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Minimax Robust Control of Structured Uncertain Time-delay Systems

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***Abstract*—The problem of minimax robust control for structured uncertain time-delay systems is dealt with. The existence conditions of minimax robust controller in the form of LMI are derived in the sense of Lyapunov theory and by the definite equivalent transform for static structured uncertain time-delay systems with multiplicative time quadratic performance cost. The convex optimization algorithm is introduced to get the minima upper bound of performance cost and the optimal parameter of minimax controller. The existence conditions of minimax robust controller are presented for time-delay systems of which structured uncertainties satisfy dynamical integral quadratic constraints (IQC). Simulation results show that the designed controller can shorten the state attenuation time effectively.**

I. Introduction

THE minimax control of uncertain systems is presented in the 1970's for the controller design to ensure quadratic stability of closed-loop uncertain systems based on parameter uncertain systems and a given performance cost. And the system performance cost are ensured to remain bounded in a certain minima bound under the condition of the worst disturbance and uncertainty. The minimax control comes in for a great deal of attention because of the exceptional application prospect. The existence conditions of minimax controller in the form of LMI are presented for the generalized linear continuous systems by Russian scholar Kogan [1], [2]. The optimal minimax controller is designed for linear stochastic systems in [3].

So far, concerning with the study of the robust control for uncertain time-delay systems, the system uncertainties are usually characterized by means of satisfying the generalized marching condition or norm-bounded condition. However, when modeling for some industry systems, some links can not be made linear and the nonlinear characters can not be described accurately besides the description of the inequalities function set. This kind of systems can be characterized by means of integral quadratic constraints (IQC) uniformly. The IQC can characterize not only the system gain information and phase information but also the structure information of system input-output [4], [5]. So IQC is the method to characterize a wide-ranging kind of nonlinear links. The problems of guaranteed performance control and stability analysis are studied for static IQC uncertain systems in [6]-[8]. The concept of dynamic IQC is presented [9] in 2001. But the mathematical expression of functional operator is adopted and the state-space expression is not supplied in [9]. The outside disturbance form characterized by dynamic IQC is presented in [10]. Based on that, the state-space expression of structured dynamic IQC uncertainties is presented completely in [11]. So the expression of uncertainties in uncertain systems is further expensed. But the research on the minimax control for structured uncertain time-delay systems is not much yet now. The minimax optimal control for discrete structured uncertain systems is studied in [12], and the key point in [12] is the system parameter optimization. The controller is designed for structured uncertain systems and the minimax dynamic game problem is discussed in [13]. The minimax optimal controller is designed for static structured uncertain systems in [14], but the time-delay factors are not considered and the existence conditions of the mentioned controller are in the Riccati form in [14].

In order to avoid difficulties of resolving, the existence conditions of the minimax robust controller in the form of LMI are presented for static structured uncertain time-delay systems with multiplicative time quadratic performance cost. The convex optimization algorithm is introduced to get the minima upper bound of performance cost and the controller optimal parameter. At the same time, based on the uncertainties satisfying dynamical IQC, the existence conditions of the minimax controller are presented.

II. Design of Minimax Controller for Static Structured Uncertain Time-delay Systems

In this section, we will discuss the existence condition of minimax controller. The following theorem addresses the optimal minimax control problem, which can be solved efficiently by convex optimization algorithms.

Consider the structured uncertain time-delay system described as follows.

$$\begin{aligned} \dot{x} &= Ax(t) + A_1x(t-d) + Bu(t) + Hp(t) \\ q(t) &= Cx(t) + Du(t) \\ p(t) &= \Delta q(t)\ ,\ \|\Delta\| \le 1 \end{aligned} \tag{1}$$

The initial conditions are as follows.

$$x(t) = \phi(t)\ ,\ \phi(\cdot) \in L_2[-d,0]\ ,\ t \in [-d,0]$$
$$u(t) = 0\ ,\ t \le 0$$

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where, $x(t)\in R^n$ is the system state, $u(t)\in R^m$ is the system control input, $p(t)$ is the system uncertain input, $q(t)$ is the system uncertain output; A, A_1, B, H, C and D are proper dimensional constant matrices. $d>0$ is the state time-delay. Δ is the uncertain matrix operator and its norm is less than 1.

Define the multiplicative time performance cost as

$$J(u,p)=\int_0^\infty e^{\alpha t}(\xi^{\mathrm T}(t)Q\xi(t)+u^{\mathrm T}(t)Ru(t)-p^{\mathrm T}(t)H^{\mathrm T}Hp(t))dt \tag{2}$$

where, $\xi=[x^T(t),x^T(t-d)]^T$, $Q=Q^T>0$, $R>0$, $\xi_0=\xi(0)$.

The key point of this section is to design the minimax state feedback robust controller $u(t)=Kx(t)$ for structured uncertain time-delay systems. The design of the controller is under the condition that the system uncertain input destroys the system stability and performance mostly. The controller is designed to make sure the asymptotic stability of the closed-loop system and to get the minima upper bound of performance cost.

By introducing the variables $\tilde\xi^{\mathrm T}(t)=e^{\alpha t}\xi^{\mathrm T}(t)$, $\tilde u(t)=e^{\alpha t}u^{\mathrm T}(t)$ and $\tilde p(t)=e^{\alpha t}p^{\mathrm T}(t)$, the performance cost becomes

$$J(u,\tilde p)=\int_0^\infty(\tilde\xi^{\mathrm T}(t)Q\tilde\xi(t)+\tilde u^{\mathrm T}(t)R\tilde u(t)-\tilde p^{\mathrm T}(t)H^{\mathrm T}H\tilde p(t))dt \tag{3}$$

Introducing the transforms $\tilde x(t)=e^{\alpha t}x^{\mathrm T}(t)$, $\tilde q(t)=e^{\alpha t}q^{\mathrm T}(t)$ yields the equivalent equations of system (1) as follows.

$$\begin{aligned}\dot{\tilde x}&=(A+\alpha I)\tilde x(t)+A_1e^{\alpha d}\tilde x(t-d)+B\tilde u(t)+H\tilde p(t)\\ \tilde q(t)&=C\tilde x(t)+D\tilde u(t)\\ \tilde p(t)&=\Delta\tilde q(t)\ ,\|\Delta\|\le 1\end{aligned} \tag{4}$$

The initial conditions are as follow.

$$\tilde x(t)=e^{\alpha t}\phi(t)\ ,\phi(\cdot)\in L_2[-d,0]\ ,t\in[-d,0]$$

Theorem 1: If there exist positive definitive symmetrical matrices P and S such that the linear matrix inequalities as follows are satisfied,

$$\begin{bmatrix}A^{\mathrm T}P+PA+\alpha P+S+C^{\mathrm T}C & PH & PB+C^{\mathrm T}D & e^{\alpha d}PA_1\\ H^{\mathrm T}P & -(I+H^{\mathrm T}H) & 0 & 0\\ B^{\mathrm T}P+D^{\mathrm T}C & 0 & -(R+D^{\mathrm T}D) & 0\\ e^{\alpha d}A_1^{\mathrm T}P & 0 & 0 & -S\end{bmatrix}<0 \tag{5}$$

$$\begin{bmatrix}A^{\mathrm T}P+PA+\alpha P+S+C^{\mathrm T}C & PH & PB+C^{\mathrm T}D & e^{\alpha d}PA\\ H^{\mathrm T}P & -\frac{1}{2}(I+H^{\mathrm T}H) & 0 & 0\\ B^{\mathrm T}P+D^{\mathrm T}C & 0 & -\Gamma & 0\\ e^{\alpha d}A_1^{\mathrm T}P & 0 & 0 & -S\end{bmatrix}<0 \tag{6}$$

then there exists the minimax state feedback control law

$$\tilde u^*=-(R+D^{\mathrm T}D)^{-1}(CD^{\mathrm T}+B^{\mathrm T}P)\tilde x(t)$$

such that the closed-loop system of system (4) is asymptotically stable and the performance cost satisfies

$$\min_{\tilde u^*}\max_{\tilde p^*}J(u,\tilde p)\le\tilde x^{\mathrm T}(0)P\tilde x(0)+\int_{-d}^0 e^{2\alpha t}\phi^{\mathrm T}(\theta)S\phi(\theta)d\theta\ ,$$

where, $\Gamma=(R+D^{\mathrm T}D)(D^{\mathrm T}D)^{-1}(R+D^{\mathrm T}D)$.

Proof: Firstly, choose the Lyapunov function as follows.

$$V(\tilde x(t))=\tilde x^{\mathrm T}(t)P\tilde x(t)+\int_{-d}^t\tilde x^{\mathrm T}(\theta)S\tilde x(\theta)d\theta \tag{7}$$

and define the quadratic function

$$W(\tilde x(t))=\tilde x^{\mathrm T}(t)P\tilde x(t)+\int_{-d}^t\tilde x^{\mathrm T}(\theta)S\tilde x(\theta)d\theta+\int_0^t(\tilde q^{\mathrm T}(t)\tilde q(t)-\tilde p^{\mathrm T}(t)\tilde p(t))d\theta \tag{8}$$

and the local check function

$$\psi=\dot W+\tilde u^{\mathrm T}(t)R\tilde u(t)-\tilde p^{\mathrm T}(t)H^{\mathrm T}H\tilde p(t) \tag{9}$$

Substituting (8) into (9) and considering $\tilde x_d=\tilde x(t-d)$, we obtain

$$\begin{aligned}\psi=&\tilde x^{\mathrm T}(t)[A^{\mathrm T}P+PA+\alpha P+S+C^{\mathrm T}C]\tilde x(t)+\tilde x^{\mathrm T}(t)PH\tilde p(t)\\&-\tilde p^{\mathrm T}(t)(I+H^{\mathrm T}H)\tilde p(t)+\tilde u^{\mathrm T}B^{\mathrm T}P\tilde x+\tilde x^{\mathrm T}PB\tilde u+\tilde u^{\mathrm T}D^{\mathrm T}C\tilde x\\&+\tilde x^{\mathrm T}C^{\mathrm T}D\tilde u+\tilde u^{\mathrm T}R\tilde u+\tilde u^{\mathrm T}D^{\mathrm T}D\tilde u+e^{\alpha d}\tilde x_d^{\mathrm T}A_1^{\mathrm T}P\tilde x+e^{\alpha d}\tilde x^{\mathrm T}PA_1\tilde x_d\\&-\tilde x_d^{\mathrm T}S\tilde x_d+\tilde p^{\mathrm T}H^{\mathrm T}P\tilde x\end{aligned} \tag{10}$$

Then maximizing (10) about $\tilde p$, we have

$$\tilde p^*=(I+H^{\mathrm T}H)^{-1}H^{\mathrm T}P\tilde x(t) \tag{11}$$

Since $\frac{\partial^2\psi}{\partial\tilde p^2}=-(I+H^TH)<0$, $\tilde p$ in (11) makes the local check function maximum. Substituting (11) into (10), we have

$$\begin{aligned}\max_{\tilde p^*}\psi=&\tilde x^{\mathrm T}(t)[A^{\mathrm T}P+PA+\alpha P+S+C^{\mathrm T}C]\tilde x(t)+\tilde x^{\mathrm T}(t)PH(I\\&+H^{\mathrm T}H)^{-1}H^{\mathrm T}P\tilde x(t)+\tilde u^{\mathrm T}B^{\mathrm T}P\tilde x+\tilde x^{\mathrm T}PB\tilde u+\tilde u^{\mathrm T}D^{\mathrm T}C\tilde x+\tilde x^{\mathrm T}C^{\mathrm T}D\tilde u\\&+\tilde u^{\mathrm T}(R+D^{\mathrm T}D)\tilde u+e^{\alpha d}\tilde x_d^{\mathrm T}A_1^{\mathrm T}P\tilde x+e^{\alpha d}\tilde x^{\mathrm T}PA_1\tilde x_d-\tilde x_d^{\mathrm T}S\tilde x_d\end{aligned} \tag{12}$$

Then minimizing (12) about $\tilde u$, we have

$$\tilde u^*=-(R+D^{\mathrm T}D)^{-1}(CD^{\mathrm T}+B^{\mathrm T}P)\tilde x(t) \tag{13}$$

It is easy to see $\frac{\partial^2\max_{\tilde p^*}\psi}{\partial\tilde u^2}=R+D^{\mathrm T}D>0$. So $\tilde u$ in (13) makes the local check function minimum.

Substituting (13) into (12), we have

$$\begin{aligned}&\min_{\tilde u^*}\max_{\tilde p^*}\psi\\&=\tilde x^{\mathrm T}(t)[A^{\mathrm T}P+PA+\alpha P+S+C^{\mathrm T}C]\tilde x(t)+\tilde x^{\mathrm T}(t)PH(I\\&\quad+H^{\mathrm T}H)^{-1}H^{\mathrm T}P\tilde x(t)-\tilde x^{\mathrm T}(PB+C^{\mathrm T}D)(R+D^{\mathrm T}D)^{-1}(B^{\mathrm T}P\\&\quad+D^{\mathrm T}C)\tilde x+e^{\alpha d}\tilde x_d^{\mathrm T}A_1^{\mathrm T}P\tilde x+e^{\alpha d}\tilde x^{\mathrm T}PA_1\tilde x_d-\tilde x_d^{T}S\tilde x_d\\&\le\tilde x^{\mathrm T}(t)[A^{\mathrm T}P+PA+\alpha P+S+C^{\mathrm T}C]\tilde x(t)+\tilde x^{\mathrm T}(t)PH(I\\&\quad+H^{\mathrm T}H)^{-1}H^{\mathrm T}P\tilde x(t)+\tilde x^{\mathrm T}(PB+C^{\mathrm T}D)(R+D^{\mathrm T}D)^{-1}(B^{\mathrm T}P\\&\quad+D^{\mathrm T}C)\tilde x+e^{\alpha d}\tilde x_d^{\mathrm T}A_1^{\mathrm T}P\tilde x+e^{\alpha d}\tilde x^{\mathrm T}PA_1\tilde x_d-\tilde x_d^{\mathrm T}S\tilde x_d\end{aligned} \tag{14}$$

Considering $\tilde\xi=[\tilde x^{\mathrm T}(t),\tilde x^{\mathrm T}(t-d)]^{\mathrm T}$, (14) becomes

$$\min_{\tilde u^*}\max_{\tilde p^*}\psi\le-\xi^{\mathrm T}(t)Q\xi(t)\ , \tag{15}$$

If the inequality (16) holds, we can ensure $Q>0$,

$$\begin{bmatrix}A^{\mathrm T}P+PA+\alpha P+S+C^{\mathrm T}C+M+N & e^{\alpha d}PA_1\\ e^{\alpha d}A_1^{\mathrm T}P & -S\end{bmatrix}<0, \tag{16}$$

where, $M=PH(I+H^{\mathrm T}H)^{-1}H^{\mathrm T}P$, $N=(PB+C^{\mathrm T}D)(R+D^{\mathrm T}D)^{-1}(B^{\mathrm T}P+D^{\mathrm T}C)$.

From the Schur complement theorem, it is obvious that the following inequality is equivalent to (16).

$$\begin{bmatrix}A^{\mathrm T}P+PA+\alpha P+S+C^{\mathrm T}C & PH & PB+C^{\mathrm T}D & e^{\alpha d}PA_1\\ H^{\mathrm T}P & -(I+H^{\mathrm T}H) & 0 & 0\\ B^{\mathrm T}P+D^{\mathrm T}C & 0 & -(R+D^{\mathrm T}D) & 0\\ e^{\alpha d}A_1^{\mathrm T}P & 0 & 0 & -S\end{bmatrix}<0$$

Substituting (11) and (13) into the derivative of $W(\tilde x(t))$ along the state trajectory of system (4), we have

$$\dot W(\tilde x(t))\le\tilde\xi^{\mathrm T}(t)\begin{bmatrix}A^{\mathrm T}P+PA+\alpha P+S+C^{\mathrm T}C+2M+N_1 & e^{\alpha d}PA_1\\ e^{\alpha d}A_1^{\mathrm T}P & -S\end{bmatrix}\tilde\xi(t) \tag{17}$$

where, $N_1=(PB+C^{\mathrm T}D)(R+D^{\mathrm T}D)^{-1}D^{\mathrm T}D(R+D^{\mathrm T}D)^{-1}(B^{\mathrm T}P+D^{\mathrm T}C)$.

From the Schur complement theorem, (17) is equivalent to the following inequality.

$$\begin{bmatrix} A^{\mathrm{T}}P+PA+\alpha P+S+C^{\mathrm{T}}C & PH & PB+C^{\mathrm{T}}D & e^{\alpha d}PA \\ H^{\mathrm{T}}P & -\frac{1}{2}(I+H^{\mathrm{T}}H) & 0 & 0 \\ B^{\mathrm{T}}P+D^{\mathrm{T}}C & 0 & -\Gamma & 0 \\ e^{\alpha d}A_1^{\mathrm{T}}P & 0 & 0 & -S \end{bmatrix}<0$$

where, $\Gamma=(R+D^{\mathrm{T}}D)(D^{\mathrm{T}}D)^{-1}(R+D^{\mathrm{T}}D)$.

If (6) is satisfied, then $\dot{W}(\tilde{x}(t))<0$. We have $\dot{V}<0$ obviously, so the asymptotically stability of system (4) is realized. Rearranging (15) and calculating the integral, as well as considering the initial conditions of the system, yields

$$\begin{aligned}\min_{\tilde{u}^*}\max_{\tilde{p}^*} J(u,\tilde{p}) &= \int_0^\infty (\tilde{\xi}^{\mathrm{T}}(t)Q\tilde{\xi}(t)+\tilde{u}^{\mathrm{T}}(t)R\tilde{u}(t)-\tilde{p}^{\mathrm{T}}(t)H^{\mathrm{T}}H\tilde{p}(t))dt \\ &\le \tilde{x}^{\mathrm{T}}(0)P\tilde{x}(0)+\int_{-d}^{0} e^{2\alpha t}\phi^{\mathrm{T}}(\theta)S\phi(\theta)d\theta \end{aligned} \tag{18}$$

Remark: If the above existence condition is satisfied, then there exists a minimax control law for system (4). From (18), we get to know that the upper bound of performance cost depends on the selection of the minimax control law. So in order to minimize the upper bound of performance of system, it is crucial that how to choose an appropriate minimax control law. By constructing and resolving the convex optimization problem, the optimal parameter of feedback controller and the minimum upper bound of performance cost will reach.

Theorem 2: For the system (3) and the performance cost (4), if the convex optimization problem

$$\min_{P,S} \mathrm{Trace}(P)+\mathrm{Trace}(e^{2\alpha t}NSN^{\mathrm{T}})$$
$$\text{s.t. (5) and (6)}$$

has the solution (P,S), then the parameter expression of the optimal minimax robust controller for system (3) can be as in (11) and

$$J\le \mathrm{Trace}(P)+\mathrm{Trace}(e^{2\alpha t}NSN^{\mathrm{T}})=J^*.$$

where, $\int_{-d}^{0}\phi^{\mathrm{T}}(\theta)\phi(\theta)d\theta=NN^{\mathrm{T}}$.

Note that the initial state of system is hard to be accurately measured in fact. So by considering the expected value of performance cost, we obtain.

$$\bar{J}=E\{J\}\le \mathrm{Trace}(P)+\mathrm{Trace}(e^{2\alpha d}NSN^{\mathrm{T}})=J^*.$$

III. Design of Minimax Controller for Dynamical IQC Uncertain Time-Delay Systems

Consider the uncertain time-delay system

$$\begin{aligned}\dot{x}&=Ax(t)+A_1x(t-d)+Bu(t)+\sum_{i=1}^{l}H_ip_i(t)\\ z(t)&=Cx(t)+Du(t)\\ q_i(t)&=E_ix(t)+F_iu(t),\ i=1,\cdots,l\end{aligned}\tag{19}$$

The initial conditions are as follows.

$$x(t)=\phi(t),\ \phi(\cdot)\in L_2[-d,0],\ t\in[-d,0]$$
$$u(t)=0,\ t\le 0$$

where, $x(t)\in R^n$ is the system state, $u(t)\in R^m$ is the system control input, $z(t)\in R^{n_1}$ is the system evaluation output, A, A_1, B, H, C, D, E_i and F_i are proper-dimensional constant matrices ($i=1,\cdots,l$). $d>0$ is the state time-delay. $p_i(t)\in R^{n_{p_i}}$ is the system uncertain input, $q_i(t)\in R^{n_{q_i}}$ is the system output. $(p_i(t),q_i(t))$ constitutes the input-output of structured uncertainties and satisfies the dynamic IQC relation [11] as follows.

$$\dot{\tilde{x}}_i(t)=\tilde{A}_i\tilde{x}_i(t)+\tilde{B}_iq_i(t)$$
$$\tilde{y}_i(t)=\tilde{C}_i\tilde{x}_i(t),\ i=1,\cdots,l$$
$$\int_0^{t^*}\begin{bmatrix}\tilde{y}_i^{\mathrm{T}}(t) & p_i^{\mathrm{T}}(t)\end{bmatrix}\begin{bmatrix}\Pi_{11}^i & \Pi_{12}^i\\ \Pi_{21}^i & \Pi_{22}^i\end{bmatrix}\begin{bmatrix}\tilde{y}_i(t)\\ p_i(t)\end{bmatrix}dt\ge 0,\ t^*\in(0,\infty)\tag{20}$$

where, $\tilde{A}_i$, $\tilde{B}_i$ and $\tilde{C}_i$ are given proper-dimensional constant matrices, and the given multiplier matrix satisfies

$$\Pi_{11}^i=\Pi_{11}^{i\ \mathrm{T}},\ \Pi_{12}^i=\Pi_{21}^{i\ \mathrm{T}},\ \Pi_{22}^i=\Pi_{22}^{i\ \mathrm{T}}$$

Define the performance cost as

$$J(u,p)=\int_0^\infty(\zeta^{\mathrm{T}}(t)Q_1\zeta(t)+u^{\mathrm{T}}(t)Ru(t)-\sum_{i=1}^{l}p_i^{\mathrm{T}}(t)H_i^{\mathrm{T}}H_ip_i(t))dt\tag{21}$$

where, $\zeta=[x^{\mathrm{T}}(t),x_d^{\mathrm{T}}(t-d),\tilde{x}^{\mathrm{T}}]^{\mathrm{T}}$, $x_d=x(t-d)$, $\tilde{x}=\begin{bmatrix}\tilde{x}_1^{\mathrm{T}},\cdots,\tilde{x}_l^{\mathrm{T}}\end{bmatrix}$, $Q_1>0$, $R>0$, $\zeta_0=\zeta(0)$.

Theorem 3: If there exist positive definitive symmetrical matrices X, S and P_1 such that the following matrix inequalities are satisfied,

$$\begin{bmatrix} XA^{\mathrm{T}}+AX-BR^{-1}B^{\mathrm{T}}-H^{\mathrm{T}}+V & A_1X & V_1\\ XA_1^{\mathrm{T}} & -V & 0\\ V_1^{\mathrm{T}} & 0 & \Theta_{33}\end{bmatrix}<0\tag{22}$$

$$\begin{bmatrix} XA^{\mathrm{T}}+AX-2BR^{-1}B^{\mathrm{T}}+\tilde{H}^{\mathrm{T}}+V & A_1X & \tilde{V}_1\\ XA_1^{\mathrm{T}} & -V & 0\\ \tilde{V}_1^{\mathrm{T}} & 0 & \tilde{\Theta}_{33}\end{bmatrix}<0\tag{23}$$

then there exists the minimax state feedback control law

$$u^*=-R^{-1}(B^{\mathrm{T}}Px+\sum_{i=1}^{l}F_i^{\mathrm{T}}\tilde{B}_i^{\mathrm{T}}S_i\tilde{x}_i)$$

such that the closed-loop system (19) is asymptotically stable and the performance cost satisfies

$$\min_{u^*}\max_{p_i^*} J(u,\sum_{i=1}^{l}p_i)\le x^{\mathrm{T}}(0)Px(0)+\sum_{i=1}^{l}\tilde{x}_i^{\mathrm{T}}(0)S_i\tilde{x}_i(0)+\int_{-d}^{0}\phi^{\mathrm{T}}(\theta)P_1\phi(\theta)d\theta.$$

where, $V=XP_1X$, $V_1=X\phi_{13}\tilde{S}$, $\tilde{V}_1=X\tilde{\phi}_{13}\tilde{S}$, $\tilde{S}=S^{-1}$, $\tilde{\Theta}_{33}=\tilde{S}^{\mathrm{T}}\tilde{\phi}_{33}\tilde{S}$, $\phi_{13}=-PZ_1-PBR^{-1}Z_2S+Z_3S$, $\tilde{\phi}_{13}=P\tilde{Z}_1-2PBR^{-1}Z_2S+Z_3S$.

Proof: Firstly, choose the Lyapunov function as follows.

$$V(x(t))=x^{\mathrm{T}}(t)Px(t)+\sum_{i=1}^{l}\tilde{x}_i^{\mathrm{T}}(\theta)S_i\tilde{x}_i(\theta)+\int_{-d}^{t}x^{\mathrm{T}}(\theta)P_1x(\theta)d\theta\tag{24}$$

and define the quadratic function

$$W(x(t))=V(x(t))+\sum_{i=1}^{l}\varepsilon_i\int_0^t\begin{bmatrix}\tilde{y}_i^{\mathrm{T}}(t) & p_i^{\mathrm{T}}(t)\end{bmatrix}\begin{bmatrix}\Pi_{11}^i & \Pi_{12}^i\\ \Pi_{21}^i & \Pi_{22}^i\end{bmatrix}\begin{bmatrix}\tilde{y}_i(t)\\ p_i(t)\end{bmatrix}ds\tag{25}$$

and the local check function

$$\psi=\dot{W}+u^{\mathrm{T}}(t)Ru(t)-\sum_{i=1}^{l}p_i^{\mathrm{T}}(t)H_i^{\mathrm{T}}H_ip_i(t)\tag{26}$$

Substituting (25) into (26), we obtain

$$\begin{aligned}\psi=&x^{\mathrm{T}}(t)(A^{\mathrm{T}}P+PA)x(t)+u^{\mathrm{T}}(t)(B^{\mathrm{T}}P+PB)u(t)+\sum_{i=1}^{l}p_i^{\mathrm{T}}H_i^{\mathrm{T}}Px+\sum_{i=1}^{l}x^{\mathrm{T}}PH_ip_i\\ &+\sum_{i=1}^{l}\tilde{x}_i^{\mathrm{T}}(\tilde{A}_i^{\mathrm{T}}S_i+S_i^{\mathrm{T}}\tilde{A}_i)\tilde{x}_i+\sum_{i=1}^{l}x^{\mathrm{T}}E_i^{\mathrm{T}}\tilde{B}_i^{\mathrm{T}}S_i\tilde{x}_i+\sum_{i=1}^{l}\tilde{x}_i^{\mathrm{T}}S_i^{\mathrm{T}}\tilde{B}_i^{\mathrm{T}}E_ix+\sum_{i=1}^{l}u^{\mathrm{T}}F_i^{\mathrm{T}}\tilde{B}_i^{\mathrm{T}}S_i\tilde{x}_i\\ &+\sum_{i=1}^{l}\tilde{x}_i^{\mathrm{T}}S_i^{\mathrm{T}}\tilde{B}_i^{\mathrm{T}}F_iu+\sum_{i=1}^{l}\varepsilon_i\tilde{x}_i^{\mathrm{T}}\tilde{C}_i^{\mathrm{T}}\Pi_{11}^i\tilde{C}_i\tilde{x}_i+\sum_{i=1}^{l}\varepsilon_ip_i^{\mathrm{T}}\Pi_{21}^i\tilde{C}_i\tilde{x}_i+\sum_{i=1}^{l}\varepsilon_i\tilde{x}_i^{\mathrm{T}}\tilde{C}_i^{\mathrm{T}}\Pi_{12}^ip_i\\ &+\sum_{i=1}^{l}p_i^{\mathrm{T}}(\varepsilon_i\Pi_{22}^i-H_i^{\mathrm{T}}H_i)p_i+x_d^{\mathrm{T}}A_1^{\mathrm{T}}Px+x^{\mathrm{T}}PA_1x_d+x^{\mathrm{T}}P_1x-x_d^{\mathrm{T}}P_1x_d+u^{\mathrm{T}}Ru\end{aligned}\tag{27}$$

Then maximizing (27) about $\tilde{p}$, we have

$$p_i^* = -(\varepsilon_i \Pi_{22}^i - H_i^{\mathrm{T}} H_i)(\varepsilon_i \Pi_{12}^{i\,\mathrm{T}} \tilde{C}_i \tilde{x}_i(t) + H_i P x(t)) \tag{28}$$

Since $\dfrac{\partial^2 \psi}{\partial p_i^2} = \varepsilon_i \Pi_{22}^i - H_i^{\mathrm{T}} H_i < 0$, p_i^* in (26) makes the local check function (27) maximum. Substituting (28) into (27), we have

$$\begin{aligned}
\max_{p_i^*} \psi &= x^{\mathrm{T}}(t)(A^{\mathrm{T}}P + PA)x(t) + u^{\mathrm{T}}(t)(B^{\mathrm{T}}P + PB)u(t) \\
&+ \sum_{i=1}^{l} \tilde{x}_i^{\mathrm{T}} (\tilde{A}_i^{\mathrm{T}} S_i + S_i^{\mathrm{T}} \tilde{A}_i)\tilde{x}_i + \sum_{i=1}^{l} x^{\mathrm{T}} E_i^{\mathrm{T}} \tilde{B}_i^{\mathrm{T}} S_i \tilde{x}_i + \sum_{i=1}^{l} \tilde{x}_i^{\mathrm{T}} S_i^{\mathrm{T}} \tilde{B}_i^{\mathrm{T}} E_i x \\
&+ \sum_{i=1}^{l} u^{\mathrm{T}} F_i^{\mathrm{T}} \tilde{B}_i^{T} S_i \tilde{x}_i + \sum_{i=1}^{l} \tilde{x}_i^{\mathrm{T}} S_i^{\mathrm{T}} \tilde{B}_i F_i u + x_d^{\mathrm{T}} A_1^{\mathrm{T}} P x + x^{\mathrm{T}} P A_1 x_d \\
&+ x^{\mathrm{T}} P_1 x - x_d^{\mathrm{T}} P_1 x_d + u^{\mathrm{T}} R u + \sum_{i=1}^{l} \varepsilon_i \tilde{x}_i^{\mathrm{T}} \tilde{C}_i^{\mathrm{T}} \Pi_{11}^i \tilde{C}_i \tilde{x}_i \\
&- \sum_{i=1}^{l} (x^{\mathrm{T}} P H_i + \varepsilon_i \tilde{x}_i^{\mathrm{T}} \tilde{C}_i^{\mathrm{T}} \Pi_{12}^i)(\varepsilon_i \Pi_{22}^i - H_i^{\mathrm{T}} H_i)^{-1} (\varepsilon_i \Pi_{12}^{i\,\mathrm{T}} \tilde{C}_i \tilde{x}_i + H_i^{\mathrm{T}} P x)
\end{aligned} \tag{29}$$

Then minimizing (29) about $\tilde{u}$, we have

$$u^* = -R^{-1}(B^{\mathrm{T}} P x + \sum_{i=1}^{l} F_i^{\mathrm{T}} \tilde{B}_i^{\mathrm{T}} S_i \tilde{x}_i) \tag{30}$$

It is easy to see $\dfrac{\partial^2 \max\limits_{p_i} \psi}{\partial u^2} = R > 0$. So u^* in (30) makes the local check function (29) minimum.

Substituting (30) into (29), we have

$$\begin{aligned}
&\min_{u^*} \max_{p_i^*} \psi \\
&= x^{\mathrm{T}}(t)(A^{\mathrm{T}}P + PA)x(t) + \sum_{i=1}^{l} \tilde{x}_i^{\mathrm{T}} (\tilde{A}_i^{\mathrm{T}} S_i + S_i^{\mathrm{T}} \tilde{A}_i)\tilde{x}_i + \sum_{i=1}^{l} x^{\mathrm{T}} E_i^{\mathrm{T}} \tilde{B}_i^{\mathrm{T}} S_i \tilde{x}_i \\
&+ \sum_{i=1}^{l} \tilde{x}_i^{\mathrm{T}} S_i^{\mathrm{T}} \tilde{B}_i^{\mathrm{T}} E_i x + x_d^{\mathrm{T}} A_1^{\mathrm{T}} P x + x^{\mathrm{T}} P A_1 x_d + x^{\mathrm{T}} P_1 x - x_d^{\mathrm{T}} P_1 x_d + \sum_{i=1}^{l} \varepsilon_i \tilde{x}_i^{\mathrm{T}} \tilde{C}_i^{\mathrm{T}} \Pi_{11}^i \tilde{C}_i \tilde{x}_i \\
&- \sum_{i=1}^{l} (x^{\mathrm{T}} P H_i + \varepsilon_i \tilde{x}_i^{\mathrm{T}} \tilde{C}_i^{\mathrm{T}} \Pi_{12}^i)(\varepsilon_i \Pi_{22}^i - H_i^{\mathrm{T}} H_i)^{-1} (\varepsilon_i \Pi_{12}^{i\,\mathrm{T}} \tilde{C}_i \tilde{x}_i + H_i^{\mathrm{T}} P x) \\
&- (x^{\mathrm{T}}(t) P B + \sum_{i=1}^{l} \tilde{x}_i^{\mathrm{T}} S_i^{\mathrm{T}} \tilde{B}_i F_i) R^{-1} (B^{\mathrm{T}} P x(t) + \sum_{i=1}^{l} F_i^{\mathrm{T}} \tilde{B}_i^{\mathrm{T}} S_i \tilde{x}_i)
\end{aligned} \tag{31}$$

Then from (14), we have

$$\min_{u^*} \max_{p_i^*} \psi \le -\zeta^{\mathrm{T}}(t) Q_1 \zeta(t) , \tag{32}$$

If the inequality followed holds, we can ensure $Q_1 > 0$,

$$\begin{bmatrix} A^{\mathrm{T}}P + PA - PBR^{-1}B^{\mathrm{T}}P - PH^{\mathrm{T}}P + P_1 & PA_1 & \phi_{13} \\ A_1^{\mathrm{T}} P & -P_1 & 0 \\ \phi_{13}^{\mathrm{T}} & 0 & \phi_{33} \end{bmatrix} < 0 , \tag{33}$$

where,

$$H = [H_1, \cdots H_l] \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & M_l \end{bmatrix} \begin{bmatrix} H_1^{\mathrm{T}} \\ \vdots \\ H_l^{\mathrm{T}} \end{bmatrix}, M_i = (\varepsilon_i \Pi_{22}^i - H_i^{\mathrm{T}} H_i)^{-1} ,$$

$$\phi_{13} = -PZ_1 - PBR^{-1} Z_2 S + Z_3 S , S = diag\left[S_1 I, \cdots, S_l I\right],$$

$$Z_1 = diag\left[\varepsilon_1 H_1 M_1 \Pi_{12}^{1\,\mathrm{T}} \tilde{C}_1, \cdots, \varepsilon_l H_l M_l \Pi_{12}^{l\,\mathrm{T}} \tilde{C}_l\right], Z_2 = diag\left[F_1^{\mathrm{T}} \tilde{B}_1^{\mathrm{T}}, \cdots, F_l^{\mathrm{T}} \tilde{B}_l^{\mathrm{T}}\right],$$

$$Z_3 = diag\left[E_1^{\mathrm{T}} \tilde{B}_1^{\mathrm{T}}, \cdots, E_l^{\mathrm{T}} \tilde{B}_l^{\mathrm{T}}\right], \ \phi_{33} = diag\left[\Sigma_1, \cdots, \Sigma_l\right],$$

$$\Sigma_i = \varepsilon_i^2 \tilde{C}_i^{\mathrm{T}} \Pi_{12}^i M_i \Pi_{12}^{i\,\mathrm{T}} \tilde{C}_i + S_i^{\mathrm{T}} \tilde{B}_i F R^{-1} F_i^{\mathrm{T}} \tilde{B}_i^{T} S_i + \varepsilon_i \tilde{C}_i^{\mathrm{T}} \Pi_{11}^i \tilde{C}_i + \tilde{A}_i^{\mathrm{T}} S_i + S_i^{\mathrm{T}} \tilde{A}_i , i = 1, \cdots, l .$$

Pre- and post-multiplying both sides of inequality (33) by $\mathrm{diag}[P^{-1} \quad P^{-1} \quad S^{-1}]$, we have

$$\begin{bmatrix} XA^{\mathrm{T}} + AX - BR^{-1}B^{\mathrm{T}} - H^{\mathrm{T}} + V & A_1 X & V_1 \\ XA_1^{\mathrm{T}} & -V & 0 \\ V_1^{\mathrm{T}} & 0 & \Theta_{33} \end{bmatrix} < 0 \tag{34}$$

where, $X = P^{-1}$, $V = XP_1X$, $V_1 = X\phi_{13}\tilde{S}$, $\tilde{S} = S^{-1}$, $\Theta_{33} = \tilde{S}^{\mathrm{T}} \phi_{33} \tilde{S}$.

Substituting (28) and (30) into the derivative of $W(x(t))$ along the state trajectory of system (19), we have

$$\begin{aligned}
\dot{W}(x(t)) &= x^{\mathrm{T}}(t)(A^{\mathrm{T}}P + PA)x(t) \\
&+ \sum_{i=1}^{l} \tilde{x}_i^{\mathrm{T}} (\tilde{A}_i^{\mathrm{T}} S_i + S_i^{\mathrm{T}} \tilde{A}_i)\tilde{x}_i + \sum_{i=1}^{l} x^{\mathrm{T}} E_i^{\mathrm{T}} \tilde{B}_i^{\mathrm{T}} S_i \tilde{x}_i + \sum_{i=1}^{l} \tilde{x}_i^{\mathrm{T}} S_i^{\mathrm{T}} \tilde{B}_i^{\mathrm{T}} E_i x \\
&+ x_d^{\mathrm{T}} A_1^{\mathrm{T}} P x + x^{\mathrm{T}} P A_1 x_d + x^{\mathrm{T}} P_1 x - x_d^{\mathrm{T}} P_1 x_d + \sum_{i=1}^{l} \varepsilon_i \tilde{x}_i \tilde{x}_i^{\mathrm{T}} \tilde{C}_i^{\mathrm{T}} \Pi_{11}^i \tilde{C}_i \tilde{x}_i \\
&+ \sum_{i=1}^{l} (x^{\mathrm{T}} P H_i + \varepsilon_i \tilde{x}_i^{\mathrm{T}} \tilde{C}_i^{\mathrm{T}} \Pi_{12}^i) Y_i (\varepsilon_i \Pi_{12}^{i\,\mathrm{T}} \tilde{C}_i \tilde{x}_i + H_i^{\mathrm{T}} P x) \\
&- 2(x^{\mathrm{T}}(t) P B + \sum_{i=1}^{l} \tilde{x}_i^{\mathrm{T}} S_i^{\mathrm{T}} \tilde{B}_i F_i) R^{-1} (B^{\mathrm{T}} P x(t) + \sum_{i=1}^{l} F_i^{\mathrm{T}} \tilde{B}_i^{\mathrm{T}} S_i \tilde{x}_i) \\
&= \zeta^{\mathrm{T}}(t) \Xi \zeta(t)
\end{aligned} \tag{35}$$

where,

$$Y_i = -2M_i + \tilde{M}_i , \ \tilde{M}_i = \varepsilon_i M_i \Pi_{22}^i M_i ,$$

$$\Xi = \begin{bmatrix} A^{\mathrm{T}}P + PA - 2PBR^{-1}B^{\mathrm{T}}P + P\tilde{H}^{\mathrm{T}}P + P_1 & PA_1 & \tilde{\phi}_{13} \\ A_1^{\mathrm{T}} P & -P_1 & 0 \\ \tilde{\phi}_{13}^{\mathrm{T}} & 0 & \tilde{\phi}_{33} \end{bmatrix}$$

$$\tilde{H} = [H_1, \cdots H_l] \begin{bmatrix} Y_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & Y_l \end{bmatrix} \begin{bmatrix} H_1^{\mathrm{T}} \\ \vdots \\ H_l^{\mathrm{T}} \end{bmatrix},$$

$$\tilde{\phi}_{13} = P\tilde{Z}_1 - 2PBR^{-1} Z_2 S + Z_3 S , \ \tilde{\phi}_{33} = diag\left[\tilde{\Sigma}_1, \cdots, \tilde{\Sigma}_l\right],$$

$$S = diag\left[S_1 I, \cdots, S_l I\right], \tilde{Z}_1 = diag\left[\varepsilon_1 H_1 Y_1 \Pi_{12}^{1\,\mathrm{T}} \tilde{C}_1, \cdots, \varepsilon_l H_l Y_l \Pi_{12}^{l\,\mathrm{T}} \tilde{C}_l\right]$$

$$\tilde{\Sigma}_i = \varepsilon_i^2 \tilde{C}_i^{\mathrm{T}} \Pi_{12}^i Y_i \Pi_{12}^{i\,\mathrm{T}} \tilde{C}_i + 2S_i^{\mathrm{T}} \tilde{B}_i F R^{-1} F_i^{\mathrm{T}} \tilde{B}_i^{\mathrm{T}} S_i + \varepsilon_i \tilde{C}_i^{\mathrm{T}} \Pi_{11}^i \tilde{C}_i + \tilde{A}_i^{\mathrm{T}} S_i + S_i^{\mathrm{T}} \tilde{A}_i , i = 1, \cdots, l .$$

It is obvious that if $\Xi < 0$, then $\dot{W}(x(t)) < 0$ and $\dot{V} < 0$, so the asymptotical stability of the closed-loop system of system (19) is realized. Pre- and post-multiplying both sides of inequality (34) by $\mathrm{diag}[P^{-1} \quad P^{-1} \quad S^{-1}]$, we have

$$\begin{bmatrix} XA^{\mathrm{T}} + AX - 2BR^{-1}B^{\mathrm{T}} + \tilde{H}^{\mathrm{T}} + V & A_1 X & \tilde{V}_1 \\ XA_1^{\mathrm{T}} & -V & 0 \\ \tilde{V}_1^{\mathrm{T}} & 0 & \tilde{\Theta}_{33} \end{bmatrix} < 0$$

where, $V = XP_1X$, $\tilde{V}_1 = X\tilde{\phi}_{13}\tilde{S}$, $\tilde{S} = S^{-1}$, $\tilde{\Theta}_{33} = \tilde{S}^{\mathrm{T}} \tilde{\phi}_{33} \tilde{S}$.

Rearranging (32) and calculating the integral, as well as considering the initial conditions of system, yields

$$\min_{u^*} \max_{p_i^*} J(u, \sum_{i=1}^{l} p_i) \le x^{\mathrm{T}}(0) P x(0) + \sum_{i=1}^{l} \tilde{x}_i^{\mathrm{T}}(0) S_i \tilde{x}_i(0) + \int_{-d}^{0} \phi^{\mathrm{T}}(\theta) P_1 \phi(\theta) d\theta$$

IV. NUMERICAL SIMULATIONS

Since the existence conditions of optimal controller in the form of LMI for static structure time-delay systems are presented in section 3, the unknown parameters are easy to obtain with LMI tool-box.

In order to confirm the results in this paper, the simulation is given based on Theorem 2. The parameter matrices of the considered system are represented by

$$A = \begin{bmatrix} 1 & 0 \\ 10 & -1 \end{bmatrix}, A_1 = \begin{bmatrix} 0.1 & 0 \\ 1 & -0.1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, D = 0.1 , R = 0.1$$

Take $\alpha = -15$ and $d = 1$, we can obtain the solutions P and S with the LMI tool-box in MATLAB.

$$P=\begin{bmatrix}2.1021 & 0.5953\\ 0.5953 & 0.6680\end{bmatrix},\quad S=\begin{bmatrix}0.1503 & -0.0074\\ -0.0074 & 0.1463\end{bmatrix}$$

Then we can get the minimax state feedback gain

$$K=\begin{bmatrix}-1.0280 & -1.0136\end{bmatrix},$$

In comparison with the H_∞ control in [15], the state response of the closed-loop system is shown in Fig. 1 and Fig. 2, where $p(t)=\sin(q(t))$. From the two figures, we can confirm that the designed controller can shorten the state attenuation time of the closed-loop system effectively.

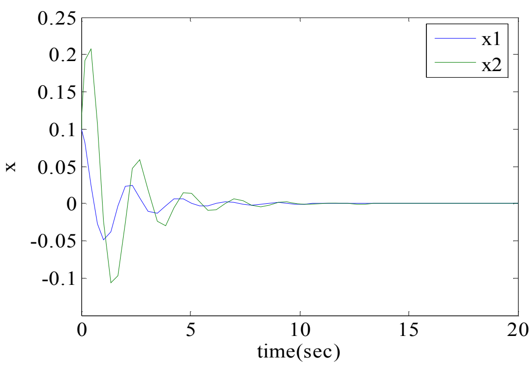

Fig.1. The state response of H_∞ control when $\alpha=-15$

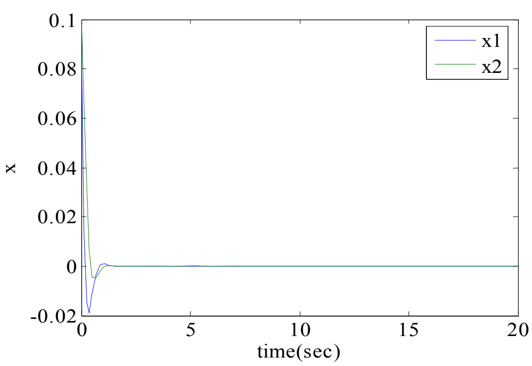

Fig.2. The state response of minimax control when $\alpha=-15$

At the same time, the minima upper bound of performance cost can be obtained with mincx tool-box as $J^*=2.7701$. If taking $\alpha=-30$, the state response of the closed-loop system is shown in Fig. 3.

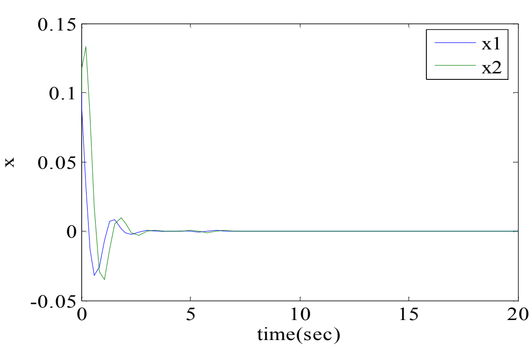

Fig.3. The state response of minimax control when $\alpha=-30$

From Fig. 3, we can see that the state attenuation time of the closed-loop system is not shortened with the reduction of α.

V. Conclusion

The minimax controller is design for time-delay systems of which the uncertainties satisfy the static and dynamical IQC respectively. The asymptotical stability of the closed-loop system and the minima upper bound existence of performance cost can be completed with the designed method. The existence conditions in the form of LMI are presented for the static condition based on the multiplicative time quadratic performance cost. The presented controller simplizes the solution process and shortens the state attenuation time of the closed-loop system effectively. Based on the dynamical IQC, the existence conditions of the matrix inequality controller are presented. It will be the further research on the transformation to the linear matrix inequality.

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