2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Global Nonlinear Control: a new geometric aspect

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***Abstract*— Many concepts of linear control theory can be extended to the nonlinear domain provided one restricts oneself to a neighborhood of the equilibrium. But what happens beyond that neighborhood? The paper addresses this question for the special case of a system defined on a compact boundaryless manifold.**

I. Introduction

An extensive literature exists on both the theoretical and the practical aspects of global nonlinear control, see e.g. [1] and [2]. A recent book chapter [3] reviews and discusses the strengths and the limitations of neural networks for the control of complex systems. One of the conclusion of the authors is that a coherent theory of nonlinear control with strong implications on practical control design is missing to date. Is it that the theory is too involved or are we still *missing* some theory which would unify the concepts developed so far? The objective of this paper is to review some of the concepts developed in geometric control theory, and, in fact, to point out a new geometric aspect. To be precise, this aspect is *geometric* in the sense that it deals with topological properties of a geometric object, a manifold, and *algebraic* in that it uses a tool from algebraic topology to describe these properties.

We begin by characterizing the natural state space of a nonlinear dynamic system. What is the essential difference between linear and nonlinear systems, from a geometric viewpoint? The state space $\mathbb{R}^n$ of a linear system is "flat" in the sense that it expands to infinity along the direction given by the vectors of a basis of $\mathbb{R}^n$. The space of nonlinear systems, on the other hand, is *curved* and is defined as the manifold M, where a point $p \in M$ if there exists an open neighborhood U of p and a homeomorphic map $\varphi : U \rightarrow \varphi(U) \subset \mathbb{R}^n$, called the (local) coordinate chart of M. In other words, the manifold "looks" locally like $\mathbb{R}^n$. This simple fact has an important consequence: Many coordinate systems may be needed to describe the global evolution of a nonlinear dynamic system. While the manifold is an abstract geometric object, the coordinate system is the physical handle on that object through which we have to interact with the system when we control it. It is important to keep this in mind when designing global nonlinear controllers. There is extensive literature on the (global and semi–global) control of specific systems having manifolds as their state–spaces, e.g [4], [5]. Also, the question of global controllability has created an important strand of literature (see e.g. [6], [2]) which, however is not addressed in this paper. We simply assume, that any point of the manifold can be reached from any other point in finite time through a suitable choice of the controls.

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II. Contractible spaces

A first hint of the importance of geometrical properties of the state space for the study of global nonlinear control is the following well known fact: Global asymptotic stabilization is impossible if the state space X is not a contractible to the origin.

A space X is said to be contractible to a point $x^* \in X$ if the identity function on X is homotopic to the constant function which maps all of X into x^*. We use here a notion from algebraic topology, homotopy, which provides us with a precise meaning of a continuous deformation of one function into another. Two maps $f_i : X \rightarrow Y$, $i = 0, 1$ are *homotopic* if there exists a family of continuous maps $f_t : X \rightarrow Y$, $t \in [0\ 1]$, varying *continuously* from f_0 to f_1. Two spaces X and Y have the same *homotopy type* if there exist maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that the composites $g \circ f : X \rightarrow X$ and $f \circ g : Y \rightarrow Y$ are homotopic to the identity map of X and Y respectively. Being of the same homotopy type is weaker than being homeomorphic since for the latter we require that $g \circ f$ and $f \circ g$ are actually *equal* to the identity map.

An important special case is when $Y \subset X$. $f : X \rightarrow Y$ is a retraction of X onto Y if $f\,|\,Y = id_Y$. If, at the same time, f is also homotopic to the identity map id_X on X, then Y is a deformation retract of X. This means that Y is the result of a continuous deformation of X which leaves all points already in Y invariant. These concepts can be transferred to a familiar situation encountered in the phase space of a dynamical system. The region of attraction $A \subseteq X$ of an asymptotically stable fixed point

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$x^* \in A$ is continuously deformed into x^* while x^* itself remains invariant. The deformation retraction is given by the flow, i.e. let $\phi_t : A \rightarrow A$ be the usual one–parameter family of maps $\phi_t(x_0) = x(t)$ from initial state x_0 to the state $x(t)$ reached at time t (under the flow). Then, for any fixed $t \geq 0$, $\phi_t(x^*) = x^*$ and ϕ_t is also homotopic to the identity map ϕ_0 on A, where the homotopy $h : A \times [0\ 1] \rightarrow A$ is given by

$$h(x,t) = \phi_{\left[\frac{1}{t}-1\right]}(x) \tag{1}$$

This is true since $\phi_t(x)$ is continuous in t. It follows that any asymptotically stable fixed point x^* may be seen as a deformation retract of its region of attraction. In particular, let the retraction be given by the flow at infinity: $\phi_\infty : A \rightarrow A$ is a constant map transferring any state to x^*. Hence, ϕ_∞ is a contraction of A to x^*. We conclude that a necessary condition of A to be the region of attraction of x^* is that A is contractible.

III. Dynamics on a cell complex

In the following we will view the state space, which we know is curved in general, slightly differently. We assume that it can be constructed from a set of base spaces of increasing dimensionality. The space is partitioned into cells where a cell of dimension n is just a "copy" of the Euclidean space $\mathbb{R}^n$, more formally, an $n-$cell, denoted e^n is the interior $D^n - \partial D^n$ of the n–dimensional disk and as such homeomorphic to $\mathbb{R}^n$.

The D^n disk itself is called a closed n– cell $\bar{e}^n$. We speak of a CW complex if the space X can be partitioned in such a way that any n–cell is attached to the $(n-1)$–skeleton formed by the union of cells of dimension $\leq n$. In other words, X is a CW complex if it is obtained from 0–cells by attaching closed cells one after another in ascending order of dimensions of cells. This construction is quite intuitive as the following example shows. If we delete a point pt (a closed 0–cell $\bar{e}_0$) from the sphere S^2 and unfold the resulting object to make it "flat" we obtain a 2–disk (a closed 2–cell). The boundary of the 2–cell is a circle and we may define the attaching map $h : \partial D^2 \rightarrow pt$ which collapses the circle back to the point, an operation which forms the sphere. We summarize:

A cell complex X is a union of cells (without boundary)

$$X = \bigcup_{p,q} e_q^p \tag{2}$$

An attaching map $h_q : \partial X^q \rightarrow X^p$ identifies all points x on the boundary of a cell complex ∂X^q with a point $h(x)$ of X^p where $p < q$. It may be that cells of some dimensions are missing but the boundary ∂X^q must always be attached to some subcomplex X^p. We have seen how the gluing of the boundaries changes the topological nature of the space. A single chart will not be sufficient to describe the entire space (as a manifold).

Example: A circle is obtained by gluing together the end points of a closed 1–cell:

$$S^1 = \bar{e}^0 \cup_{h_1} \bar{e}^1 \tag{3}$$

i.e. the boundary of $\bar{e}_1$ is attached to a point pt. While $\bar{e}^1 = D^1$ may be described quite naturally as a subspace of $\mathbb{R}$, the construction in (3) cannot be identified with a subspace of the flat Euclidean space. At least to charts a needed to capture the geometry of S^1: A candidate chart $\varphi^{-1} : \mathbb{R} \rightarrow S^1$, $\varphi^{-1}(\theta) = [\cos\theta\ \sin\theta]^T$ is a homeomorphism on $(-\pi,\, \pi)$ but not on $[-\pi,\, \pi]$ since in the latter case it is not one-to-one ($\theta = -\pi$ and $\theta = \pi$ are mapped to the same point on the manifold). Hence we have to exclude 2π-periodic points in the definition of the neighborhood on which the chart is defined. We define $N_1 = (-\pi/4;\ \pi{+}\pi/4)$ and $N_2 = (-\pi{-}\pi/4;\ \pi/4)$. We obtain the coordinate charts (N_1, φ^{-1}) and (N_2, φ^{-1}) which together show that S_1 is indeed a 1-manifold. The number of neighborhoods U necessary to describe the manifold provides a rough measure of its geometric complexity.

IV. One step further: Homology Theory

There is also a purely algebraic means of characterizing the geometric complexity of spaces. In doing so, one focuses on the connectivity of the space. It turns out that the set of arc–wise connected components of X provides a basis of a certain Abelian group associated with X. A homology theory assigns to any topological space a sequence of Abelian groups $H_0(X), H_1(X), \ldots, H_n(X)$ and to any continuous map $f : X \rightarrow Y$ a sequence of homomorphisms $f_* : H_n(X) \rightarrow H_n(Y)$, $n \in \mathbb{N}$. The structure of the groups $H_n(X)$ depends only on the topological type of X. If f is a homotopy equivalence then f_* is an isomorphism. Thus, it is the homotopy type of X that determines the structure of $H_n(X)$. $H_0(X)$ has a basis in one–to–one correspondence with the arc components of X. Similarly, $H_n(X)$, $n > 0$, expresses what may be called the higher connectivity properties of X [7].

In order to see how such an algebraization may be realized we define an orientation on each cell contained in X (i.e. the corresponding Euclidean space has an

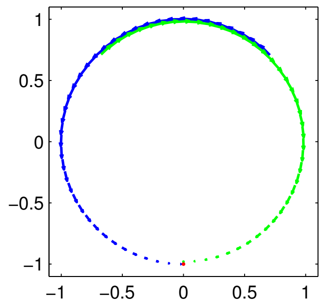

Fig. 1. Combination of two vectorfields on S^1

ordered basis). We may consider a formal sum with integer coefficients of these cells:

$$c = a_1\langle e_1^q\rangle + a_2\langle e_2^q\rangle + \cdots + a_m\langle e_m^q\rangle \tag{4}$$

is called a $q-$ chain of X. The set of all q–chains of X forms a free abelian group $C_q(X) \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}$ (m copies of $\mathbb{Z}$). Next, we define a boundary homomorphism

$$\partial_q : C_q(X) \to C_{q-1}(X) \tag{5}$$

Let $\langle\bar{e}_i^q\rangle$ be an oriented q–cell in X. This cell is attached to the cell complex Y of (at least) one dimension less than X by an attaching map

$$h : \partial\bar{e}_i^q \to Y \tag{6}$$

If X does not contain a $(q-1)$–cell, then $C_q(X) = \{0\}$. A sequence consisting of chain–groups of a cell complex X and boundary homomorphisms

$$\cdots \to C_{q+1}(X) \xrightarrow{\partial_q+1} C_q(X) \xrightarrow{\partial_q} C_{q-1}(X) \xrightarrow{\partial_{q-1}} \cdots \xrightarrow{\partial_1} C_0(X) \to 0 \tag{7}$$

is called a chain complex of X. Finally we define the q–dimensional homology group

$$H_q(X) = Z_q(X)/B_q(X) \tag{8}$$

$Z_q(X) = \ker\ \partial_q$ is called the cycle group and contains all "closed" chains with no boundary $\partial c = 0$. $B_q(X) = \mathrm{Im}\ \partial_{q+1}$, in turn, is the boundary group of X which contains all chains of dimension q that are the boundary of some chain of dimension $q+1$ i.e. if there is a $c' \in C_{q+1}(X)$ such that $\partial c' = c$ then c is in the boundary group of X. The cell complex is constructed in such a way that every boundary is also a cycle, i.e. we have

$$\partial_{q-1} \circ \partial_q = 0 \tag{9}$$

Homology measures how many q–dimensional cycles exist which are not the boundary of a higher–dimensional cell. Let us refer to these cycles as *proper* cycles. The homology groups $H_q(X)$ of a (finite) cell complex X are finitely generated Abelian groups and, hence, can be written as

$$H_q(X) \cong \mathbb{Z}^p \oplus \mathbb{Z}_{T_1} \oplus \mathbb{Z}_{T_2} \oplus \cdots \oplus \mathbb{Z}_{T_N} \tag{10}$$

$T_1, T_2, \ldots, T_N$ are the orders of the subgroup generators of $H_q(X)$ also referred to as the *torsion* coefficients –for the elements of $H_q(X)$ of finite order form its torsion subgroup. The elements of infinite order form the *free* part whose rank p is called the q–dimensional Betti number of X. We will see that these formal definitions have geometric interpretations. In particular, the order of the cyclic generators of $H_q(X)$ may be thought of as the number of rounds one can take along a cycle without bounding a region of higher dimension. This will have a direct consequence on the way a global stabilizer can be defined on the manifold represented by X.

V. Global Control Design

The job of the control designer is to make sure that an asymptotically stable flow (with the origin being a fixed point) can live on a given manifold. If we speak of local control, it is quite straightforward –at least from this abstract perspective– to define a controller using the "usual toolset" provided by linearization. This is because, apart from possible constraints due to non–controllability (that we assume away in this paper) one does not look beyond the boundaries of a local chart and, as a consequence, one does not encounter the limitations that the global analysis brings about. A vector field on a manifold M is a smooth assignment of a vector tangent to M at any point. Our objective is to define a vector–field on the manifold which has a single singular point at zero and provides a covering of the manifold with tangent vectors which –loosely speaking– all point to the remaining fixed point at zero.

It is a well–known fact that the existence of a smooth vector–field on a compact boundaryless manifold implies the existence of a certain number of critical points. As an example, the number of equilibrium points on a surface depend on the genus (number of holes) of the surface. A surface of genus k has one source, one sink, and $2k$ saddles. This is a consequence of the fact that the vector–field has to be compatible with the geometry of the space on which it is defined. The celebrated Poincaré–Hopf theorem is a much deeper statement of this basic fact [8].

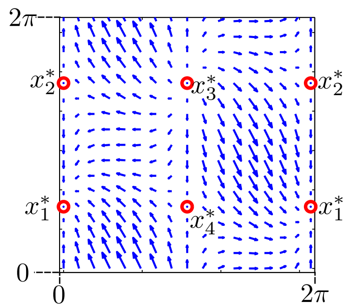

Fig. 2. The open–loop vector–field on the torus in local coordinates. The space covered by the neighborhoods is $[0,\ 2\pi] \times [0,\ 2\pi]$

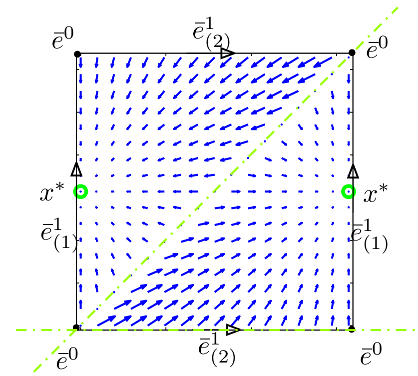

Fig. 3. Cell decomposition and closed–loop vector field on $S^1 \times S^1$

The appearance of other equilibria is of course against our goal of globally stabilizing a single equilibrium point x^* on M. A key realization is that the complication arises when attaching the boundary of the highest–dimensional cell to a subcomplex which itself is attached to yet a smaller dimensional subcomplex and so on. Because of the gluing, it is no longer possible to "move away" a point on the former boundary and approach it to x^*. In the language of dynamical systems, the *set of image points of the attaching map h is invariant*.

Let us consider a cell e^n before the attachment. Since it is homeomorphic to $\mathbb{R}^n$ it can be contracted. In other words, the whole cell can be shrunk to a point inside the cell (assume that this point is the origin) under the action of the flow generated by the closed–loop system. On the boundary of the closed cell $\bar{e}^n$, all tangent vectors point inside. When attaching the cell to e^m where $m < n$, the boundary gets mapped onto a lower–dimensional space and disappears. The vectors at the missing boundary point in opposite directions. During the construction of the cell–complex, cells having no boundaries (cycles) play an important role since they may appear as the boundary of a higher–dimensional cell. If this boundary disappears (due to some further gluing) the cycle still remains and becomes *proper* cycle in the sense defined above. These proper cycles are what concerns the control theorist since the spaces in which they appear cannot be contracted.

The only place where the direction of the vectors of a field may undergo a radical change is the neighborhood of an equilibrium point. If we want the equilibrium point to disappear and still realize the directional change of the tangent vectors we need more than just one vector–field. In order to design a globally stabilizing controller, switching among multiple vector fields becomes necessary. The minimum number of such vector fields can be determined from the homology groups of the space X on which the dynamics takes place. We explore this point further by considering two basic examples of compact, connected 2–manifolds. It will be seen that the information relevant to the design of a global stabilizer is contained in the first homology groups of these manifold.

A. The torus T^2

A torus may be represented as the union of two closed one–cells $\bar{e}^1_{(1)}$ and $\bar{e}^1_{(2)}$ which bound a closed 2–cell $\bar{e}^2$ but are itself attached to a zero–cell $\bar{e}^0$:

$$T^2 : \left(\bar{e}^0 \cup_{h_1} (\bar{e}^1_{(1)} \cup \bar{e}^1_{(2)})\right) \cup_{h_2} \bar{e}^2 \qquad (11)$$

Figure (3) displays this construction where $\bar{e}^2$ corresponds to the familiar square that results after cutting the torus twice. The fact that all cell boundaries are ultimately attached to the zero–cell $\bar{e}^0$ shows that the resulting space, is in fact boundaryless. The homology theory of the torus is given by [9]

$$H_0(T^2) \cong \mathbb{Z}, \quad H_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}, \quad H_2(T^2) = \mathbb{Z}.$$

We focus on the first homology group $H_1(T^2)$ since this describes the way the space is attached to the skeleton of lower–dimensional cells. $H_1(T^2)$ is a free Abelian group of rank 2 which, in geometric terms, means that there exist two proper one–dimensional cycles. One may go round each one these cycles an infinite number of times without bounding a region of higher dimension. This comes as no surprise as it is well known that the torus may be represented as the product of two circles $T^2 = S^1 \times S^1$.

We first examine the situation on a single circle S^1. Using the coordinate charts defined previously the system

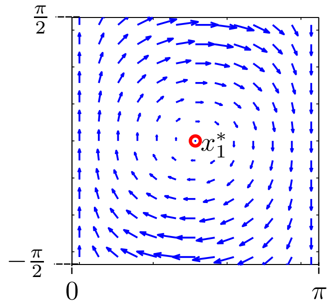

Fig. 4. The open–loop vector–field on the projective plane in local coordinates. The space covered by the neighborhoods is $[0,\ \pi] \times [-\frac{\pi}{2},\ \frac{\pi}{2}]$

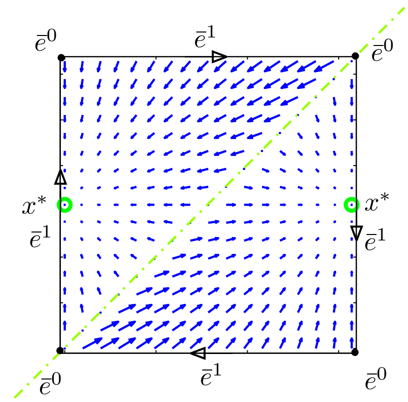

Fig. 5. Cell decomposition and closed–loop vector field on $P^2(\mathbb{R})$

can be described on a subspace of $\mathbb{R}$.

$$\dot{\phi} = -\sin\phi + u \tag{12}$$

where $\phi \in [-\pi,\ \pi]$. It is easily seen that the origin is asymptotically stable by itself (i.e. for $u = 0$), but the point $\phi^* = \pm\pi$ is excluded from the region of attraction in $A \subset S^1$. In order to make this (unstable) invariant point disappear, we simply choose a controller $u = -\phi$ which "extends" the vector–field beyond ϕ^*. At first sight, there is nothing special about this control law which is even linear. The important point to note, however, is that the effect of u is quite different depending on the coordinate chart one operates in. When the chart (N_1, φ^{-1}) is applied, the flow follows the right arc of the circle and for (N_2, φ^{-1}) the left arc. In a neighborhood of ϕ^*, the charts overlap and this creates a discontinuity since a decision has to be made which of the two charts to applied.

On the torus, the situation is slightly more involved, but the arguments are the same. First, observe that –having genus 1– (Euler characteristic 0), there is either none (as in the case of a Hamiltonian system) or 4 equilibria. A dynamical system that is consistent with the geometry of the torus may be described in local coordinates by:

$$\dot{\phi} = -\sin\phi + u_1 \tag{13}$$

$$\dot{\psi} = \cos\psi + \sin\phi + u_2 \tag{14}$$

where ϕ, ψ are both contained in the interval $[0,\ 2\pi]$. The system is obviously globally controllable. The open–loop system has four equilibria x_1^*: locally asymptotically stable, x_2^*: saddle, x_3^*: unstable and x_4^*: saddle displayed in figure 2. In order to make $x^* = (0, \pi)$ (chosen arbitrarily) the only invariant point, and moreover, a stable and globally attracting one, we may define a vectorfield as in figure 3. The same figure also indicates the construction of the torus from a square: opposite sides having the same orientation are identified. While the identification of the two vertical sides does not cause any discontinuities (the sides form an invariant set for the closed–loop flow) the horizontal sides cannot be identified without interrupting the vector–field. From the geometry it is clear that another discontinuity is needed in order to orient all tangent vectors towards the equilibrium point x^*. This discontinuity is defined along the diagonal of the square in figure 3. We see that the two ways of going round the torus on a one–cycle are each an independent source of discontinuity in the vector–field of the global stabilizer. The one–dimensional Betti–number (= 2) of the torus determines the number of cuts that have to be made in order to contract the whole state space to x^*.

B. *The real projective plane* $P^2(\mathbb{R}^2)$

Suppose a symmetric rigid body (e.g. a handle) has to be oriented in space $\mathbb{R}^3$ then a natural state–space is given by the rotation group $SO(3)$ modulo the antipodal map contained in $O(1)$. In other words, only the angles of the body are controlled and any points that lie on the same line through the origin of $\mathbb{R}^3$ are identified — we obtain the real projective plane $P^2(\mathbb{R}) \cong SO(3)/O(1)$. Its topology can be understood by noting that each such a line intersects the unit sphere S^2 in a pair of diametrically opposite points. The projective plane is obtained from S^2 by subtracting the southern hemisphere and identifying opposite points on the equator. This results in a Möbius strip being sewed to the edge of the upper hemisphere thereby closing the resulting space.

The cell complex reflects this situation:

$$P^2(\mathbb{R}) : \left(\bar{e}^0 \cup_{h_1} \bar{e}^1\right) \cup_{h_2} \bar{e}^2 \tag{15}$$

The attaching map $h_2 : \partial\bar{e}^2 \cong S^1 \to \left(\bar{e}^0 \cup_{h_1} \bar{e}^1\right) \cong S^1$ sends the boundary $\partial\bar{e}^2$ of the 2–cell around the 1–cycle $\left(\bar{e}^0 \cup_{h_1} \bar{e}^1\right)$ twice. This is exactly the nature of the boundary of the Möbius strip which after two rounds comes back to the starting point. An algebraization of this construction is given by the homology theory of the projective plane (see e.g. [9]):

$$H_0(P^2(\mathbb{R})) \cong \mathbb{Z}, \quad H_1(P^2(\mathbb{R})) \cong \mathbb{Z}_2, \quad H_2(P^2(\mathbb{R})) = 0$$

As before, we focus on the first homology group $H_1(P^2(\mathbb{R})) \cong \mathbb{Z}_2 = \mathbb{Z}/2Z$ which is a torsion subgroup with coefficient 2. As in the case of the torus, the manifold contains a boundaryless curve but after two rounds this curve bounds a higher–dimensional region, namely the 2–cell corresponding to the upper hemisphere in the above construction. $H_1(P^2(\mathbb{R})$ is a finite group since $1 + 1 \ (mod)\, 2 = 0$, or "two times the antipodal map gives the identity map". More importantly, we find that there is only a single independent way to go round the projective plane. This has an important consequence on the design of a global stabilizer. A dynamical system on the projective plane may assume the following form in local coordinates:

$$\dot{\phi} = 0.1\,\psi \cdot \sin\phi + u_1 \tag{16}$$

$$\dot{\psi} = -\sin(\phi - \pi/2) + u_2 \tag{17}$$

where $\phi \in [0,\ \pi]$ and $\psi \in [-\frac{\pi}{2},\ \frac{\pi}{2}]$. Again, the system is trivially controllable. Its open–loop equilibrium is at $x_1^* = (\frac{\pi}{2},\ 0)$. Figure 4 displays the open–loop vector–field together with the cells contained in the cell–complex of the projective plane. Notice the reversed orientations of the sides (one–cells). When identifying opposite sides, one of them must be twisted in order to make the orientations coincide. If the same closed–loop dynamics is defined as in the example of the torus the discontinuity at the horizontal side of the square disappears. Figure 5 displays the closed–loop vector–field on the projective–plane. It has the exact same form (in local coordinates) than the one on the torus, except that there is only a single discontinuity along the diagonal. This is reflected by the fact that the first homology group has only a single (finite) generator.

Figures 3 and 5 contain the main statement of the paper. They demonstrate in very simple geometric terms that the number of vector–fields required to define a global stabilizer on a compact manifold depends on the underlying cell–complex. Figures 6 displays a sample trajectory of the closed–loop system on the projective plane using an embedding in $\mathbb{R}^3$ (the corresponding situation on the torus is visualized easily by thinking of a doughnut).

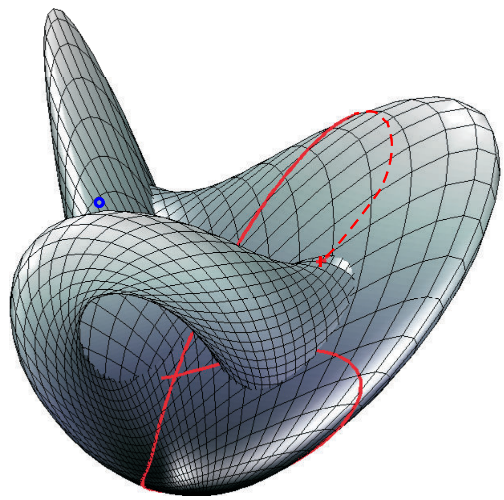

Fig. 6. Global stabilization of an equilibrium point (marked by a **+**) on the projective plane. The open–loop equilibrium is marked as **o**.

VI. Conclusion

The paper introduces homology to the study of global nonlinear control problems. Multiple vector fields are required to cover a compact boundaryless state space globally. In the examples provided, the minimum number of such vector–fields is given by the number of (finite or infinite) generators of the first homology groups.

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