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Optimal Stable State-Space Partitioning for Piecewise Linear Planar Systems

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Abstract

A method of state-space partitioning that stabilizes piecewise linear systems is developed. Specifically, given available system matrices, partition of the two-dimensional state space is determined in a graphical manner. Finally, we show several illustrative examples to demonstrate efficacy of the proposed approach.

1. Introduction

Stabilization of piecewise linear and affine systems have been attracting much attention in the literature (see, for example, [1–4]). In particular, even 2-dimensional piecewise linear systems have rich characteristics. In this paper, first we assume that we are given (2×2)-dimensional system matrices that have complex eigenvalues and provide methodology of partitioning the state space that yields stable system trajectories with respect to the origin. Next, we extend the framework to the case where the system matrices contain real eigenvalues. In particular, two different kinds of optimality are discussed in terms of the convergence rates of stabilizable piecewise linear systems by state-space partition.

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^n$ denotes the set of $n\times 1$ real column vectors. Furthermore, we write $(\cdot)^{\mathrm{T}}$ for transpose, $\mathrm{spec}(A)$ for the spectrum of the matrix A, and $\|\cdot\|$ for the Euclidean vector norm.

2. Mathematical Preliminaries

In this section we introduce notation, several definitions, and some key results concerning 2-dimensional linear dynamical systems that are necessary for developing the main results of this paper. Specifically, consider the planar linear dynamical system given by

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad t \geq 0, \tag{1}$$

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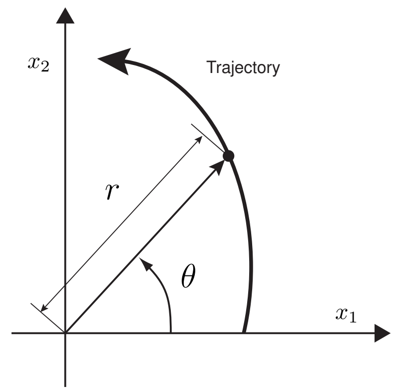

Figure 2.1: Polar form

where $x(t) = [x_1(t), x_2(t)]^{\mathrm{T}} \in \mathbb{R}^2$ is the state vector and $A \in \mathbb{R}^{2\times 2}$.

Now, consider the trajectory of (1) at the point x in the state space. Furthermore, consider the polar form (r,θ) of the coordinate (x_1, x_2) as shown in Figure 2.1, where r is the distance of x from the origin and θ is the angle (phase) from the positive x_1-axis in the counterclockwise direction.

2.1. Rotational Direction of Trajectories

The rotational direction of the trajectory of (1) at x can be determined by examining the sign of $\mathrm{d}\theta/\mathrm{d}t$; that is, $\mathrm{d}\theta/\mathrm{d}t > 0$ (resp., $\mathrm{d}\theta/\mathrm{d}t < 0$) implies that the trajectory of (1) is moving in the counterclockwise (resp., clockwise) direction at x. In fact, since

$$\begin{aligned}
\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\tan^{-1}\frac{x_2}{x_1}\Big) = \frac{-\dot{x}_1 x_2 + x_1\dot{x}_2}{x_1^2 + x_2^2} \\
&= \frac{1}{\|x\|^2}\det\begin{bmatrix} x_1 & \dot{x}_1 \\ x_2 & \dot{x}_2 \end{bmatrix} \\
&= \frac{1}{r^2}\det[x, Ax] \\
&= \det[\eta(\theta), A\eta(\theta)],
\end{aligned} \tag{2}$$

where $\eta(\theta) \triangleq [\cos\theta, \sin\theta]^{\mathrm{T}}$, the rotational direction of the trajectory of (1) at x can be determined by examining the sign of $\det[\eta(\theta), A\eta(\theta)]$. It is important to note that the sign of $\det[\eta(\theta), A\eta(\theta)]$ depends solely on

θ but r. Furthermore, in the case where A has complex conjugate eigenvalues, it follows that the sign of $\det[\eta(\theta), A\eta(\theta)]$ is invariant over θ.

2.2. Radial Growth Rate of Trajectories

First, note that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{x^{\mathrm{T}}x} = \frac{1}{r}x^{\mathrm{T}}\dot{x} = \frac{1}{r}x^{\mathrm{T}}Ax = r\eta^{\mathrm{T}}(\theta)A\eta(\theta). \quad (3)$$

Then it follows from (2) and (3) that the radial growth rate of the trajectories of (1) at x is characterized by

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\frac{\mathrm{d}r}{\mathrm{d}t}}{\frac{\mathrm{d}\theta}{\mathrm{d}t}} = \frac{r\eta^{\mathrm{T}}(\theta)A\eta(\theta)}{\det[\eta(\theta), A\eta(\theta)]}. \quad (4)$$

Since the rate of radial growth with respect to θ is proportional to the distance r from the origin, it follows that the 'normalized' radial growth rate with respect to θ defined by

$$\rho(\theta) \triangleq \frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\eta^{\mathrm{T}}(\theta)A\eta(\theta)}{\det[\eta(\theta), A\eta(\theta)]}, \quad (5)$$

depends solely on θ but r. Note that the function $\rho(\theta)$ is a periodic of period π, that is, $\rho(\theta+\pi) = \rho(\theta)$.

2.3. Integral of Radial Growth Rate

By integrating the radial growth rate given by (5) from θ_0 to θ_{f}, it can be examined how the distance of the trajectory of (1) is changed over $\theta_{\mathrm{f}} - \theta_0$. Specifically, suppose that the matrix A in (1) has complex conjugate eigenvalues and that the rotational direction of the trajectories is in the counterclockwise direction. In this case, assuming the initial distance of the trajectories is given by r_0, it follows from (5) that the distance r_{f} when the trajectory first intersects the semi-infinite straight line with phase θ_{f} satisfies

$$\int_{\theta_0}^{\theta_{\mathrm{f}}} \rho(\theta)\mathrm{d}\theta = \log\frac{r_{\mathrm{f}}}{r_0}. \quad (6)$$

If this value is positive (resp., negative), then it implies $\log r_{\mathrm{f}} > \log r_0$ (resp., $\log r_{\mathrm{f}} < \log r_0$) and hence the distance r_{f} from the origin is larger (resp., smaller) than the original point. Hence, assuming that $\theta_{\mathrm{f}} = \theta_0 + 2\pi$ and taking account of the fact that $\rho(\theta)$ is a periodic function of period π, it follows that examining the integral

$$\int_{\theta_0}^{\theta_0+2\pi} \rho(\theta)\mathrm{d}\theta = \int_0^{2\pi} \rho(\theta)\mathrm{d}\theta = \int_0^{2\pi} \frac{\eta^{\mathrm{T}}(\theta)A\eta(\theta)}{\det[\eta(\theta), A\eta(\theta)]}\mathrm{d}\theta, \quad (7)$$

leads us to determine stability of (1). Of course, the real part of the complex conjugate eigenvalue of A is negative (resp., positive), then (7) is negative (resp., positive).

Next, the following lemma states the duration when the trajectories travel from $\theta = 0$ to $\theta = 2\pi$.

Lemma 2.1. Consider the linear system given by (1). Then the time (period) T for the trajectories of (1) to travel from $\theta = \theta_0$ to $\theta = \theta_0 + 2\pi$ is given by

$$T = 2\int_0^{\pi} \frac{\mathrm{d}\theta}{\det[\eta(\theta), A\eta(\theta)]}, \quad (8)$$

which is independent of r.

Proof. The proof is immediate from (2) and

$$\begin{aligned} T &= \int_{\theta_0}^{\theta_0+2\pi} \frac{\mathrm{d}t}{\mathrm{d}\theta}\mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{\mathrm{d}t}{\mathrm{d}\theta}\mathrm{d}\theta \\ &= 2\int_0^{\pi} \frac{\mathrm{d}t}{\mathrm{d}\theta}\mathrm{d}\theta. \end{aligned} \quad (9)$$

□

Note that the period T does not depend on the initial position of the trajectory.

Finally, we present one of the ways to check whether the linear system (1) is stable.

Theorem 2.1. Consider the linear system given by (1), where A has complex conjugate eigenvalues and satisfy $\det[\eta(\theta), A\eta(\theta)] > 0$, $\theta \in [0, 2\pi)$. Furthermore, consider

$$\gamma \triangleq \int_0^{2\pi} \rho(\theta)\mathrm{d}\theta. \quad (10)$$

Then the following statements hold:

- If $\gamma < 0$, then (1) is globally exponentially stable;
- If $\gamma = 0$, then (1) is marginally stable and the trajectory of (1) constitutes a closed orbit;
- If $\gamma > 0$, then (1) is unstable.

This variable γ represents how far the trajectory is going to be from the origin when the trajectory travels and makes one round from a point with phase θ_0 back to another point with the same phase θ_0. Note that γ does not depend on θ_0.

3. Problem Formulation

In this section we introduce a problem of partitioning the state space in order for the piecewise linear systems of dimension 2 to be stable. Specifically, consider the piecewise linear system given by

$$\dot{x}(t) = A_i x(t), \quad x(t) \in \mathcal{D}_i, \quad x(0) = x_0, \quad t \geq 0, \quad i = 1, \ldots, k, \tag{11}$$

where $x(t) \in \mathbb{R}^2$ is the state vector, $A_i \in \mathbb{R}^{2\times 2}$, $i = 1, \ldots, k$, are system matrices that we are allowed to assign to the domain $\mathcal{D}_i$, and k is the number of domains (modes) which the state space is partitioned into. Here the domains $\mathcal{D}_i$, $i = 1, \ldots, k$, are assumed to satisfy

$$\bigcup_{i=1}^{k} \mathcal{D}_i = \mathbb{R}^n, \quad \mathrm{int}(\mathcal{D}_i \cap \mathcal{D}_j) = \emptyset, \quad i, j = 1, \ldots, k, \ i \neq j. \tag{12}$$

Furthermore, for each mode $\mathcal{D}_i$ the matrix A_i is assigned.

In this paper, we assume that the state space is partitioned by semi-infinite straight lines such that

$$\mathcal{D}_i = \bigcup_{j=1}^{m_i} \mathcal{S}_{ij}, \quad i = 1, \ldots, k, \tag{13}$$

$$\mathcal{S}_{ij} \triangleq \{x \in \mathbb{R}^2 \mid C_{ij} x \geq 0\}, \quad j = 1, \ldots, m_i, \tag{14}$$

where $C_{ij} \in \mathbb{R}^{2\times 2}$ characterizes the slopes of the semi-infinite straight lines.

In the following section, when the candidates of A_i's are given, we construct a framework to place the semi-infinite lines in the state space so that the resulting piecewise linear system is stable.

4. Piecewise Linear Systems with Complex Conjugate Spectrum

In this section we characterize the way we partition the state space and assign system dynamics to each of the partitioned domains. Specifically, in this section we assume that the available systems matrices A_i, $i = 1, \ldots, k$, have all complex conjugate eigenvalues. Furthermore, we assume that the rotational direction of the trajectories is in the counterclockwise direction so that we avoid intricate issues concerning existence and uniqueness of solutions of the piecewise linear system (11).

4.1. Stability Analysis

As discussed in Section 2, stability of simple linear systems can be determined by checking the sign of the integral (7). In this section we apply and extend the same idea to the piecewise linear system given by (11).

Let $\rho_i(\theta)$ denote the radial growth rate of the trajectory with respect to the system matrix A_i. Furthermore, let $I(\theta)$ be the function of the phase representing which mode is active, that is,

$$I(\theta) = i \quad \text{if} \quad \eta(\theta) \in \mathcal{D}_i. \tag{15}$$

Then the radial growth rate $\rho(\theta)$ for the piecewise linear system (11) is given by

$$\rho(\theta) = \rho_{I(\theta)}(\theta). \tag{16}$$

As in the simple linear systems case, if $\int_0^\pi \rho_{I(\theta)}(\theta)\mathrm{d}\theta < 0$ (resp., $\int_0^\pi \rho_{I(\theta)}(\theta)\mathrm{d}\theta > 0$), then the system (11) is exponentially stable (resp., unstable). Therefore, designing state-space partitioning and assigning system dynamics reduces to the problem of finding the function $I(\theta)$ such that

$$\int_0^\pi \rho_{I(\theta)}(\theta)\mathrm{d}\theta < 0. \tag{17}$$

Now we state the main result of the section.

Theorem 4.1. Consider the piecewise linear system given by (11), where A_i, $i = 1, \ldots, k$, have complex conjugate eigenvalues and satisfy $\det[\eta(\theta), A_i\eta(\theta)] > 0$, $i = 1, \ldots, k$. Then the piecewise linear system given by (11) is exponentially stable if and only if

$$\int_0^\pi \min_{I(\theta)}\{\rho_{I(\theta)}(\theta)\}\mathrm{d}\theta < 0. \tag{18}$$

Proof. When (18) holds, taking $\rho(\theta) = \min_{I(\theta)}\{\rho_{I(\theta)}(\theta)\}$ it follows that $\int_0^\pi \rho(\theta)\mathrm{d}\theta < 0$. Alternatively, when (18) does not hold, it follows that

$$\min_{I(\theta)}\left\{\int_0^\pi \rho_{I(\theta)}(\theta)\mathrm{d}\theta\right\} = \int_0^\pi \min_{I(\theta)}\{\rho_{I(\theta)}(\theta)\}\mathrm{d}\theta \geq 0, \tag{19}$$

which implies that for any $I(\cdot)$ we have $\int_0^{2\pi} \rho(\theta)\mathrm{d}\theta \geq 0$. Hence, it follows from Theorem 2.1 that there is no state-space partitioning such that resulting piecewise linear system is exponentially stable. □

5. State-Space Partitioning with Unstable System Matrices

In the preceding section we assume that the given system matrices A_i, $i = 1, \ldots, k$, have complex conjugate eigenvalues and their rotational direction matches with each other. In this section we relax these assumptions and extend the results to the case where the given system matrices consists of general 2×2 matrices.

One of the difficulties that may arise when we deal with matrices with real eigenvalues is the fact that the rotational direction (sign of $\frac{\mathrm{d}\theta}{\mathrm{d}t}$) is not invariant over the state space. Hence, it is required to predetermine which direction the trajectory is moving depending on the regions in the state space. As discussed in Section 2, the rotational direction of the trajectories of (11) is characterized by $\det[\eta(\theta), A_i\eta(\theta)]$.

First, consider the linear differential equation given by

$$\dot{x}(t) = A_i x(t), \quad x(0) = x_0, \quad t \geq 0, \tag{20}$$

and define a function which represents the direction of rotation given by

$$\xi_i(\theta) = \begin{cases} 1, & \text{if } \det[\eta(\theta), A_i\eta(\theta)] > 0, \\ -1, & \text{if } \det[\eta(\theta), A_i\eta(\theta)] < 0, \\ 0, & \text{if } \det[\eta(\theta), A_i\eta(\theta)] = 0. \end{cases} \tag{21}$$

Using the function $I(\theta)$ defined by (15), it follows that the necessary and sufficient condition for the trajectory of the piecewise linear system (11) to travel in the counterclockwise direction is given by

$$\xi_{I(\theta)}(\theta) = 1, \quad \theta \in [0, 2\pi). \tag{22}$$

5.1. Stabilizing State-Space Partitioning

In this section we present a procedure of determining how to partition the state space so that the piecewise linear system (11) is exponentially stable. Specifically, here we assume that we are given a set of system matrices A_i, $i = 1, \ldots, k$, which possess eigenvalues in the real and the complex fields. The following is the step-by-step procedure to construct a stable piecewise linear system. Once again, we assume that the available system matrices A_i, $i = 1, \ldots, k$, do not contain stable ones.

Step 1. For the matrices that have complex conjugate eigenvalues, classify them into the counterclockwise and the clockwise dynamics. For the matrices with real eigenvalues, determine the domains for each matrix that corresponds to the counterclockwise and the clockwise dynamics. It is important to note that as shown in Figure 5.1, rotating directions for a given linear systems with real eigenvalues are characterized by the eigenspaces of the system matrices A_i.

Step 2. Check whether there exists $I(\theta)$ such that (22) holds. If there is no such $I(\theta)$, then it is not possible to construct an exponentially stable piecewise linear system for (11).

Step 3. For each A_i, plot $\rho_i(\theta)$ for the domain associated with the counterclockwise rotational direction. Then consider $\min_{I(\theta)} \rho_{I(\theta)}(\theta)$. If there exists $\rho_{I(\theta)}(\theta)$ such that $\gamma \triangleq \int_0^{\pi} \rho_{I(\theta)}(\theta)\mathrm{d}\theta < 0$, then the corresponding state-space partitioning guarantees exponential stability of the piecewise linear system (11). If there does not exist such $\rho_{I(\theta)}(\theta)$, then examine **Steps 1–3** for the clockwise direction.

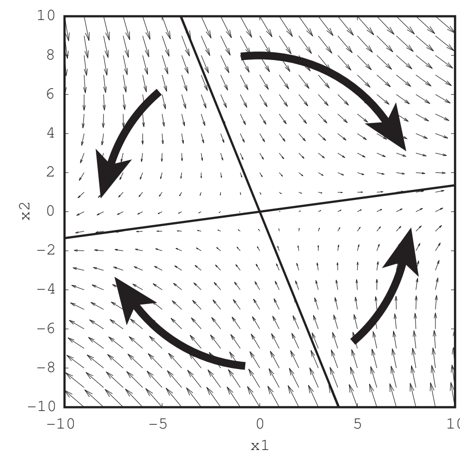

Figure 5.1: Vector field of a linear system with real eivenvalues. Straight lines represent the eigenspaces of the vector field. Between the eigenspaces, direction of rotation around the origin is uniform.

Step 4. If there does not exist $\rho_{I(\theta)}(\theta)$ which satisfies the condition in **Step 3**, then it is not possible to construct an exponentially stable piecewise linear system for (11).

6. Characterization of Convergence Rate of Piecewise Homogeneous Systems

By the procedure shown in the previous section, we may construct an exponentially stable piecewise linear systems. However, the proposed steps do not take into account any factor in terms of convergence rates. In this section, we characterize the speed of convergence of exponentially stable piecewise linear systems.

For the linear system $\dot{x}(t) = A_i x(t)$, let

$$\tau_i(\theta) \triangleq \frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{1}{\det[\eta(\theta), f_i(\eta(\theta))]}, \tag{23}$$

be the time growth rate at the phase θ. Note that this rate $\tau_i(\cdot)$ does not depend on time. Furthermore, let

$$\begin{aligned} \beta &\triangleq \frac{\log \gamma}{\int_0^T \mathrm{d}t} \\ &= \frac{\int_0^{2\pi} \rho_i(\theta)\mathrm{d}\theta}{\int_0^{2\pi} \tau_i(\theta)\mathrm{d}\theta}, \end{aligned} \tag{24}$$

where T is the time the trajectory takes to travel from phase 0 to 2π given by (8), is constant irrespective of initial conditions. With this β it follows that there exists $\alpha > 0$ such that the solution of (8) satisfies

$$\|x(t)\| \leq \alpha \|x_0\| e^{-\beta t}. \tag{25}$$

Now, consider an exponentially stable piecewise linear system (11). In this case, there exists $\beta > 0$ such that the piecewise linear system

$$\dot{x}(t) = \tilde{A}_i x(t), \quad x(t) \in \mathcal{D}_i, \quad x(0) = x_0, \quad t \geq 0, \quad i = 1, \ldots, k, \tag{26}$$

where $\tilde{A}_i \triangleq A_i + \beta I_2$, is marginally stable. The following theorem shows that such β is unique for a given exponentially stable piecewise linear system.

Lemma 6.1. Consider the piecewise linear system given by (11). Let $I(\theta)$ and β be such that

$$\int_0^{2\pi} (\rho_{I(\theta)}(\theta) + \beta\tau_{I(\theta)}(\theta))\mathrm{d}\theta = 0. \tag{27}$$

If β in (27) is positive, then the zero solution of (11) is exponentially stable and there exists $\alpha > 0$ such that (25) is satisfied.

Theorem 6.1. Consider the piecewise linear system given by (11). Then the function

$$q(\beta) \triangleq \int_0^{2\pi} \min_i(\rho_i(\theta) + \beta\tau_i(\theta))\mathrm{d}\theta, \tag{28}$$

is monotone increasing. Furthermore, if $q(0) < 0$, then there exists $\beta_{\max} > 0$ such that $q(\beta_{\max}) = 0$ and it is largest attainable $\beta > 0$ such that (27) is satisfied.

7. Illustrative Numerical Examples

In this section we present several numerical examples to demonstrate the utility of the proposed framework.

7.1. The Case for Matrices with Complex Conjugate Eigenvalues

Assume that we are given the following system matrices

$$A_1 = \begin{bmatrix} 0.4 & 1.0 \\ -5.0 & 1.0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.4 & 2.0 \\ -1.0 & 0.4 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0.1 & 1.0 \\ -1.0 & 0.1 \end{bmatrix},$$

where the eigenvalues of A_1, A_2, A_3 are $7.0 \pm 2.2i$, $4.0 \pm 1.4i$, and $1.0 \pm 1.2i$, respectively. Note that A_1, A_2, A_3 are all unstable. Furthermore, since $\det[\eta(\theta), A_i\eta(\theta)] > 0$, $i = 1, 2, 3$, it follows from (21) that

$$\xi_i(\theta) = 1, \quad \theta \in [0, 2\pi), \quad i = 1, 2, 3. \tag{29}$$

In addition, A_1, A_2, A_3 yield trajectories moving in the clockwise direction.

Next, plot $\rho_1(\theta)$, $\rho_2(\theta)$, and $\rho_3(\theta)$ as in Figure 7.1. Now, for each $\theta \in [0, \pi)$ choose the index from 1, 2,

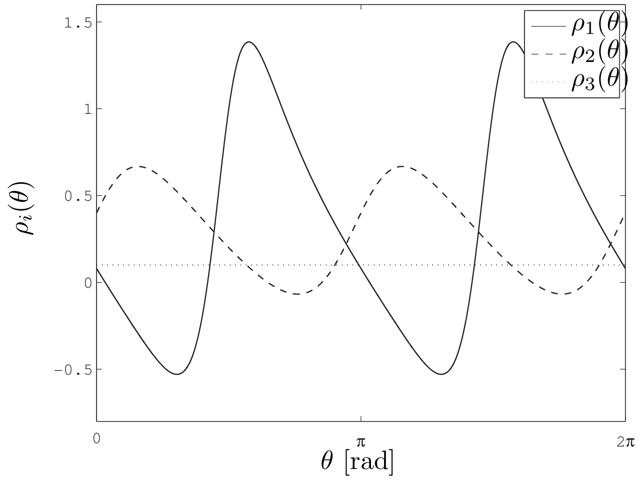

Figure 7.1: Normalized radial growth rate of A_1, A_2, and A_3

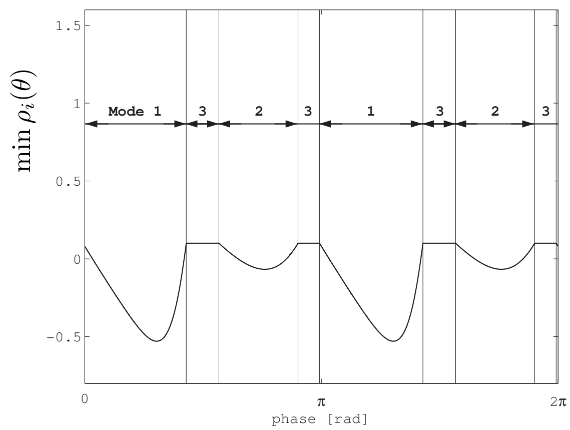

Figure 7.2: $\rho(\theta) = \min_{I(\theta)} \rho_{I(\theta)}(\theta)$

or 3; that is, $I(\theta) = \arg\min_{i=1,2,3}\rho_i(\theta)$. In this case, Figure 7.2 shows $\rho(\theta) = \min_{I(\theta)} \rho(\theta)$. Hence, since

$$\int_0^{\pi} \min_{I(\theta)}\{\rho_{I(\theta)}(\theta)\}\mathrm{d}\theta = -0.33 < 0, \tag{30}$$

it follows from Theorem 4.1 that the piecewise linear system shown in Figure 7.3 is exponentially stable.

7.2. The Case for Matrices with Real and Complex Conjugate Eigenvalues

Assume that we are given the following system matrices

$$A_1 = \begin{bmatrix} -1.1 & 1.4 \\ -10.7 & 8.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.0 & 8.0 \\ -2.0 & 3.0 \end{bmatrix},$$

where the eigenvalues of A_1 are 1.0, 6.0 and those of A_2 are $1.5 \pm 3.7i$. Note that A_1, A_2 are both unstable. Now, consider the domain in which the trajectories of A_1, A_2 moves in the clockwise direction. In this

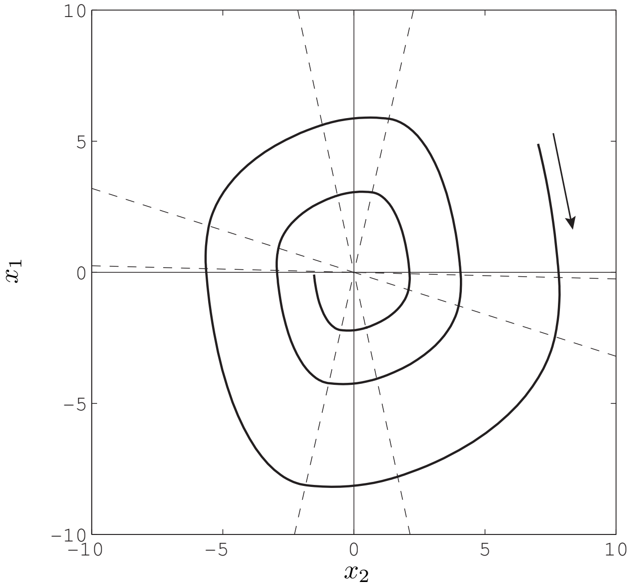

Figure 7.3: Phase portrait

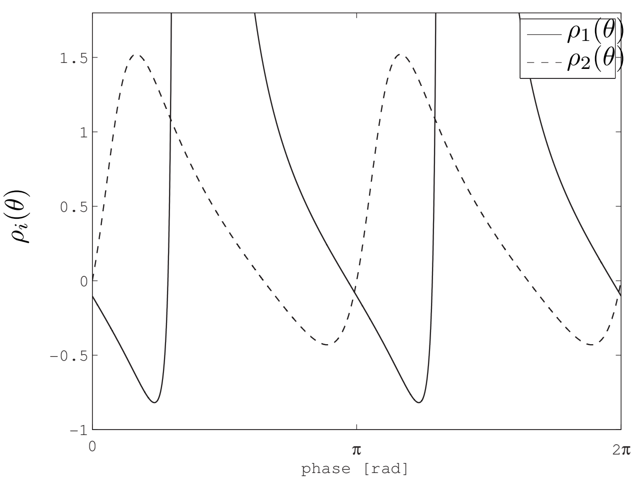

Figure 7.4: Normalized radial growth rate of A_1 and A_2

case, plot $\rho_1(\theta)$ and $\rho_2(\theta)$ as in Figure 7.4. Now, for each $\theta \in [0, \pi)$ choose the index from 1, 2, or 3; that is, $I(\theta) = \text{aug}\min_{i=1,2,3}\rho_i(\theta)$. In this case, Figure 7.5 shows $\rho(\theta) = \min_{I(\theta)} \rho(\theta)$. Hence, since

$$\int_0^{\pi} \min_{I(\theta)}\{\rho_{I(\theta)}(\theta)\}\mathrm{d}\theta = -0.21 < 0, \tag{31}$$

it follows from Theorem 4.1 that the piecewise linear system shown in Figure 7.6 is exponentially stable.

8. Conclusion

In this paper we propose the way to partition the state space so that with given available system matrices we construct an globally exponentially stable piecewise linear system. Future extensions include the stable state-space partitioning for piecewise affine systems.

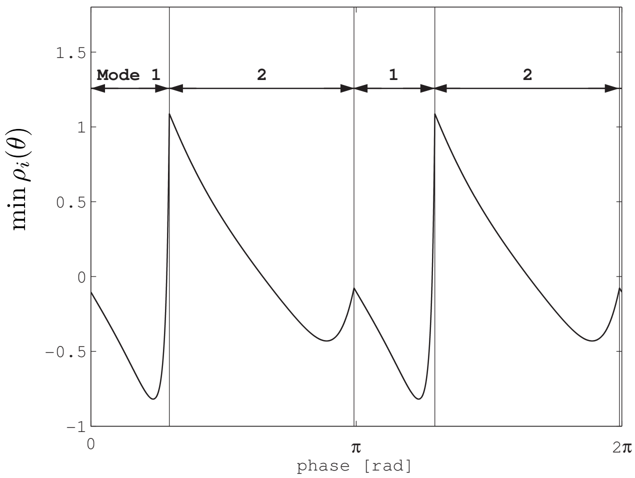

Figure 7.5: $\rho(\theta) = \min_{I(\theta)} \rho(\theta)$

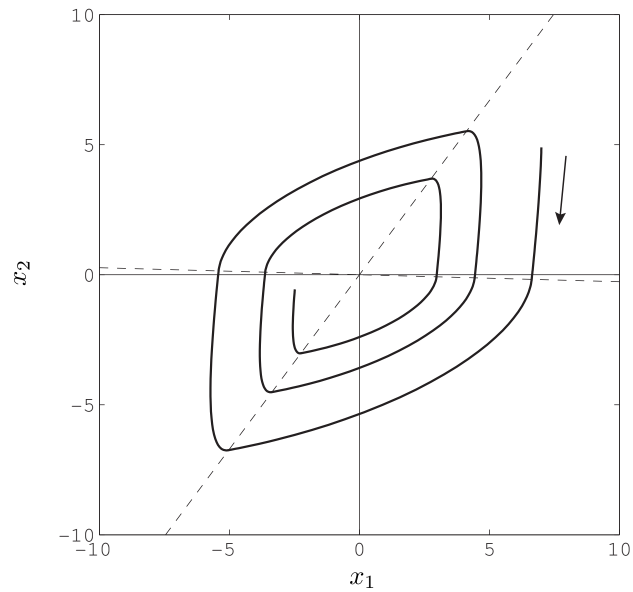

Figure 7.6: Phase portrait

References

[1] B. G. Romanchuk, "Computing regions of attraction with polytopes: Planar case," *Automatica*, vol. 32, no. 12, pp. 1727–1732, 1996.

[2] F. Giannakopoulos and K. Pliete, "Planar systems of piecewise linear differential equations with a line of discontinuity," *Nonlinearity*, vol. 14, no. 6, pp. 1611–1632, 2001.

[3] L. Rodrigues, "Stability analysis of piecewise-affine systems using controlled invariant sets," *Sys. Contr. Lett.*, vol. 53, no. 2, pp. 157–169, 2004.

[4] Y. Iwatani and S. Hara, "Stability tests and stabilization for piecewise linear systems based on poles and zeros of subsystems," *Automatica*, vol. 42, no. 10, pp. 1685–1695, 2006.