y,z) = (0{,}0{,}0.5)m$ and $(\phi,\theta,\psi) = (0{,}0{,}0.5)rad$. The values of the model parameters used for simulations are the following: $m = 0.74kg$, $l = 0.21m$, $g = 9.81m/s^2$ and $I_{xx} = I_{yy} = 0.004kg.m^2$, $I_{zz} = 0.0084kg.m^2$. In the simulations a $\pm 20\%$ uncertainty in the inertia parameters was used.

In the simulation external disturbances on the aerodynamic moments were considered. Maintained steps were applied: at $t = 5s$ the step $A_r = 0.5Nm$ was introduced, at $t = 15s$ the disturbance $A_p = 1Nm$ affects the system and at $t = 25s$ the last disturbance with an amplitude of $A_q = 1Nm$ was applied.

The backstepping controller parameters were adjusted as follows: $\alpha_1 = 50$, $\alpha_2 = 3$, $\alpha_3 = 9$, $\alpha_4 = 3$, $\alpha_5 = 9$ y $\alpha_6 = 3$. The nonlinear $\mathscr{H}_\infty$ controller gains were syntonized with the following values: $\omega_1 = 1.5$, $\omega_2 = 0.5$, $\omega_3 = 3.0$ y $\omega_u = 0.6$. The following results were obtained:

Fig. 2 to 4 show the simulation results of the path tracking of the quadrotor helicopter. The results illustrate the robust performance provided by the controller in the case of parametric uncertainty in the inertia terms. Besides, the figures present a perfect tracking of the reference trajectory when external disturbance originated by aerodynamic moments are considered.

## VI. CONCLUSIONS

In this paper a robust control strategy to solve the path tracking problem for a quadrotor helicopter has been presented. The proposed strategy was designed in consideration of external disturbances like aerodynamic moments. The movement dynamic equations by Lagrange-Euler formalism have been developed. This model is divided in two subsystem to perform the control laws. A robust control based on nonlinear $\mathscr{H}_\infty$ theory is used for the helicopter stabilization. It can be shown that the applied moment disturbances were

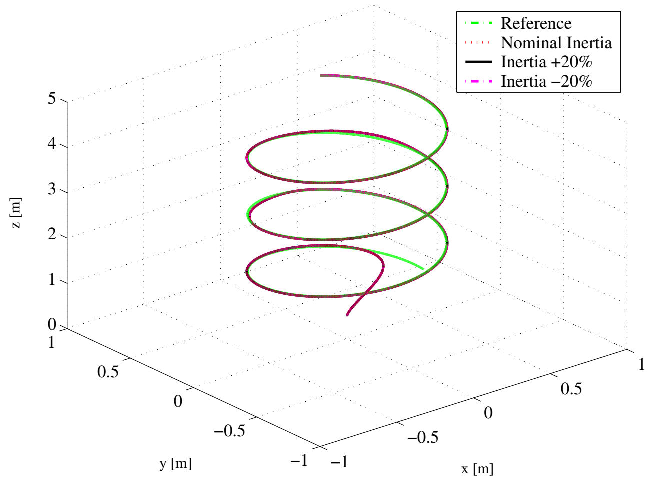


Fig. 2. Path following with external disturbances.

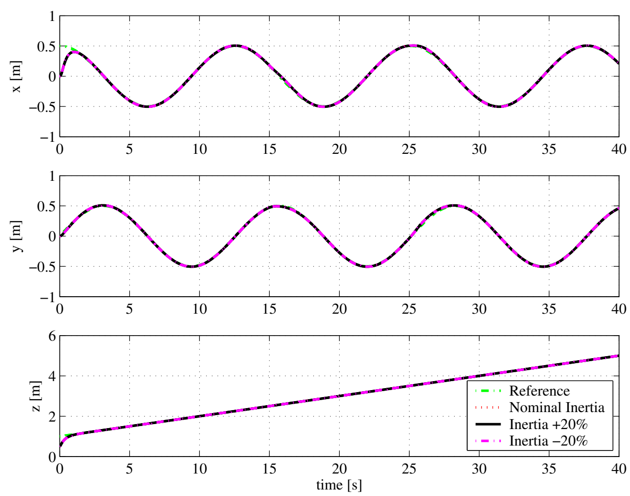


Fig. 3. Position $(x, y, z)$ with external disturbances.

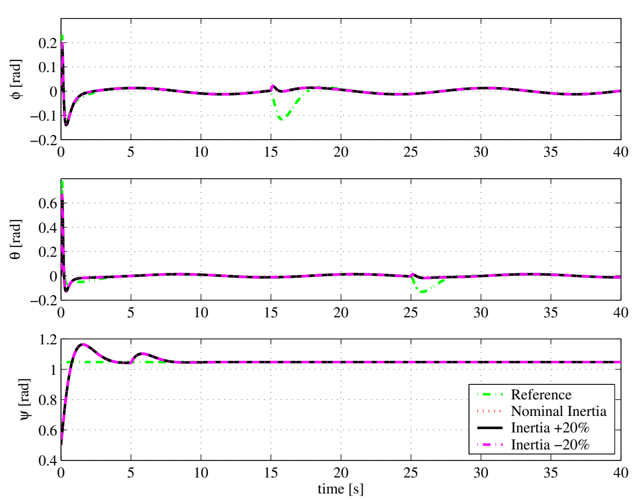


Fig. 4. Orientation $(\phi, \theta, \psi)$ with external disturbances.

rejected by this controller. Through the backstepping control for the linear movements a good performance in the reference tracking is achieved. The $\mathscr{H}_\infty$ controller robustness can be observed when uncertainty in the inertia terms is considered. Finally, the robustness of the presented control strategy has been corroborated in simulations.

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