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Stability Robustness Conditions for Market Power Analysis in Industrial Organization Networks

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***Abstract*— Business networks provide one of the most compelling environments to study the conflicting effects of competition and cooperation on multi-agent dynamical systems. While firms engage various merger and divestiture strategies to create the desired cooperative environment that enhances their market power, governmental regulatory agencies enforce antitrust measures that protect competition as a means to limit the market power of these organizations. Merger simulation has subsequently evolved in recent years as a mechanism to study the impact of different organizational structures on the market. Nevertheless, typical economic models can often lead to competition dynamics that arbitrarily lose stability when considering different organizational structures. This work provides stability robustness conditions with respect to coalition structure for profit-maximizing dynamical systems with network demand, and partially convex utility. In particular, we show that stability of the coalition of all agents is sufficient to guarantee stability of all other coalition structures. These conditions are then leveraged to provide a systematic methodology for estimating a rich variety of demand systems that guarantee sensible stability results regardless of the structure of cooperation in the marketplace.**

I. Background: Firms, Market Power, and Merger Simulation

One of the most well-studied multi-agent systems is the marketplace. Market dynamics are governed by competition, nevertheless one of the most interesting features of the market is the spontaneous emergence of cooperation structures we call firms. Firms represent coalitions of agents that offset the computational limitations of individual agents to better compete for scarce resources. They orchestrate policies that attempt to drive profit-generating dynamics in the face of considerable uncertainty, both from the consumer market and from the competitive forces of other firms. Piloting a firm is one of the most interesting and difficult control problems we've encountered.

One way firms cope with market uncertainty is through growth. As firms deploy successful policies, they acquire capital that enable them to attract the cooperation of more agents in the marketplace. This can happen organically through the hiring of employees and the natural expansion of the firm's existing operations, or it can happen suddenly through mergers and acquisitions. Either way, such growth attempts to mitigate uncertainty by either entrenching the firm in the market niche known to have been previously successful, or by offsetting risk by diversifying the types of products or services the firm uses to compete for profits.

As firms generate wealth, they distribute a portion of it to their stake holders, who then engage the marketplace as consumers or investors of one kind or another. The ability of consumers to translate this wealth into an improved quality of life, however, depends significantly on the balance of power between firms in the marketplace. When firms are too strong, they do not have the incentive to innovate, and they can restrict the flow of existing goods and services to consumers unless premium prices are paid. When firms are too weak, they do not have the ability to innovate, nor do they generate the wealth their stake holders might otherwise have had to participate more fully as consumers or further investors in the marketplace. As a result, governments control the growth and strength of firms, either by stopping proposed mergers or by forcing firms to divide. This maintains competition as an effective force to limit the market power of firms, and it ideally creates resonance between the welfare of consumers and the welfare of investors that fuel growth.

At the heart of both the firm's growth strategy and the government's regulation strategy, then, lies the ability to measure a firm's market power. In 1997 the US Department of Justice and the Federal Trade Commission's released guidelines governing the regulation of mergers within the United States [1]. This, in turn, precipitated growing interest in the use of "merger simulations" to estimate the effects of proposed mergers or acquisitions [6], [7], [13], [2] and [3].

Merger simulations predict post-merger prices based on a demand model of the relationship between prices charged and quantities sold by the firms under investigation in the relevant market. Assumptions or models about supply issues are also incorporated into the simulation. Under a Bertrand model of pricing, every firm sets the prices of its brands to maximize its profits. Equilibrium results when no firm can unilaterally change its prices to improve its profits. Simulations compare pre-merger prices and profits with post-

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merger prices and profits to analyze the impact of the merger. “Reverse” simulations compare prices and profits of an existing firm with those resulting from the division of the firm into constitutive components, thereby measuring the “Value of Cooperation” achieved by the strategic positioning of the firm as the coalition of those particular components within the context of the larger market [9], [8], and [10].

In this way, Value of Cooperation can be viewed as a quantification of market power, and merger simulation can be thought of as a Value of Cooperation measurement on the post-merger firm. The presence of market power alone, however, is not necessarily illegal, nor is it sufficient to give the firm monopolistic power, as the firm would also need to create barriers of entry to prevent new firms from competing. Likewise, there may be other measures used to quantify the impact of market structure or industrial organization on market conditions. Nevertheless, such measures typically compare a property of an equilibrium of one market structure with that resulting from a different market structure, and are thus comparative static analyses that typically ignore dynamic issues.

Often, however, the demand models used in such simulations can lead to unstable equilibria, or even conditions where no equilibria exist at all for some market structures [3]. Such results are generally not the foreshadows of pending market doom should the right conspiracy be formed, but rather are simply dynamic limitations resulting from mathematical technicalities of the these models. None of the demand models typically used in economics, i.e. linear, log-linear (constant elasticity), logit, AIDS, and PCAIDS, guarantee the existence and stability of equilibria for all possible market structures.

Viewing the marketplace as a profit-maximizing multi-agent dynamical system (Section II), this work resolves these issues by providing stability robustness conditions with respect to coalition structure for such systems when these systems have a particular network demand structure (Section III). These conditions are then leveraged to provide a systematic methodology for empirically estimating a rich variety of AIDS-like demand systems from market data, using standard convex-optimization tools, that guarantee sensible stability results regardless of the structure of cooperation in the marketplace (Section IV).

II. MARKETS AS MULTI-AGENT SYSTEMS

Consider a market consisting of n products, each produced and controlled by a single product division. These product divisions are the constitutive agents in our multi-agent system, N, and they are arbitrarily ordered and numbered 1 to n. Following a Bertrand model of pricing, each agent has complete authority and control to price its product as it sees fit. The prices for all the products are public knowledge, known at any given time by all the agents, and denoted by the vector $x \in \mathbb{R}^n$. For convenience, we will assume that the prices are in units relative to the unit cost of production for each product. That is, x_i is the markup for product i.

We suppose that the aggregate effect of consumers in the market is given by a demand function, $q(x) : \mathbb{R}^n \to \mathbb{R}^n$, which characterizes how the quantity sold for each product varies with prices. Note that the demand, $q_i(x) : \mathbb{R}^n \to \mathbb{R}$, for product i depends, in general, not only on its own price, but on the prices of all the other products as well.

Each agent is equipped with a utility function that scores its reward as a function of the decisions of all the agents in the system. This utility function is a component of the market utility and is given by each product division’s profits:

$$U_i(x) = x_i q_i(x). \tag{1}$$

A firm, F, is a coalition of agents, represented as a subset of N. We allow the market to coalesce into $m \leq n$ firms, where every agent belongs to one and only one firm. Thus, the market structure, or industrial organization, $\mathscr{F} = \{F_1, F_2, ..., F_m\}$, is a partition of N. We will write $\mathscr{F}^{-1}(i)$ for the firm to which agent i belongs.

We associate with each firm an objective or profit function given by the sum of the utility functions of the agents belonging to the firm,

$$U_F(x) = \sum_{i \in F} U_i(x) = \sum_{i \in F} x_i q_i(x). \tag{2}$$

By associating with a firm, an agent agrees to adjust the prices of its product to maximize the total profits or objective of the firm, rather than simply maximize its own utility. Thus, all agents belonging to the same firm adopt a common objective and effectively surrender their pricing authority to the firm, allowing the firm to lose money by underpricing in one division in order to induce a greater demand and profit in another division.

Each agent therefore changes its price in the direction of the gradient of the objective of the firm to which it belongs;

$$\dot{x}_i = \frac{\partial U_F}{\partial x_i}(x) = \frac{\partial \left[\sum_{i \in F} U_i\right]}{\partial x_i} = \sum_{i \in F} \frac{\partial U_i}{\partial x_i}(x). \tag{3}$$

Substituting from (1) for the profit structure of an agent’s utility and writing them in vector notation, these dynamics become

$$\dot{x} = V_{\mathscr{F}}(x) = \left[D_{\mathscr{F}}\left(J_q^T(x)\right)\right] x + q(x), \tag{4}$$

where $J_q(x)$ is the Jacobian of the function $q(x)$, A^T denotes transpose of a matrix A, and $D_{\mathscr{F}}(A)$ is defined as: a) $d_{ij} = a_{ij}$ if $j \in \mathscr{F}^{-1}(i)$, and b) $d_{ij} = 0$ otherwise. Thus, if $\mathscr{F} = \{(1,2), 3\}$ and A were given by

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \text{then} \quad D_{\mathscr{F}}(A) = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

Given a market structure and a demand function, Equation (4) thus represents the profit-maximizing dynamics of the multi-agent system and becomes the central focus of our analysis.

Our stability robustness problem, then, is to find conditions under which we can guarantee existence, uniqueness and stability of the equilibrium of Equation (4) for all market structures $\mathscr{F} \in \Delta$, where Δ is the set of all partitions of N.

Example 1: Consider a market with three products with consumer demand given by:

$$\begin{bmatrix} q_1(x) \\ q_2(x) \\ q_3(x) \end{bmatrix} = \begin{bmatrix} -3 & -5 & 4 \\ -4 & -4 & 3 \\ 1 & 2 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 80 \\ 90 \\ 80 \end{bmatrix} \quad (5)$$

Note that the demand is linear, and based on the signs of coefficients in the demand function, we can see that Products (1 and 2) are compliments, while (1 and 3) and (2 and 3) are substitutes. That is to say, an increase in the price of Product 1 results in decreased sales of both Products 1 (as you would expect) and 2 (i.e it is a compliment to Product 1), but an increase of sales of Product 3 (i.e. it is a substitute for Product 1).

The utility functions of the constitutive agents, meaning the three product divisions that each control a single product, are thus given by

$$\begin{aligned} U_1 &= (-3x_1 - 5x_2 + 4x_3 + 80)x_1 \\ U_2 &= (-4x_1 - 4x_2 + 3x_3 + 90)x_2 \\ U_3 &= (x_1 + 2x_2 - 15x_3 + 80)x_3 \end{aligned} \quad . \quad (6)$$

Moreover, given any market structure $\mathscr{F}$, the profit-maximizing dynamics of this multi-agent system then become

$$\dot{x} = D_{\mathscr{F}}\left(\begin{bmatrix} -3 & -4 & 1 \\ -5 & -4 & 2 \\ 4 & 3 & -15 \end{bmatrix}\right)x + q(x). \quad (7)$$

Now, lets compare the market dynamics for two different industrial organizations. First, we will consider the organization where every product division is its own firm, $\mathscr{F} = \{1,2,3\}$. In this case, the dynamics become:

$$\dot{x} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -15 \end{bmatrix} x + q(x)$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -6 & -5 & 4 \\ -4 & -8 & 3 \\ 1 & 2 & -30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 80 \\ 90 \\ 80 \end{bmatrix} \quad (8)$$

It is easy to verify that this system has a stable equilibrium at $x = (8.91, 8.11, 3.50)$ dollars. The demand at this point becomes $q = (26.72, 32.42, 52.63)$ units sold per unit time, and the profits for each firm are $U = (238.07, 262.93, 184.21)$ dollars per unit time.

Now let's consider the organization where Divisions 1 and 2 merge to form a single firm. This market structure is given by $\mathscr{F} = \{(1,2),3\}$, and the corresponding dynamics become:

$$\dot{x} = \begin{bmatrix} -3 & -4 & 0 \\ -4 & -4 & 0 \\ 0 & 0 & -15 \end{bmatrix} x + q(x)$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -6 & -9 & 4 \\ -8 & -8 & 3 \\ 1 & 2 & -30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 80 \\ 90 \\ 80 \end{bmatrix} \quad (9)$$

We see that with these dynamics the system has an equilibrium at $x = (6.40, 6.08, 3.29)$ dollars, corresponding to the demand of $q = (43.56, 49.95, 49.21)$ units sold per unit time and profits for the two firms of $U = (582.48, 161.90)$ dollars per unit time. Nevertheless, this equilibrium point is unstable. As a result, these equilibrium values are never really attainable, the profits of \$582.48 for the merged firm can not actually be realized, because even small changes in prices will lead, according to this model, to a never ending price war that never converges. Note that there is no way to detect a priori that this particular market structure would be unstable with this particular demand system. The merger of Divisions 1 and 3, for example, corresponding to market structure $\mathscr{F} = \{(1,3),2\}$, is stable.

■

III. Stability Robustness Conditions

Example 1 demonstrates how otherwise reasonable models of market dynamics can fail when considering industrial organization issues. The demand model, which is of sufficient fidelity to address questions such as the complimentary/substitutive relationship between products, drives the prediction that one possible merger will result in prices going to infinity. In reality, such a merger would not result in continually increasing prices; this result is simply an artifact of the model we have chosen. As a result, we see that this model is simply inadequate to describe market dynamics under changes in market structure, at least for some structures.

Nevertheless, if a model breaks down for some market structures by predicting unstable equilibria (or the lack of any equilibria, as happens for constant-elasticity models), can it be trusted to yield accurate results for any market structure? Whatever simplifications in the model cause it to drastically fail for some market structures might degrade its representation of the true dynamics under other market structures. The only safe course is to identify models that have sufficient fidelity to yield sensible results for every possible market structure.

Note that verifying the fidelity of a proposed model by checking the stability properties for all possible market structures is intractable; the number of possible market structures grows worse than exponential with n, the number of products, and real markets can involve thousands of products. As a result, we need tractable robustness conditions that can guarantee existence, uniqueness and stability of equilibria regardless of market structure.

To generate such conditions, we begin by defining the quantities we will use to check stability robustness of the system (4). For notational convenience let $F(i) = \mathscr{F}^{-1}(i)$ denote the firm to which the i^{th} agent belongs. When introducing the lemmas, we will write $M_{m,n}(\mathbb{F})$ for the set of all $m \times n$ matrices whose entries are elements of the field $\mathbb{F}$, and we will abbreviate to $M_n(\mathbb{F})$ in the case of square matrices. For any square matrix $A \in M_n(\mathbb{C})$, we will denote its numerical range as $W(A) = \{x^*Ax \mid \|x\|_2 = 1\}$, and its spectrum as $\sigma(A)$. For a subset S of a vector space, we will write $\mathrm{co}(S)$ to denote its convex hull. For two subsets A and B of a group $(G,+)$, we write $A+B=\{x+y \mid x\in A, y\in B\}$.

Lemma 1: Given the system (4), the Jacobian of the system dynamics, $V_{\mathscr{F}}$, decomposes as:

$$J_{V_{\mathscr{F}}}(x)=\left[A(x)+D_{\mathscr{F}}(A^T(x))\right]+B_{\mathscr{F}}(x)+C_{\mathscr{F}}(x), \quad (10)$$

where $A(x)$, $B_{\mathscr{F}}(x)$, and $C_{\mathscr{F}}(x)$ are given as follows:

$$A(x): \quad A_{ii}(x)=\tfrac{1}{2}\textstyle\sum_{j=1}^n \frac{\partial^2 U_j}{\partial x_i^2}(x), \quad A_{ij}(x)=\frac{\partial^2 U_i}{\partial x_i \partial x_j}(x) \quad (11)$$

$$B_{\mathscr{F}}(x): \quad B_{ii}(x)=0, \quad B_{ij}(x)=\textstyle\sum_{k\in F(i)\setminus\{i,j\}} \frac{\partial^2 U_k}{\partial x_i\partial x_j}(x) \quad (12)$$

$$C_{\mathscr{F}}(x): \qquad C_{ii}(x)=-\textstyle\sum_{j\notin F(i)} \frac{\partial^2 U_j}{\partial x_i^2} \quad C_{ij}(x)=0 \quad (13)$$

Proof: The diagonal entries of $J_{V_{\mathscr{F}}}(x)$ are given by,

$$\begin{aligned} J_{ii}(x) &= \frac{\partial V_i}{\partial x_i}(x)=\sum_{j\in F(i)} \frac{\partial^2 U_j}{\partial x_i^2}(x) \\ &= \sum_{j=1}^n \frac{\partial^2 U_j}{\partial x_i^2}(x) - \sum_{j\notin F(i)} \frac{\partial^2 U_j}{\partial x_i^2}(x) \\ &= 2A_{ii}(x)+C_{ii}(x)=2A_{ii}(x)+B_{ii}(x)+C_{ii}(x). \quad (14)\end{aligned}$$

For $j\neq i$, the off-diagonal $J_{ij}(x)$ is given by,

$$\begin{aligned} J_{ij}(x) &= \frac{\partial V_i}{\partial x_j}(x)=\sum_{k\in F(i)} \frac{\partial^2 U_k}{\partial x_i\partial x_j}(x) \\ &= \sum_{k\in F(i)\cap\{i,j\}} \frac{\partial^2 U_k}{\partial x_i\partial x_j}(x)+\sum_{k\in F(i)\setminus\{i,j\}} \frac{\partial^2 U_k}{\partial x_i\partial x_j}(x) \\ &= \sum_{k\in F(i)\cap\{i,j\}} \frac{\partial^2 U_k}{\partial x_i\partial x_j}(x)+B_{ij}(x). \quad (15)\end{aligned}$$

When $j\in F(i)$, we then have

$$\begin{aligned} J_{ij}(x) &= \frac{\partial^2 U_i}{\partial x_i\partial x_j}(x)+\frac{\partial^2 U_j}{\partial x_i\partial x_j}(x)+B_{ij}(x) \\ &= A_{ij}(x)+A_{ji}(x)+B_{ij}(x) \\ &= A_{ij}(x)+A_{ji}(x)+B_{ij}(x)+C_{ij}(x). \quad (16)\end{aligned}$$

Otherwise, when $j\notin F(i)$, we then have

$$\begin{aligned} J_{ij}(x) &= \frac{\partial^2 U_i}{\partial x_i\partial x_j}(x)+B_{ij}(x) \\ &= A_{ij}(x)+B_{ij}(x)+C_{ij}(x) \quad (17)\end{aligned}$$

Therefore,

$$J_{V_{\mathscr{F}}}(x)=\left[A(x)+D_{\mathscr{F}}(A^T(x))\right]+B_{\mathscr{F}}(x)+C_{\mathscr{F}}(x). \quad (18)$$

■

Definition 1: The market structure consisting of a single firm, $\mathscr{F}=\{(1,2,...,n)\}$, that is, where all agents belong to the same coalition, is called the *Grand Structure*, denoted $\mathscr{G}$, and the associated firm is called the *Grand Coalition*, denoted G.

Lemma 2: Given by (13) and Definition 1, $C_{\mathscr{G}}(x)=0$.

Proof: This follows directly from the definition of $C_{\mathscr{F}}$ in (13), where the only nonzero elements are on the diagonal, and the diagonal elements become zero for the Grand Structure since all agents belong to the same firm. ■

Definition 2: A function $h(x):\mathbb{R}^n\to\mathbb{R}^m$ is said to have *network structure* if there exist functions $f_{ij}:\mathbb{R}^2\to\mathbb{R}$ such that

$$h_i(x)=\sum_{j=1}^n f_{ij}(x_i,x_j), \quad i=1,...,n. \quad (19)$$

Lemma 3: A demand function, $q(x):\mathbb{R}^n\to\mathbb{R}^n$, with network structure induces network structure on the market utility function, given by (1), and the objective function of each firm in the market, given by (2).

Lemma 4: If the utility function, $U(x)$, associated with system (4) has network structure, then $B_{\mathscr{F}}(x)=0$ for all market structures $\mathscr{F}$.

Proof: Network structure of $U(x)$ implies there exist functions $f_{ij}:\mathbb{R}^2\to\mathbb{R}$ such that $U_i(x)=\sum_{j=1}^n f_{ij}(x_i,x_j), \quad i=1,...,n$. Hence, for $k\notin\{i,j\}$,

$$\begin{aligned} \frac{\partial^2 U_k}{\partial x_i\partial x_j}(x) &= \sum_{l=1}^n \frac{\partial^2 f_{kl}(x_k,x_l)}{\partial x_i\partial x_j} \\ &= \frac{\partial^2 f_{ki}(x_k,x_i)}{\partial x_i\partial x_j}+\frac{\partial^2 f_{kj}(x_k,x_j)}{\partial x_i\partial x_j} \\ &= \frac{\partial}{\partial x_i}\frac{\partial f_{ki}(x_k,x_i)}{\partial x_j}+\frac{\partial}{\partial x_j}\frac{\partial f_{kj}(x_k,x_j)}{\partial x_i}=0.\end{aligned}$$

Therefore, following from (12), $B_{\mathscr{F}}(x)=0$ for all market structures $\mathscr{F}$. ■

Definition 3: A utility function $U(x):\mathbb{R}^n\to\mathbb{R}^n$ is said to be *partially convex* if,

$$\frac{\partial^2 U_j}{\partial x_i^2}(x)\geq 0 \quad \forall j\neq i, \quad \forall x\in\mathbb{R}^n. \quad (20)$$

Lemma 5: When utility functions of the system (4) are partially convex, $C_{\mathscr{F}}(x)$ is a negative semidefinite diagonal matrix.

Definition 4: An n-product market with profit-maximizing dynamics given by (4), with demand function $q(x):\mathbb{R}^n\to\mathbb{R}^n$ that has network structure, and with partially convex utility is said to be an *industrial organization network* for any market structure $\mathscr{F}$.

Definitions 1 through 4 equip the models we will use to represent market dynamics with the technical structure we will need to guarantee stability robustness for all industrial organizations. In particular, industrial organization networks provide a model class with sufficient fidelity to explore questions involving changes in market structure. The following lemma comes from various parts in [4].

Lemma 6: Let $A,B\in M_n(\mathbb{C})$.

(*i*) $W(A)$ is compact and convex.
(*ii*) $\mathrm{co}\big(\sigma(A)\big)\subseteq W(A)$.
(*iii*) $W(A+B)\subseteq W(A)+W(B)$.
(*iv*) A is normal $\Rightarrow \mathrm{co}\big(\sigma(A)\big)=W(A)$.

Lemma 7: For $A\in M_n(\mathbb{R})$,

$$\max W(A+A^T)=\max\mathfrak{Re}W(A)+\max\mathfrak{Re}W(A^T).$$

Proof: Essentially follows from the definition of numerical range,

$$\begin{aligned}
&\max W(A+A^T) = \max_{\|x\|_2=1, x\in\mathbb{C}^n} x^*(A+A^T)x \\
&= \max_{\|x\|_2=1, x\in\mathbb{C}^n} \left(x^*Ax + x^*A^Tx\right) = \max_{\|x\|_2=1, x\in\mathbb{C}^n} \left(x^*Ax + \overline{x^*Ax}\right) \\
&\qquad = 2\max_{\|x\|_2=1, x\in\mathbb{C}^n} \mathfrak{Re}(x^*Ax) = 2\max \mathfrak{Re}W(A).
\end{aligned}$$

Following the same reasoning, $\max W(A + A^T) = 2\max \mathfrak{Re}W(A^T)$, hence $\max W(A+A^T) = \max \mathfrak{Re}W(A) + \max \mathfrak{Re}W(A^T)$. ■

The following lemma is from [12].

Lemma 8: Given $f : \mathbb{R}^n \to \mathbb{R}^n$, the equation $f(x) = y$ will have exactly one root for each y if there exist positive $\varepsilon, R \in \mathbb{R}$ such that for all $x \in \mathbb{R}^n$, $\|x\|_2 > R$,

$$z^T \frac{\partial f}{\partial x}(x) z \le -\varepsilon \|z\|_2^2 \quad \forall z \in \mathbb{R}^n.$$

Corollary 1: Given $f : \mathbb{R}^n \to \mathbb{R}^n$, the equation $f(x) = y$ will have exactly one root for each y if there exists positive $\varepsilon \in \mathbb{R}$ such that,

$$\max \mathfrak{Re}W\left(\frac{\partial f}{\partial x}(x)\right) \le -\varepsilon \quad \forall x \in \mathbb{R}^n.$$

Proof: For all $z \in \mathbb{R}^n, \frac{z^T}{\|z\|_2}\frac{\partial f}{\partial x}(x)\frac{z}{\|z\|_2} \in W\left(\frac{\partial f}{\partial x}(x)\right)$, and also, $\frac{z^T}{\|z\|_2}\frac{\partial f}{\partial x}(x)\frac{z}{\|z\|_2} \in \mathbb{R}$, hence

$$\frac{z^T}{\|z\|_2}\frac{\partial f}{\partial x}(x)\frac{z}{\|z\|_2} \in W\left(\frac{\partial f}{\partial x}(x)\right) \cap \mathbb{R} \subseteq \mathfrak{Re}W\left(\frac{\partial f}{\partial x}(x)\right).$$

Thus, if $\max \mathfrak{Re}W\left(\frac{\partial f}{\partial x}(x)\right) \le -\varepsilon$ then,

$$\begin{aligned}
\frac{z^T}{\|z\|_2}\frac{\partial f}{\partial x}(x)\frac{z}{\|z\|_2} &\le \max \mathfrak{Re}W\left(\frac{\partial f}{\partial x}(x)\right) \le -\varepsilon \\
&\Rightarrow z^T \frac{\partial f}{\partial x}(x) z \le -\varepsilon \|z\|_2^2,
\end{aligned}$$

which satisfies the condition of Lemma 8. ■

Lemma 9: For matrix $A \in M_n(\mathbb{R})$, $W\big(D_{\mathscr{F}}(A)\big) \subseteq W(A)$.

Proof: For $F \subseteq \mathscr{F} = \{F_1, F_2, \ldots F_m\}$, let $I_F = \mathrm{diag}_{i=1}^n(\chi_F(i))$, with $\chi_F(\cdot)$ being the membership function of F. Note that

$$D_{\mathscr{F}}(A) = \sum_{k=1}^m I_{F_k} A I_{F_k}.$$

Let $w \in W\big(D_{\mathscr{F}}(A)\big)$ and let $x \in \mathbb{C}^n$ such $\|x\|_2 = 1$ and $w = x^* D_{\mathscr{F}}(A)x$. Since $\sum_{k=1}^m I_{F_k} = I$, $\sum_{k=1}^m I_{F_k}x = x$, hence

$$\begin{aligned}
1 = \|x\|_2^2 = x^*x &= \left(\sum_{k=1}^m I_{F_k}x\right)^* \sum_{l=1}^m I_{F_l}x \\
&= \sum_{k=1,l=1}^m x^* I_{F_k} I_{F_l} x = \sum_{k=1}^m x^* I_{F_k}^2 x = \sum_{k=1}^m \|I_{F_k}x\|_2^2.
\end{aligned}$$

Let $\mathscr{F}^+ = \{F \in \mathscr{F} : I_F x \neq 0\}$. For $F \in \mathscr{F}^+$, let $y_F = \frac{I_F x}{\|I_F x\|_2}$. Therefore $\|y_F\|_2 = 1$ and $I_F x = \|I_F x\|_2 y_F$.

$$\begin{aligned}
w = x^* D_{\mathscr{F}}(A)x &= x^*\left(\sum_{F\in\mathscr{F}} I_F A I_F\right)x = \sum_{F\in\mathscr{F}} x^* I_F A I_F x \\
&= \sum_{F\in\mathscr{F}} (I_F x)^* A (I_F x) = \sum_{F\in\mathscr{F}^+} (I_F x)^* A (I_F x) \\
&= \sum_{F\in\mathscr{F}^+} (\|I_F x\|_2 y_F)^* A (\|I_F x\|_2 y_F) \\
&= \sum_{F\in\mathscr{F}^+} \|I_F x\|_2^2 (y_F^* A y_F),
\end{aligned} \tag{21}$$

while $\sum_{F\in\mathscr{F}^+} \|I_F x\|_2^2 = \sum_{F\in\mathscr{F}} \|I_F x\|_2^2 = 1$. Therefore, w is a convex combination of $y_F^* A y_F$, which are in $W(A)$ because $\|y_F\|_2 = 1$. $W(A)$ is convex (Lemma 6) $\Rightarrow w \in W(A)$. ■

These lemmas demonstrate intermediate results that will enable us to provide stability robustness conditions for profit-maximizing dynamics under any coalition structure. In particular, Lemma 8 and Corollary 1 provide the machinery used to guarantee existence and uniqueness of an equilibrium for every market structure. To demonstrate stability of these equilibria using Lyapunov's indirect method, Lemma 1 provides a decomposition of the Jacobian of the system dynamics that simplify under certain technical assumptions. Lemmas 2-5 then invoke these technical assumptions to characterize an industrial organization network and simplify the expression for the Jacobian of its dynamics. Finally, Lemmas 6, 7, and 9 then yield the machinery to demonstrate how a simple check on the stability of the Grand Structure dynamics will guarantee stability for all other market structures. We now state and prove the stability robustness theorem.

Theorem 1: Consider an n-product market with agent set $N = \{1, 2, ..., n\}$ and an industrial organization network characterized by (4). Let the Grand Coalition, G, of this network be given as in Definition 1, with objective function, U_G, as specified in (2). Under these conditions, then (4) will have a unique and stable equilibrium for all $\mathscr{F} \in \Delta$, where Δ is the set of all partitions of N, if there exists positive $\varepsilon \in \mathbb{R}$ such that

$$\max \sigma\big(H(x)\big) \le -\varepsilon \quad \forall x \in \mathbb{R}^n, \tag{22}$$

where $H(x)$ is the Hessian matrix of the objective function $U_G(x)$.

Proof: Let $\mathscr{F}$ be an arbitrary market structure in Δ. Let $J_{V_{\mathscr{F}}}(x)$ be the Jacobian matrix of $V_{\mathscr{F}}(x)$ given by (4). Following from Lemma 1,

$$J_{V_{\mathscr{F}}}(x) = \big[A(x) + D_{\mathscr{F}}\big(A^T(x)\big)\big] + B_{\mathscr{F}}(x) + C_{\mathscr{F}}(x).$$

The network structure of demand, $q(x)$, and thus also of utility, $U(x)$, then imply that $B_{\mathscr{F}}(x) = 0$ as shown in Lemma 4. In the case that $\mathscr{F}$ is the Grand Structure, we know from Lemma 2 that $C_{\mathscr{G}}(x) = 0$. Thus, $J_{V_{\mathscr{G}}}(x) = A(x) + A^T(x) = H(x)$. In general, however, we have $J_{V_{\mathscr{F}}} = A(x) + D_{\mathscr{F}}(A^T(x)) + C_{\mathscr{F}}(x)$. From Lemma 6 this yields,

$$\begin{aligned}
W\left(J_{V_{\mathscr{F}}}(x)\right) &= W\left(A(x) + D_{\mathscr{F}}\left(A^T(x)\right) + C_{\mathscr{F}}(x)\right) \\
&\subseteq W\left(A(x)\right) + W\left(D_{\mathscr{F}}\left(A^T(x)\right)\right) + W(C_{\mathscr{F}}(x)).
\end{aligned}$$

From Lemma 9, $W\left(D_{\mathscr{F}}\left(A^T(x)\right)\right) \subseteq W\left(A^T(x)\right)$, hence

$$W\left(J_{V_{\mathscr{F}}}(x)\right) \;\subseteq\; W\left(A(x)\right) \,+\, W\left(A^T(x)\right) \,+\, W\left(C_{\mathscr{F}}(x)\right).$$

As a result,

$$\begin{aligned}\max \Re\mathfrak{e} W\left(J_{V_{\mathscr{F}}}(x)\right) \leq \max \Re\mathfrak{e} W\left(A(x)\right) + \Re\mathfrak{e} W\left(A^T(x)\right) \\ + \max \Re\mathfrak{e} W\left(C_{\mathscr{F}}(x)\right).\end{aligned}$$

Due to Lemma 5, $W\left(C_{\mathscr{F}}(x)\right) \leq 0$. Also, from Lemma 7,

$$\begin{aligned}\max \Re\mathfrak{e} W\left(A(x)\right) + \Re\mathfrak{e} W\left(A^T(x)\right) = \max W\left(A(x) + A^T(x)\right) \\ = \max W\big(H(x)\big) = \max \sigma(H(x)).\end{aligned}$$

Following that,

$$\max \Re\mathfrak{e} W\left(J_{V_{\mathscr{F}}}(x)\right) \leq \max \sigma(H(x)) \leq -\varepsilon.$$

By Corollary 1, we can conclude that the equation $V_{\mathscr{F}}(x) = 0$ has exactly one solution x_e. Hence the market structure $\mathscr{F}$ yields exactly one equilibrium x_e. Moreover, since the Jacobian evaluated at the equilibrium point,$J_{V_{\mathscr{F}}}(x_e)$, satisfies,

$$\max \Re\mathfrak{e}\sigma(J_{V_{\mathscr{F}}}(x_e)) \leq \max \Re\mathfrak{e} W(J_{V_{\mathscr{F}}}(x_e)) \leq -\varepsilon < 0,$$

then the equilibrium x_e is locally stable due to Lyapunov's indirect method. ■

IV. Demand Estimation for Industrial Organization Networks

This section shows how we apply the stability robustness condition in Theorem 1 to a class of AIDS-like demand models. We will begin to cover first our main tool, semidefinite programming [11] [5], used in finding the model parameters that best fit the data, while meeting the sufficient condition given in Theorem 1.

A. Semidefinite Programming

In semidefinite programming, one minimizes a convex function subject to the constraint that an affine combination of symmetric matrices is positive semidefinite. As the authors of [11] noted, such a constraint is nonlinear and nonsmooth, but convex. In fact, it is shown in [11] that although semidefinite programs are much more general than linear programs, they are not much harder to solve. Most interior-point methods for linear programming have been generalized to semidefinite programs. As in linear programming, these methods have polynomial worst-case complexity, and perform very well in practice.

Let us show the canonical form of a semidefinite program,

$$\begin{aligned}&\text{minimize} \quad f_0(x) \\ &\text{subject to} \quad \sigma\left(\Psi_0 + \sum_{i=1}^{n} \Psi_i x_i\right) \leq 0,\end{aligned} \tag{23}$$

where $f_0(x)$ is convex and Ψ_i are symmetric for $i = 0, 1, \ldots, n$.

B. Demand Estimation with Stability Robustness Constraint

Now we will show our methodology applying to a class of demand models. Let us first do so by describing our model, after which we shall show that both the requirements given in Definition 2 and Definition 3 are met. This demand model is based on the concept of effective price: we recognize that changing prices from different price ranges will yield different effects on demand. Therefore, let $f_i(x_i)$ be a function representing the effective price of product i, our demand function will be,

$$q = Pf(x) + b, \text{where } f(x) = \big(f_1(x_1), f_2(x_2), \ldots, f_n(x_n)\big). \tag{24}$$

1) Use demand functions given by Equation (24): Let us show that this demand model given in (24) satisfies all the assumptions of an industrial organization network. First, it can be shown that the network assumption in Definition 2 is met, because

$$q_i(x) = \sum_{j=1}^{n} p_{ij} f_j(x_j). \tag{25}$$

Also the partially convex requirement, as defined in Definition 3, is met. For $j \neq i$,

$$\begin{aligned}\frac{\partial^2 U_j}{\partial x_i^2}(x) = \frac{\partial^2 [x_j q_j(x)]}{\partial x_i^2} = x_j \frac{\partial}{\partial x_i} \frac{\partial q_j(x)}{\partial x_i} \\ = x_j \frac{\partial}{\partial x_i}\left(p_{ji} \frac{\partial f_j(x_j)}{\partial x_i}\right) = 0 \geq 0.\end{aligned} \tag{26}$$

2) Use splines to design the effective price functions, f(x), in the demand model : These functions should be monotone and will serve as basis functions in a nonlinear regression when fitting P and b from data. The choice of f(x) can be guided by data or use professional expertise to characterize price sensitivity in the market.

3) Substitute the desired effective price functions to build a semidefinite program. Note that this program samples $H(x)$ to try to enforce that $\sigma(H(x)) \leq -\varepsilon$ everywhere: This is the most important step in the process. Let us be detailed in showing how it is carried out. Assuming that we are given K data points $(\mathbf{q_i}, \mathbf{x_i})$, $i = 1, 2, \ldots K$, where $\mathbf{q_i} \in \mathbb{R}^n$ are quantity demanded at a price setting $\mathbf{x_i} \in \mathbb{R}^n$, our objective is to minimize the regression error. For example, if the regression error is measured by the l_2 norm, then we have a least square regression problem,

$$\begin{aligned}&\text{find} \quad P \in M_n(\mathbb{R}), b \in \mathbb{R}^n \quad \text{to} \\ &\text{minimize} \quad \sum_{i=1}^{K} \|P\mathbf{x_i} + b - \mathbf{q_i}\|_2^2.\end{aligned} \tag{27}$$

In addition, we need to ensure that the condition (22) is met. This condition needs to be held for an infinite number of $x \in \mathbb{R}^n$. However, by looking carefully at,

$$\begin{aligned}H(x) = \frac{\partial^2 U_G}{\partial^2 x} = \left[P \operatorname{diag}\left(\frac{df_i}{dx_i}(x)\right) + \operatorname{diag}\left(\frac{df_i}{dx_i}(x)\right) P^T\right] \\ + \operatorname{diag}\left(\sum_j p_{ij} x_j \frac{d^2 f_i}{dx_i^2}(x)\right),\end{aligned} \tag{28}$$

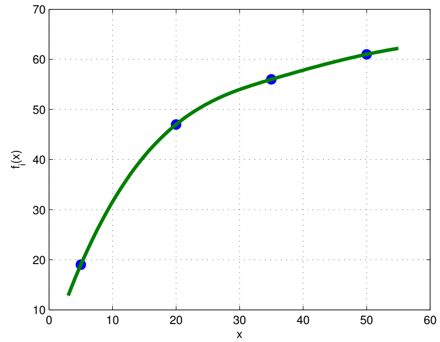

Fig. 1. Plot showing price sensitivity: a spline going through points $(5, 19)$, $(20, 47)$, $(35, 56)$, and $(50, 61)$.

we recognize that if we require that the effective price functions are linear for $x \in \mathscr{C}_n(R) = \{\mathbb{R}^n, \|x\|_2 > R\}$ for some radius R, then, $H(x)$ is unchanged for $x \in \mathscr{C}_n(R)$. Therefore we only need to meet the constraint (22) for a compact ball $x \in \mathscr{B}_n(R) = \{x \in \mathbb{R}^n \mid \|x\|_2 \le R\}$. In fact, we will make one step further by sampling the points in this ball, so that the number of points to check is finite. This is often done in practice. So, let $S = \{\mathbf{s_j}\}$ be a finite sample of $x \in \mathscr{B}_n(R)$, constraint (22) can be approximated by,

$$\max \sigma\big(H(\mathbf{s_j})\big) \le -\varepsilon \quad \forall \mathbf{s_j} \in S, \tag{29}$$

If we let $\mathbf{y} = \begin{bmatrix} p_{11} & \dots p_{1n} & b_1 \dots p_{n1} & \dots & p_{nn} & b_n \end{bmatrix}^T$, $\Pi \in M_{Kn,n^2+n}(\mathbb{R})$, $\Pi = \operatorname{diag}(\Sigma, \Sigma, \dots, \Sigma)$, where $\Sigma \in M_{K,n+1}$,

$$[\Sigma]_i = \begin{bmatrix} f_1(x_1) & \dots f_n(x_i) & 1 \end{bmatrix},$$

$\mathbf{z} = \begin{bmatrix} q_{11} & \dots & q_{K1} & \dots q_{1n} & \dots q_{Kn} \end{bmatrix}^T$, and $l = n^2 + n$, then the regression objective becomes,

$$\begin{aligned} \text{find} \quad & \mathbf{y} \in \mathbb{R}^l \quad \text{to} \\ \text{minimize} \quad & \|\Pi \mathbf{y} - \mathbf{z}\|_2^2. \end{aligned} \tag{30}$$

Also, let $\Phi_{ij}(\mathbf{s}) \in M_n(\mathbb{R})$, $\Phi_{ij}(\mathbf{s}) = \operatorname{diag}_{i=1}^n(\sum s_j \frac{\partial^2 f_i}{\partial x_i}(s_i))$, $\Theta_{ij}(\mathbf{s}) \in M_n(\mathbb{R})$ having two non-zero $(i,j)^{th}$ and $(j,i)^{th}$ entries with value $\frac{\partial f_i}{\partial x_i}(s_i)$, $\Psi_{ij} \in M_{n\|S\|}(\mathbb{R})$, $\Psi_{ij} = \operatorname{diag}_{\mathbf{s_k} \in \mathbf{S}}[\Phi_{ij}(\mathbf{s_k}) + \Theta_{ij}(\mathbf{s_k})]$, and $\Psi_0 \in M_{n\|S\|}(\mathbb{R}), \Psi_0 = \operatorname{diag}(\varepsilon, \varepsilon, \dots \varepsilon)$, then the regression constraint becomes

$$\text{subject to} \quad \max \sigma \left[\Psi_0 + \sum_{i=1,j=1}^{n} \Psi_{ij} y_{(n+1)i+j} \right] \le 0. \tag{31}$$

(30) and (31) together constitute a semidefinite program.

4) Solve for **y** *- or equivalently - P and b :* Solving the least square semidefinite program in (30) and (31) yields the network demand function, $q(x)$, that best fits the data, and guarantees that the conditions from Theorem 1 on $H(x)$ that guarantee stability robustness for all market structures are met.

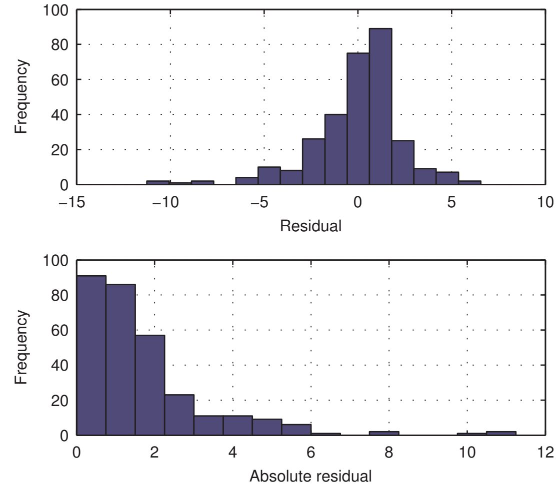

Fig. 2. Plot showing the histogram of residuals. The top one is the histogram residual values, while the bottom shows a histogram of absolute residuals. These results indicate that the industrial organization network model fits the data extremely well.

C. Numerical Experiment

Consider 100 data points generated by the log-linear model,

$$\log q(x) = \begin{bmatrix} -0.57 & 0.10 & -0.12 \\ 0.20 & -1.00 & 0.11 \\ -0.02 & 0.06 & -0.68 \end{bmatrix} \log x + \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix} + w, \tag{32}$$

where w is white noise with standard deviation 1. We choose $f_i(\cdot)$ to be the same function for each dimension: a spline going through $(5, 19)$, $(20, 47)$, $(35, 56)$, and $(50, 61)$ (we chose these points by looking at the generated data, and roughly estimating the effects of different price ranges on demand.) A plot showing this spline is shown in Figure 1.

Based on this spline, we perform a semidefinite regression to fit the demand function $q = Af(x) + b$ while meeting the robustness condition. The optimal parameters become:

$$q(x) = \begin{bmatrix} -5.70 & 0.96 & -1.23 \\ 1.96 & -10.00 & 1.17 \\ -0.24 & 0.59 & -6.82 \end{bmatrix} f(x) + \begin{bmatrix} 481.22 \\ 636.45 \\ 563.00 \end{bmatrix}. \tag{33}$$

These matrices do not look quite the same as the matrices in the original model because our regression model is not in logarithm scale. To see how our model fits the demand, we plot of percentage difference of demands between our regression model and the log-linear model in Figure 2. Since we have 100 data points, and each data point reflects the demand of three different products, we show in our plot the histogram of 300 differences, and the histogram of 300 absolute error. While the demanded quantities range between 150 and 400 units, the differences range between 0 and 12 units. For 90% of the data points, the difference is less than 1.5 percent. The maximal difference is about 3.5 percent. Our model fits the data quite well, but more importantly, it guarantees existence, uniqueness, and stability of equilibriums under all market structures.

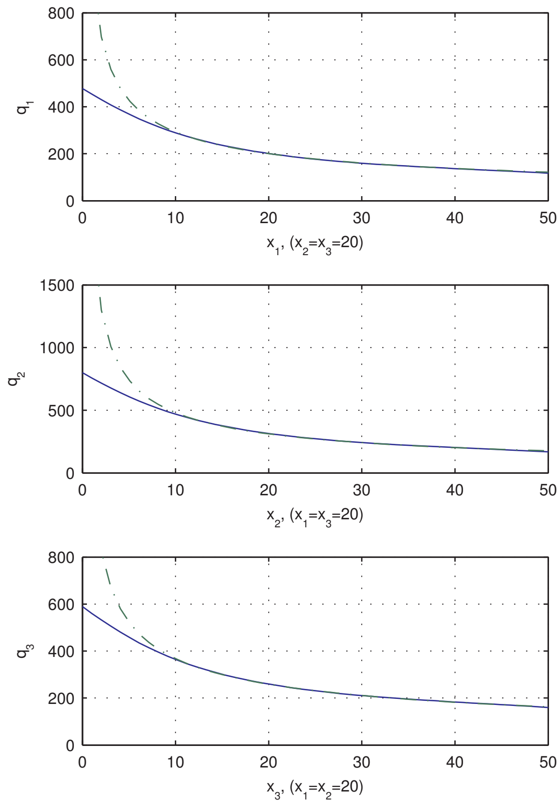

Fig. 3. Demand plots of each product with respect to its own price, fixing the other two prices at 20. The solid lines plot our demand functions, and the dashed lines plot the loglinear demand functions.

Note also see that the complimentary/substitutive relationships between different products are also preserved. In the log-linear model, we see that the pairs of products 1 and 2, and 2 and 3 are substitutes, while products 1 and 3 are complements. This is also reflected by the sign of elements of P.

Finally, we show how our demand model reflects own-price demand by plotting q_i with respect to x_i, while fixing both other two prices at 20. The shape looks quite realistic (Figure 3), as it shows a decreasing function that gets flatter when price increases, reflecting the law of diminishing returns. These results suggest the method is quite practical.

We also show the log linear demand function in the same plot. The difference between our demand function and the log linear demand function is when price is close to 0, and due to nature of logarithm, log-linear demand increases exponentially fast.

V. Conclusion

In this paper we demonstrated stability robustness conditions with respect to coalition structure for a class of profit-maximizing nonlinear systems. These conditions were then leveraged to provide a systematic methodology for estimating a rich variety of demand systems from data that guarantee sensible stability results regardless of the structure of cooperation within the marketplace.

The importance of these results emerges from the ability for regulators and managers alike to reliably conduct market power analyses using merger simulation and reverse merger simulation techniques. In such studies one can compute, for example, the value of cooperation of a firm as a measure of its market power.

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