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Improvements in Direct Lyapunov Stabilization of Underactuated, Mechanical Systems

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Abstract— **A Lyapunov direct method is presented for the stabilization of underactuated, mechanical systems. The Lyapunov approach provides the tools for control law design. This work represents a continued development of previously published techniques. The major contribution of the presentation is that a method is demonstrated for assuring the positive definiteness of certain matrices associated with the formulation. A stabilization example using the rotary inverted pendulum is included.**

I. Introduction

Underactuated mechanical systems form a challenging application area for control research. Examples of underactuated systems consist of aerial rockets, overhead crane cargo transport, hovercraft, and underwater vehicles. The control design difficulty stems from the nonlinear nature of the system dynamics and the underactuation which is characterized as having fewer actuators than mechanical degrees of freedom. Furthermore, some familiar control design schemes such as feedback linearization that work well for fully actuated systems are unsuccessful for underactuated systems. Linearization with pole placement or LQR work well in a neighborhood of the set point or equilibrium however, the basin of attraction may be limited.

Recent developments of control design approaches for underactuated systems include the Hamiltonian/Lagrangian based approaches. These developments as well as the results of this paper are applied to the stabilization of an unstable equilibrium. Owing to the similarity of the approach presented in this paper to Lagrangian/Hamiltonian methods, the literature will be reviewed in regard to work being done in this area. Notable contributions in the study of stabilizing underactuated mechanical systems have been made by Bloch, Leonard, and Marsden (2000, 2001) with their controlled Lagrangian design scheme, Auckly, Kapitanski, and White (2000) with the λ method, and Ortega, Spong, Gómez-Estern, Blankenstein (2002) in addition to Acosta, Ortega, Astolfi and Mahindrakar (2005) with interconnection damping assignment – passivity based control (IDA-PBC). Recent contributions to the IDA-PBC literature include Gómez-Estern, Van der Schaft, and Acosta (2004) together with Laila and Astolfi (2006). Another recent study is the Direct Lyapunov Approach (DLA) of White, Foss, and Guo (2006, 2007) which also contain more extensive literature reviews. All of these approaches rely on a matching equation solution method. The DLA application has shown promise in that its range of application appears to be larger than the Hamiltonian/Lagrangian methods which become intractable for systems having complicated dynamics or several degrees of freedom. The merits and limitations of these approaches are discussed further in the two cited DLA papers. The main concentration of the control design methods just mentioned as well as this paper is the stabilization of holonomic, underactuated systems.

The present paper continues the development of the DLA for more complicated systems. The DLA was first presented in White, Foss, and Guo (2006) and was applied to the stabilization of a class of systems characterized by dynamic equations where the nonlinearities depended on only one generalized coordinate and generalized velocity. The applications consisted of the inverted pendulum cart and the inertia wheel pendulum. Applying DLA to more complicated systems showed that certain matrices used in the formulation did not return to the original form after equilibrium was reached, a difficulty that altered the system dynamics during subsequent disturbances. This difficulty was addressed in White, Foss, and Guo (2007) where the formulation was changed so that a matrix associated with the kinetic energy ($\boldsymbol{K}_D$, defined in the Section II as the product of a matrix $\boldsymbol{P}$ and the dynamic system mass matrix $\boldsymbol{M}$) was made to return to the same form as equilibrium was approached. The resulting formulation was successfully applied to the stabilization of the ball and beam system. The formulation was such that the matrix $\boldsymbol{K}_D$ essentially stayed constant during the stabilization period.

Further testing of the approach showed that the procedure used to make $\boldsymbol{K}_D$ return to a nominal form also had the tendency to drive the rate of change of the candidate Lyapunov function to zero and in some cases even positive and thus limiting the basin of attraction for stabilization of the system. A better formulation of the problem addressed in White, Foss, and Guo (2007) will be presented in this paper where it will be seen that certain parameters will be

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introduced that preserve the sign of the candidate Lyapunov function rate of change.

This paper will present derivation of the DLA and the innovation used to improve the performance. Proofs of the viability of the approach are included in the development. An example involving a rotary, inverted pendulum is presented that illustrates the veracity of the formulation.

II. The Lyapunov Formulation and Matching Conditions

The application of the Lyapunov direct method to stabilization of mechanical systems possessing n degrees of freedom is done so that the state vector $\boldsymbol{x(t)} \in \Re^{2n}$ is driven to the state space origin, i.e. the zero vector (result derived from, perhaps, suitable axis translation). The derivation of the new controller follows, in many respects, that presented by White, Foss, and Guo (2007) resulting in three matching conditions, the solution of which determine the stabilizing controller. It will be seen in the development that the first matching condition differs from previous versions of this approach.

The motion of the mechanical system is governed by

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}+\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}+\boldsymbol{C}_D\dot{\boldsymbol{q}}+\boldsymbol{G}(\boldsymbol{q})=\boldsymbol{\tau} \tag{1}$$

where $\boldsymbol{q}\in \Re^n$ is the vector of generalized coordinates for the mechanical system, $\boldsymbol{M}(\boldsymbol{q}) \in \Re^{n\times n}$ is the symmetric, positive definite mass matrix, $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \in \Re^n$ consists of centripetal and corilis forces and/or torques, $\boldsymbol{C}_D \in \Re^{n\times n}$ is the symmetric, positive semi-definite, viscous damping coefficient matrix, and $\boldsymbol{G(q)} \in \Re^n$ consists of forces and/or moments stemming from gradients of conservative fields.

The candidate Lyapunov function is stated as

$$V(\boldsymbol{q},\dot{\boldsymbol{q}})=\frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{K}_D\dot{\boldsymbol{q}}+\Phi(\boldsymbol{q}) \tag{2}$$

where $V(\boldsymbol{q},\dot{\boldsymbol{q}})$: $\Re^{2n} \rightarrow \Re$ is the candidate Lyapunov function, $\Phi(\boldsymbol{q})$: $\Re^n \rightarrow \Re$ is a potential function, and $\boldsymbol{K}_D \in\Re^{n\times n}$ is a symmetric, positive definite matrix defined as

$$\boldsymbol{K}_D=\boldsymbol{P}(t)\boldsymbol{M}(\boldsymbol{q}) \tag{3}$$

where $\boldsymbol{P}(t) \in\Re^{n\times n}$ is a matrix chosen such that $\boldsymbol{K}_D$ has the stated properties. Lyapunov's equation (Chen, 1999) can be used to show that all of the eigenvalues of $\boldsymbol{P}$ have positive real parts. This important result will be used in the second matching condition.

The time derivative of the candidate Lyapunov function is computed as

$$\dot{V}=\dot{\boldsymbol{q}}^T\boldsymbol{K}_D\ddot{\boldsymbol{q}}+\frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{K}}_D\dot{\boldsymbol{q}}+\dot{\boldsymbol{q}}^T\nabla\Phi(\boldsymbol{q})=-\dot{\boldsymbol{q}}^T\boldsymbol{K}_V\dot{\boldsymbol{q}} \tag{4}$$

where the matrix $\boldsymbol{K}_V \in \Re^{n\times n}$ is symmetric and at least positive semi-definite and $\nabla\Phi(\boldsymbol{q})$ is the gradient of the potential with respect to the generalized positions. Owing to the nature of the right hand side of (4), LaSalle's theorem, as discussed by Khalil (2002), will be necessary to demonstrate asymptotic stability, however the right side of (4) is similar to the Hamiltonian formulations cited earlier. The result in (4) shows that $\dot{V}(\boldsymbol{q},\dot{q})$ is a non-positive function. Substituting for $\ddot{\boldsymbol{q}}$ from (1) into (4) produces

$$\begin{aligned}\dot{V}= {} & \dot{\boldsymbol{q}}^T\boldsymbol{K}_D\boldsymbol{M}(\boldsymbol{q})^{-1}\left(\left(-\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})-\boldsymbol{C}_D\right)\dot{\boldsymbol{q}}-\boldsymbol{G}(\boldsymbol{q})+\boldsymbol{\tau}\right)\\ & +\frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{K}}_D\dot{\boldsymbol{q}}+\dot{\boldsymbol{q}}^T\nabla\Phi(\boldsymbol{q}_d)=-\dot{\boldsymbol{q}}^T\boldsymbol{K}_v\dot{\boldsymbol{q}}.\end{aligned} \tag{5}$$

The strategy in solving (5) is through a matching equation approach by breaking (5) into three separate equations. The three matching conditions are developed in the coming sections.

Examination of (5) shows that there are two classes of terms that occur excluding the input. The first involves those terms that are pre and post multiplied by the generalized velocities. The other terms are pre-multiplied by a generalized velocity and involve a vector function of generalized coordinates (gravity terms and potential gradient) and these terms will give rise to the third matching condition. The first class of terms can be further divided into terms that are a function of the generalized velocities such as $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$ (these terms are found in the first matching condition) and terms that are constant such as $\boldsymbol{K}_v$ or $\boldsymbol{C}_D$ (terms comprising the second matching condition). The input vector $\boldsymbol{\tau}$ will be broken into three parts, one for each matching condition. Following these descriptions, (5) is written as three separate equations or matching conditions. The first matching condition is

$$\dot{\boldsymbol{q}}^T\boldsymbol{K}_D\boldsymbol{M}(\boldsymbol{q})^{-1}\left(-\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}+\begin{bmatrix}\boldsymbol{F}_1\\0\end{bmatrix}\right)+\frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{K}}_D\dot{\boldsymbol{q}}=0. \tag{6}$$

The second matching condition is given by

$$\dot{\boldsymbol{q}}^T\boldsymbol{K}_D\boldsymbol{M}(\boldsymbol{q})^{-1}\left(-\boldsymbol{C}_D\dot{\boldsymbol{q}}+\begin{bmatrix}\boldsymbol{F}_2\\0\end{bmatrix}\right)=-\dot{\boldsymbol{q}}^T\boldsymbol{K}_v\dot{\boldsymbol{q}}. \tag{7}$$

Finally, the third matching condition is provided by

$$\dot{\boldsymbol{q}}^T\boldsymbol{K}_D\boldsymbol{M}(\boldsymbol{q})^{-1}\left(-\boldsymbol{G}(\boldsymbol{q})+\begin{bmatrix}\boldsymbol{F}_3\\0\end{bmatrix}\right)+\dot{\boldsymbol{q}}^T\nabla\Phi(\boldsymbol{q})=0. \tag{8}$$

In (6) – (8), the input vector $\boldsymbol{\tau}$ has been broken into three terms given by

$$\boldsymbol{\tau}=\begin{bmatrix}\boldsymbol{F}_1\\0\end{bmatrix}+\begin{bmatrix}\boldsymbol{F}_2\\0\end{bmatrix}+\begin{bmatrix}\boldsymbol{F}_3\\0\end{bmatrix} \tag{9}$$

where the $\boldsymbol{F}_i$ is used in the i$^{\text{th}}$ matching condition. In (9), the m nonzero inputs will be placed in the first m rows of $\boldsymbol{\tau}$ while the last n-m rows will correspond to unactuated mechanical degrees of freedom. It should be realized that the sum of (6) – (8) observing (9) is the same as (5).

Each of the matching conditions and its corresponding solution will be treated in the following sections.

A. The First Matching Condition

Before developing a solution for the first matching condition, consideration is given to nature of this solution. The goal in solving the first matching condition is the determination of both the matrix $\boldsymbol{K}_D$ such that the matrix is symmetric and positive definite and the control input $\boldsymbol{F}_1$. Likewise, the goal of the second matching condition solution is both the matrix $\boldsymbol{K}_v$ such that it is symmetric and at least positive semi-definite and the control law contribution $\boldsymbol{F}_2$. The overall goal in satisfying the third matching condition is the determination of the potential Φ and the final portion of $\boldsymbol{\tau}$, namely $\boldsymbol{F}_3$. In examining (7) and (8), it is seen that the matrix $\boldsymbol{K}_D$ plays a role in satisfying the second and third matching conditions. Suppose a given system is at equilibrium and is subjected to a disturbance. If the matrix $\boldsymbol{K}_D$ does not return to the same form after equilibrium has been restored, then the response of the system will be different should the system be subjected to the same disturbance. That $\boldsymbol{K}_D$ must return to the same form as equilibrium is attained is a necessary requirement for the first matching condition.

The requirement concerning the limiting form of $\boldsymbol{K}_D$ as equilibrium is approached is satisfied by the inclusion of additional terms in both the first and second matching conditions. So that these additional terms do not alter (5), they will be subtracted from the first matching condition and added to the second matching condition. By including these new terms, (6) and (7) become

$$\dot{\boldsymbol{q}}^T \boldsymbol{K}_D \boldsymbol{M}(\boldsymbol{q})^{-1}\left(\left(-\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})-\overline{\boldsymbol{C}}_D-\boldsymbol{C}_D^{'}\right)\dot{\boldsymbol{q}}+\begin{bmatrix}\boldsymbol{F}_1\\0\end{bmatrix}\right) + \frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{K}}_D\dot{\boldsymbol{q}}=0 \qquad (10)$$

and

$$\dot{\boldsymbol{q}}^T \boldsymbol{K}_D \boldsymbol{M}(\boldsymbol{q})^{-1}\left(\left(-\boldsymbol{C}_D+\overline{\boldsymbol{C}}_D+\boldsymbol{C}_D^{'}\right)\dot{\boldsymbol{q}}+\begin{bmatrix}\boldsymbol{F}_2\\0\end{bmatrix}\right)=-\dot{\boldsymbol{q}}^T\boldsymbol{K}_v\dot{\boldsymbol{q}}, \qquad (11)$$

respectively, where $\boldsymbol{C}_D' \in \Re^{n\times n}$ and $\overline{\boldsymbol{C}}_D \in \Re^{n\times n}$ are symmetric matrices. Notice that the sum of (10) and (11) is the same as the sum of (6) and (7). The matrices $\overline{\boldsymbol{C}}_D$ and $\boldsymbol{F}_1$ will be used to drive $\boldsymbol{K}_D$ back to a specified form as equilibrium is attained. The matrix $\boldsymbol{C}_D$' is used with the second matching condition and further discussion of $\boldsymbol{C}_D$' will be deferred until that time.

By defining the first matching condition control input as

$$\begin{bmatrix}\boldsymbol{F}_1\\0\end{bmatrix}=\begin{bmatrix}\boldsymbol{F}_{m1}\\0\end{bmatrix}\dot{\boldsymbol{q}} \qquad (12)$$

where $\boldsymbol{F}_{m1} \in \Re^{m\times n}$ is a coefficient matrix yet to be determined, it can be seen that each term in the first matching condition is pre and post multiplied by the vector of generalized velocities. By introducing (12) into (10) and stripping off the generalized velocities, the first matching condition becomes

$$\boldsymbol{K}_D\boldsymbol{M}(\boldsymbol{q})^{-1}\left(-\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})-\overline{\boldsymbol{C}}_D-\boldsymbol{C}_D^{'}+\begin{bmatrix}\boldsymbol{F}_{m1}\\0\end{bmatrix}\right)+\frac{1}{2}\dot{\boldsymbol{K}}_D=0 \qquad (13)$$

which is a linear, differential equation for the matrix $\boldsymbol{K}_D$. When the leftmost matrix product of (13) (term in parentheses) is pre and post multiplied by the generalized velocity vector, the skew-symmetric part of that matrix vanishes. By setting the symmetric part of (13) to zero, then (10) will be automatically satisfied. Performing this operation provides

$$\begin{aligned}&\dot{\boldsymbol{K}}_D-\boldsymbol{K}_D\boldsymbol{M}(\boldsymbol{q})^{-1}\left(\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{C}_D^{'}\right)-\left(\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{C}_D^{'}\right)^T\boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{K}_D+\\&\boldsymbol{K}_D\boldsymbol{M}(\boldsymbol{q})^{-1}\left(-\overline{\boldsymbol{C}}_D+\begin{bmatrix}\boldsymbol{F}_{m1}\\0\end{bmatrix}\right)+\left(-\overline{\boldsymbol{C}}_D+\begin{bmatrix}\boldsymbol{F}_{m1}\\0\end{bmatrix}\right)^T\boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{K}_D=0.\end{aligned} \qquad (14)$$

The strategy for driving $\boldsymbol{K}_D$ to the same form as equilibrium is approached is to choose the elements of $\boldsymbol{F}_{m1}$ and $\overline{\boldsymbol{C}}_D$ so that the last two terms of (14) will equal

$$\begin{aligned}&\boldsymbol{K}_D\boldsymbol{M}(\boldsymbol{q})^{-1}\left(-\overline{\boldsymbol{C}}_D+\begin{bmatrix}\boldsymbol{F}_{m1}\\0\end{bmatrix}\right)+\\&\left(-\overline{\boldsymbol{C}}_D+\begin{bmatrix}\boldsymbol{F}_{m1}\\0\end{bmatrix}\right)^T\boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{K}_D=-\beta\left(\boldsymbol{K}_D-\boldsymbol{K}_{Df}\right)\end{aligned} \qquad (15)$$

where β is a negative constant and $\boldsymbol{K}_{Df}$ is the final form of the matrix $\boldsymbol{K}_D$. The matrix $\boldsymbol{K}_{Df}$ is a user supplied quantity. In solving (15), only the minimum number of elements in the two unknown matrices on the left of (15) is used. In order to satisfy (15), $n(n+1)/2$ equations can be written to determine the same number of unknowns. In (15), there are a total of $n(n+1)/2 + nm$ unknowns in the matrices $\overline{\boldsymbol{C}}_D$ and $\boldsymbol{F}_{m1}$. In fact, the matrix $\overline{\boldsymbol{C}}_D$ can be used exclusively to solve (15), however there is some advantage in using both of the arrays $\overline{\boldsymbol{C}}_D$ and $\boldsymbol{F}_{m1}$ to solve (15). By utilizing as much of the matrix $\boldsymbol{F}_{m1}$ as possible to build the right side (15), then fewer terms of $\overline{\boldsymbol{C}}_D$ (a term shared with the second matching condition) are needed. Substituting (15) into (14) yields

$$\begin{aligned}&\dot{\boldsymbol{K}}_D-\boldsymbol{K}_D\boldsymbol{M}(\boldsymbol{q})^{-1}\left(\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{C}_D^{'}\right)-\\&\left(\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{C}_D^{'}\right)^T\boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{K}_D-\beta\left(\boldsymbol{K}_D-\boldsymbol{K}_{Df}\right)=0\end{aligned}. \qquad (16)$$

In order for (16) to have the correct steady state solution, the term containing the matrices $\boldsymbol{C}_D$' and $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$ must vanish as equilibrium is approached. Because $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$ is linear in the generalized velocities, this is easily accomplished. In order for the vanishing of this term to be completely accomplished, $\boldsymbol{C}_D$' will have to be defined so that it approaches zero as equilibrium is approached. In the presentation of the second matching condition, this requirement will be included.

There are two advantageous parts of (16), the first of which is that the matrix $\boldsymbol{K}_D$ can be evaluated numerically as part of the feedback. The other advantageous part is that the matrix $\boldsymbol{K}_D$ can be made essentially constant by a suitably ample choice for the constant β. If $\boldsymbol{K}_D$ is constant, then the matrix $\boldsymbol{P}$ shown in (3) becomes a function of $\boldsymbol{q}$ alone as seen by

$$\boldsymbol{P}(\boldsymbol{q})=\boldsymbol{K}_D\boldsymbol{M}(\boldsymbol{q})^{-1}. \qquad (17)$$

In the sequel, the constant β will be assumed to be large enough so that (17) is true.

B. The Second Matching Condition

This section will present the solution of the second matching condition and in so doing will provide a definition for the matrix $\boldsymbol{C}_D$'. Let the forcing term of (11) be written as

$$\begin{bmatrix}\boldsymbol{F}_2\\0\end{bmatrix}=\begin{bmatrix}\boldsymbol{Fm}_2\\0\end{bmatrix}\dot{\boldsymbol{q}} \qquad (18)$$

where $\boldsymbol{Fm}_2$ is a $m \times n$ real matrix. Substituting (18) into (11), removing the pre and post multiplication by the generalized velocity vector, and simplifying shows

$$\boldsymbol{P}(\boldsymbol{q})\left(-\boldsymbol{C}_D+\overline{\boldsymbol{C}}_D+\boldsymbol{C}_D^{'}\right)+\boldsymbol{P}(\boldsymbol{q})\begin{bmatrix}\boldsymbol{Fm}_2\\0\end{bmatrix}=-\boldsymbol{K}_v. \qquad (19)$$

Because the quantity $\overline{\boldsymbol{C}}_D$ was used with $\boldsymbol{F}_{m1}$ in the first matching condition to construct $\beta(\boldsymbol{K}_D - \boldsymbol{K}_{Df})$, it has no intended purpose in the second matching condition. The matrix $\boldsymbol{C}_D$' is used for two purposes, the first of which is to remove $\overline{\boldsymbol{C}}_D$

from the second matching condition and the second purpose is to introduce "virtual" viscous damping. The intended purpose of the viscous damping is to facilitate energy removal and to vanish as equilibrium is attained, i.e. the dynamics of the system remain unaltered. Let the matrix C_D" contain the virtual damping where

$$C_D'' = -\left(\overline{C}_D + C_D'\right). \tag{20}$$

The matrix C_D" nominally consists of diagonal entries. Once C_D" is defined, (20) then provides a definition for C_D'.

In order to solve (19), we choose

$$\begin{bmatrix} Fm_2 \\ 0 \end{bmatrix}_1 = -P(q)^{-1} K_{v1} \tag{21}$$

for which the solution is

$$K_{v1} = \sum_{i=1}^{m} \alpha_i P_i P_i^T \tag{22}$$

where the α_i are constants chosen so that K_{v1} is positive semi-definite and P_i is the i[th] column of $P(q)$. Note that the matrix on the left side of (21) (part of the control law) is evaluated by extraction of the first m rows of the matrix product on the right side of (21). Applying (20) and (21) to (19) shows that

$$P(q)\left(-C_D - C_D''\right) - K_{v1} = -K_v. \tag{23}$$

The product of $P(q)$ and the viscous damping matrices is not symmetric, however, the pre and post multiplication by $\dot{q}$ extracts the symmetric portion of the product matrices. Thus, we require

$$\frac{1}{2}\left[P(q)\left(-C_D - C_D''\right) + \left(-C_D - C_D''\right)P(q)^T\right] = -K_{v2} \tag{24}$$

where $\mathbf{K}_{v2}$ is a symmetric, real matrix with non-negative eigenvalues. This last result in (24) can be recognized as Lyapunov's equation. From Section 5.4 of Chen (1999), if all of the eigenvalues of $P(q)$ has positive real parts and if the viscous damping matrices are symmetric and with non-negative eigenvalues, then $\mathbf{K}_{v2}$ is symmetric with non-negative eigenvlaues. The matrix $\mathbf{K}_v$ on the right side of (19) is always positive definite (semi-definite) and is given by

$$K_{v1} + K_{v2} = K_v. \tag{25}$$

C. The Third Matching Equation

From (8) we have

$$-P(t)G(q) + P(t)\begin{bmatrix} F_3 \\ 0 \end{bmatrix} + \nabla\Phi(q) = 0. \tag{26}$$

The first m equations in (26) are used to determine the control law contribution F_3 while the last $n - m$ rows of the equation provide linear, first order partial (ordinary) differential equations for the potential Φ. The last result in (26) is identical to previous DLA analyses.

In taking the time derivative of the candidate Lyapunov function, the potential Φ is assumed to be a function of the generalized positions q alone. In examining (18), it is seen that $P(t)$ appears in the equation leading to the conclusion that Φ also depends upon $P(t)$. If K_D is constant then (17) applies and P is a function of q alone. When viewed in this light, it is seen that the time derivative of the candidate Lyapunov function was correctly calculated.

III. The Evolution of K_D

The equations of motion can be expressed as

$$M\ddot{q} + (C + C_D)\dot{q} + G = F_1\dot{q} + F_2 + F_3 \tag{27}$$

where the three control law terms have been written in a more compact format. As before, the lower $n - m$ rows of the control law vectors are zero.

Define the candidate Lyapunov function as

$$V(\dot{q}, q) = \left[K_D(t)\dot{q}\right]\cdot\dot{q} + \Phi(q). \tag{28}$$

Then

$$\begin{aligned} V(\dot{q}, q) &= \frac{1}{2}\left[K_D\dot{q}\right]\cdot\dot{q} + \\ &\left\{K_D M^{-1}\left(\left[F_1 - C - C_D\right]\dot{q} + F_2 + F_3 - G\right)\right\}\cdot\dot{q} \\ &+ \frac{\partial}{\partial q}\Phi(q)\cdot\dot{q}. \end{aligned} \tag{29}$$

The first matching condition is

$$\begin{aligned} &\dot{K}_D - K_D M^{-1}\left(C + C_D'\right) - \left(C + C_D'\right)^T M^{-1} K_D^T \\ &+ K_D M^{-1}\left(-\overline{C}_D + F_1\right) + \left(-\overline{C}_D + F_1\right)^T M(q)^{-1} K_D^{\ T} = 0. \end{aligned} \tag{30}$$

We show that if K_D is positive definite at $t = 0$, then it is always positive definite.

Lemma 1. *Suppose that* $K_D : [0, +\infty) \to \Re^{n\times n}$ *is continuously differentiable and satisfies* $K_D(0) = K_D(0)^T$. *If the 1[st] matching condition holds for all* $t \in (0, +\infty)$, *then* K_D *is positive definite for all* $t \in (0, +\infty)$ *provided that* $K_D(0)$ *is positive definite.*

Proof. If K_D satisfies the 1[st] matching condition it is clearly symmetric.

Let $x : [0, +\infty) \to \Re^n$ be a normalized eigenvector of K_D with $\lambda : [0, +\infty) \to \Re$ the corresponding eigenvalue. Because K_D is symmetric and continuously differentiable, we may assume that x and λ are continuously differentiable.

Because $K_D(t)x(t) = \lambda(t)x(t)$ for all $t \in [0, +\infty)$, we find that

$$\frac{d}{dt}\left[K_D x\right] = \dot{K}_D x + K_D\dot{x} = \dot{\lambda}x + \lambda\dot{x}. \tag{31}$$

Also, $K_D(t)$ is symmetric and $\left\|x(t)\right\| = 1$ for all $t \in [0, +\infty)$, so

$$x\cdot\dot{K}_D x = \dot{\lambda}\left\|x\right\|^2 + \lambda x\cdot\dot{x} - x\cdot K_D\dot{x} = \dot{\lambda}\left\|x\right\|^2. \tag{32}$$

Multiplying the 1[st] matching condition by x, taking the inner product of the resulting vector with x and using the symmetry of K_D we deduce that

$$\begin{aligned} &x\cdot\dot{K}_D x - x\cdot K_D M^{-1}\left(C + C_D'\right)x - x\cdot\left(C + C_D'\right)^T M^{-1} K_D^T x + \\ &x\cdot K_D M^{-1}\left(-\overline{C}_D + F_1\right)x + x\cdot\left(-\overline{C}_D + F_1\right)^T M(q)^{-1} K_D^{\ T} x = 0 \\ &\Rightarrow \dot{\lambda} - 2\lambda x\cdot M^{-1}\left(C + C_D' + \overline{C}_D - F_1\right)x = 0. \end{aligned} \tag{33}$$

Define the function $\varphi : [0, +\infty) \to \Re$ by

$$\varphi(t) := 2\int_0^t x(r)\cdot M^{-1}\left(C + C_D' + \overline{C}_D - F_1\right)x(r)\,dr. \tag{34}$$

The general solution to the result of (33) may be written as

$$\lambda(t) = \lambda(0)e^{\varphi(t)}. \tag{35}$$

Because $\boldsymbol{K}_D(0)$ is positive definite, we necessarily have that $\lambda(0) > 0$, so we may conclude that $\lambda(t) > 0$ for all $t \in [0, +\infty)$. Hence, $\boldsymbol{K}_D(t)$ is positive definite provided that $\boldsymbol{K}_D(0)$ is. □

IV. Rotary Inverted Pendulum Example

The control law design method was applied to a rotary pendulum system. The system geometry together with the dynamic equations of motion are shown in Figure 1 with definitions of the physical parameters. The potential $\Phi(\theta,\phi)$ was found to be

$$\Phi(\theta,\phi) = g\ln\left(-2l\mathbf{K}_{D11} - 3r_o\mathbf{K}_{D12}\cos(\phi)\right)\frac{|\mathbf{K}_D|}{\mathbf{K}_{D12}} + \cosh\left(\theta + \frac{\mathbf{K}_{D22}}{\mathbf{K}_{D12}}\phi - 4l\tanh^{-1}\left(\frac{(-2l\mathbf{K}_{D11} + 3r_o\mathbf{K}_{D12})\tan\left(\frac{\phi}{2}\right)}{Den}\right)\frac{|\mathbf{K}_D|}{Den}\right) \tag{36}$$

where

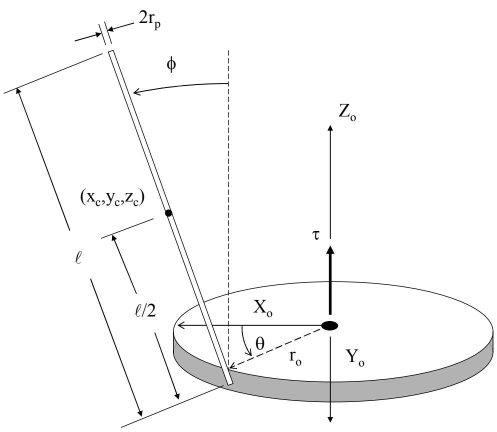
$$Den = \mathbf{K}_{D12}\sqrt{-4\mathbf{K}_{D11}{}^2 + 9r_o\mathbf{K}_{D12}{}^2}\,. \tag{37}$$

The matrix $C_D^{'}$ was defined as

$$\mathbf{C}_D^{'} = -\overline{\mathbf{C}}_D - \mathbf{C}_D^{''} = -\begin{bmatrix} 0 & 0 \\ 0 & \overline{\mathbf{C}}_{D22} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 10^{-8}\left(\dot{\mathbf{q}}^T\dot{\mathbf{q}} + \mathbf{q}^T\mathbf{q}\right) \end{bmatrix} \tag{38}$$

where

$$\overline{\mathbf{C}}_{D22} = -\frac{1}{2}\frac{\mathbf{P}_{2,1}^2\beta(\mathbf{K}_{D11} - \mathbf{K}_{D11})}{\mathbf{P}_{1,1}|\mathbf{P}|} + \frac{\mathbf{P}_{2,1}\beta(\mathbf{K}_{D12} - \mathbf{K}_{D12})}{|\mathbf{P}|} - \frac{1}{2}\frac{\mathbf{P}_{1,1}\beta(\mathbf{K}_{D22} - \mathbf{K}_{D22})}{|\mathbf{P}|}. \tag{39}$$



$$\mathbf{M}\begin{bmatrix}\ddot{\theta}\\ \ddot{\phi}\end{bmatrix} + (C + C_D)\begin{bmatrix}\dot{\theta}\\ \dot{\phi}\end{bmatrix} + \mathbf{G} = \begin{bmatrix}\tau\\ 0\end{bmatrix}$$

$$\mathbf{M} \equiv \begin{bmatrix} m_p r_o^2 + I_D + \frac{1}{3}m_p l^2\cos(\phi)^2 + \frac{1}{2}m_p r_p^2\cos(\phi)^2 & -\frac{1}{2}m_p l\cos(\phi)r_o \\ -\frac{1}{2}m_p l\cos(\phi)r_o & \frac{1}{3}m_p l^2 \end{bmatrix}$$

$$\boldsymbol{C} \equiv \begin{bmatrix} \sin(\phi)\cos(\phi)\dot{\phi}m_p\frac{1}{2}(l^2 - r_p^2) & \frac{1}{2}m_p r_o l\sin(\phi)\dot{\phi} + \sin(\phi)\cos(\phi)\dot{\theta}m_p(\frac{1}{3}l^2 - \frac{1}{2}r_p^2) \\ -\sin(\phi)\cos(\phi)\dot{\theta}m_p(\frac{1}{3}l^2 - \frac{1}{2}r_p^2) & 0 \end{bmatrix}$$

$$\boldsymbol{G} \equiv \begin{bmatrix} 0 \\ -\frac{m_p g\sin(\phi)l}{2} \end{bmatrix}$$

Figure 1 : Rotary Inverted Pendulum

The values of the physical parameters are r_p= .01 m, r_o= .2159 m, ℓ=.430 m, g= 9.81 m/s^2, m_r=.2 kg, m_p =.14 kg, $\boldsymbol{K}_{D11}$=.41622, $\boldsymbol{K}_{D12}$=-.74191, $\boldsymbol{K}_{D22}$=1.81207, β = -1000, and α_1=.0142 with the closed loop linearized system poles located at -10,-2,-3,-4.

Figures 2-7 show the response of the system at rest to an initial displacement of $\phi(0)$ = 0.65 radians. The chosen example could not be stabilized by the linear controller. The $\boldsymbol{K}_D$ matrix was evaluated through numerical integration of (16). Figure 4 shows that the elements of $\boldsymbol{K}_D$ are remaining essentially constant. Figure 5 shows that the $\boldsymbol{K}_v$ matrix is either positive definite or positive semi-definite. The behavior shown in Figures 6 and 7 demonstrate the validity of the Lyapunov candidate function.

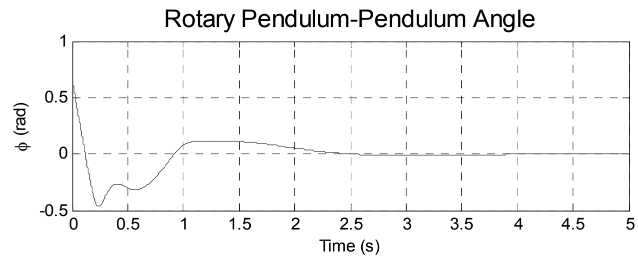


Figure 2 : Pendulum Position

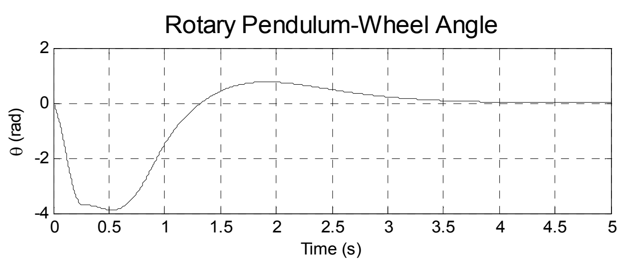


Figure 3 : Wheel Position

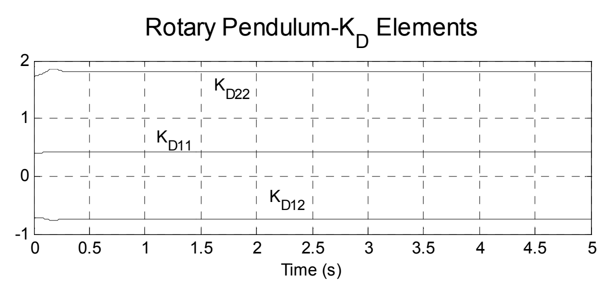


Figure 4 : Elements of KD Matrix

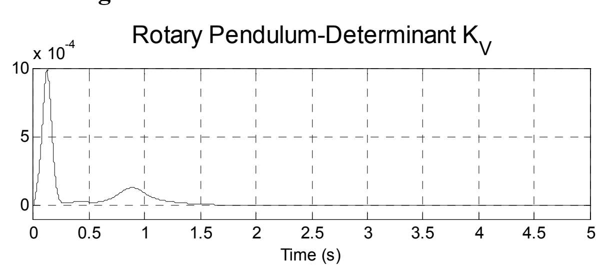


Figure 5 : Determinant of Kv Matrix

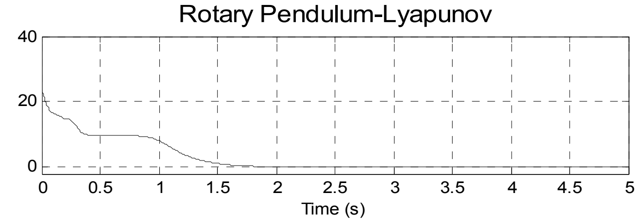


Figure 6 : Lyapunov Time History

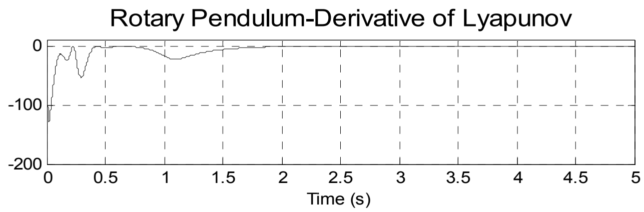


Figure 7 : Time Derivative of Lyapunov

V. CONCLUSION

A direct Lyapunov method for underactuated systems has been presented. The matching method results in three equations. The first matching condition results in linear ODEs for the matrix $\boldsymbol{K}_D$ that can be solved as part of the feedback and it was mathematically proved that the eigenvalues of $\boldsymbol{K}_D$ remain positive and bounded for all time. Also, the eigenvalues of $\boldsymbol{P}(\boldsymbol{q})$ all have positive real parts. In this formulation as was done in the previous one, $\boldsymbol{K}_D$ approaches a constant matrix. The first matching equation improvement also results in a modified second matching condition that still consists of linear algebraic equations. The major difference between this formulation and its predecessor is in the way the matrix $\boldsymbol{K}_D$ is made to approach its final value and in the way that the second matching condition is solved.

The second matching condition is formulated and solved in such a way that the matrix $\boldsymbol{K}_v$ always remains at least positive semi-definite. By Lyapunov's equation the inclusion of virtual positive viscous damping on the un-actuated axes provides a $\boldsymbol{K}_v$ matrix that is positive definite. In the previous formulation of the DLA, the way in which the $\boldsymbol{K}_D$ matrix was made to approach its final form could contribute to the $\boldsymbol{K}_v$ matrix becoming indeterminate, possibly leading to instability. In cases where there was no viscous damping present on the un-actuated axes, the previous formulation resulted in one eigenvalue of $\boldsymbol{K}_v$ exhibiting a small oscillation about zero. The formulation presented in this paper does not suffer such drawbacks. The presented simulation demonstrates expected performance.

The third matching condition produces $n - m$ linear PDEs for the potential Φ. The third matching condition presents the limitation on the basin of attraction. The basin of attraction in the example is limited by a singularity in the potential function, a singularity determined by the choice of constants for $\boldsymbol{K}_{Df}$. Further study of the third matching condition is necessary so that the basin of attraction can be expanded.

VI. REFERENCES

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