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Modeling and Analysis for a Temperature System Based on Resource Dynamics and the Ideal Free Distribution

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Abstract— **Bioinspired design approaches seek to exploit nature in order to construct robust and optimal solutions. One useful concept from behavioral ecology is the ideal free distribution (IFD). Here, we relax the IFD main assumptions using the standing crop idea to introduce dynamics to the resources. Using the IFD, the standing crop, and the replicator dynamics concepts, we make some analogies with a temperature control in order to get a maximum uniform temperature. We analytically show that the equilibrium point of the system (i.e., the IFD) is a locally asymptotically stable point, and it is a globally asymptotically stable point for one special case.**

I. INTRODUCTION

The concept of ideal free distribution (IFD) was originally introduced in [1]. For many years, this concept has been used to analyze how animals distribute themselves across different habitats. These habitats have different characteristics (e.g., one habitat might have more food than another), but animals tend to reach an equilibrium point where each has the same correlate of fitness such as consumption rate. The term "ideal" means that the animals can sense the quality of all habitats and seek to maximize the suitability of the habitat they are in, and the term "free" means that the animals can go to any habitat. In [2], [3] the authors survey the various extensions to the IFD (e.g., the interference model [4] and the standing crop idea [5]), and overview the experimental biological evidence that supports these models. In the original formulation of the IFD, each habitat has unlimited resources (i.e., the consumption rate is constant for all time). However, this assumption is not true. Lessels in [5] adds dynamics to the resources, modeling each habitat with standing crop conditions. This model predicts relationships between resource density, animal density and mortality rate. She also provides some ideas in order to see the relationships between "continuous-input" models [6], [7] and the interference model [8]-[9]. In [5] four cases are presented: i) only exploitation competition, no alternative mortality; ii) exploitation and interference competition, no alternative mortality; iii) only exploitation competition, plus alternative mortality; and iv) exploitation and interference competition, plus alternative mortality. In this paper, we only consider one of these cases.

In the other hand, it has been proved that the IFD has applications to control systems in [10]-[14]. In [11] the authors show that the IFD is a global optimum point, and also it was shown that the IFD is an asymptotically stable point when the dynamics are modeled using the replicator dynamics [15]-[17]. In [11], [12], [13] the authors illustrate their ideas through a multizone temperature control problem. In this case, we use ideas in [5] to model one temperature zone.

The paper is organized as follows. In Section II we state the conditions to model a temperature zone, using Lessels's ideas. It is shown that an IFD emerges, and two cases are analyzed. First, we state the conditions for a "open loop" system, and we show that the IFD is a globally asymptotically stable (GAS) equilibrium point. Also, it is shown that the zones reach a uniform temperature that is given by the characteristics of the system. However, it is clear that a system that has some type of feedback behavior is preferred. For that, in Section III we show how the temperature zones can be modeled and controlled using a feedback type of control. In order to prove that the IFD is an stable equilibrium point for this case, we use the Sum of Squares (SOS) techniques [18], [19], [20] for a limited number of zones.

II. BIOLOGICAL MODEL

In order to model a temperature system based on the standing crop idea [5], first we introduce Lessels's concepts, and then we show that the connection with the replicator dynamics can be seen as a open loop model with a globally asymptotically stable equilibrium point.

A. Lessels' Model

Suppose that there is a set $H = \{1, 2, \dots, N\}$ of N disjoint habitats in an environment that are indexed by $i = 1, 2, \dots, N$. Let the continuous variable $x_i(t) \in \mathbb{R}_+$ be the amount of animals in the i^{th} habitat at time $t \geq 0$, where $\mathbb{R}_+ = [0, \infty)$. Let $x = [x_1, x_2, \dots, x_N]^\top \in \mathbb{R}_+^N$. Suppose that $\sum_{i=1}^{N} x_i = P$, where $P > 0$ is a constant for all time t, i.e., the amount of animals in the environment is constant. Let $R_i > 0$ be the rate at which the resources are input into the i^{th} habitat. Let $q_i(t) \in \mathbb{R}_+$ be the standing crop for the i^{th} habitat. Let $d_i(q_i)$ be a function that represents the resource loss, or an "alternative mortality" [5]. Finally, let $g_i(q_i, x_i)$ be the consumption rate per predator at habitat i.

Using these variables, Lessels introduces in [5] the model

$$\dot{q}_i = R_i - x_i g_i(q_i, x_i) - d_i(q_i) \tag{1}$$

It is assumed that the animals are "free" to move to any habitat, and that they know all parameters described before.

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The resources are depleted because the "competition" between animals. In this case "competition" can be: "exploitation competition," "interference competition," or both. "Exploitation competition" results when the individuals in any given habitat decrease the resources due to the consumption (indirect effect). In other words, if $q_i(t)$ increases, then the consumption rate per animal increases too. Mathematically we say that there exists a function $g_i(q_i, x_i)$ that is monotonically increasing in q_i, i.e.,

$$\frac{\partial g_i(q_i, x_i)}{\partial q_i} > 0, \quad \forall x_i \geq 0 \tag{2}$$

"Interference competition" is caused by the presence of other animals in a given habitat (direct effect). It is assumed either that there is no interference, represented by

$$\frac{\partial g_i(q_i, x_i)}{\partial x_i} = 0, \quad \forall x_i, q_i$$

or that there is interference, which implies that if the number of animal increases, then the consumption rate per animal decreases, i.e.,

$$\frac{\partial g_i(q_i, x_i)}{\partial x_i} < 0, \quad \forall x_i, q_i \tag{3}$$

If we let Equation (1) equal to zero, we obtain that the consumption rate per animal in the i^{th} habitat (when there is no "alternate mortality") is given by

$$\frac{R_i}{x_i} = g_i(q_i^*, x_i) \tag{4}$$

where q_i^* is an equilibrium standing crop. For the case when there is resource loss, we have that the equilibrium is

$$R_i = x_i^* g_i(q_i^*, x_i^*) + d_i(q_i^*) \tag{5}$$

B. Open Loop System Analogy

Let us choose $g_i(q_i, x_i)$ in such a way that Equations (2) and (3) are satisfied, whenever $d_i(q_i) = 0$. By the previous assumptions, if we take for all $i = 1, 2, \ldots, N$,

$$g_i(q_i, x_i) = \frac{q_i}{x_i} \tag{6}$$

the dynamics of the standing crop are given by

$$\dot{q}_i = R_i - q_i$$

where there is not influence of x_i, that could be seen as the input to the system.

We can see that Equations (2) and (3) are satisfied, because $\frac{\partial g_i(q_i,x_i)}{\partial q_i} = \frac{1}{x_i} > 0$, and $\frac{\partial g_i(q_i,x_i)}{\partial x_i} = -\frac{q_i}{x_i^2} < 0$. These assumptions are valid for the case when $x_i, q_i > 0$.

At the equilibrium, we have from Equation (1) that

$$0 = R_i - x_i^* \frac{q_i^*}{x_i^*}$$

$$q_i^* = R_i \tag{7}$$

This equation means that the standing crops for different habitats will be different if the rates at which the resources are input into the i^{th} habitat are different (i.e., $q_i = q_j$ for all i, j only if $R_i = R_j$). For this case, it is assumed that $g_i(q_i^*, x_i^*)$ is constant across patches, i.e.,

$$g_i(q_i^*, x_i^*) = g_j(q_j^*, x_j^*) = C$$

where C is a positive constant value. Using $g_i(q_i, x_i)$ in Equation (6), we obtain

$$\frac{q_i^*}{x_i^*} = \frac{q_j^*}{x_j^*} = C$$

In the literature [21], [9], when q_i^* and q_j^* are constants, the previous equation is known as the habitat matching rule. This habitat matching rule is equivalent to the input matching rule [2], [11], and hence we can predict the distribution of predators across the N habitats. This distribution is given by,

$$x_i^* = \frac{q_i^* P}{\sum_{j=1}^N q_j^*} = \frac{P R_i}{\sum_{j=1}^N R_j} \tag{8}$$

where $\sum_{j=1}^N x_j = P$.

In the next section we introduce the replicator dynamics, we show that the animals distribute according to Equation (8), and that this equilibrium point is globally asymptotically stable.

1) Replicator Dynamics: The replicator dynamics are a simple model of how selection via differential fitness affects the proportions of animals using different strategies [16], [17]. Here, we show how equilibria of one class of replicator dynamics are related to the IFD. These are not the standard replicator dynamics that are developed based on random pairings of two individuals in what is called a "linear game." Here, we extend such standard formulations in [15], [16] to represent a nonlinear game.

Each animal has N pure strategies, which correspond to choosing which habitat to live in for its entire life, and that the number of animals is constant and $\sum_{i=1}^N x_i = P$ for some $P > 0$ and all $t \geq 0$. Let

$$p_i = \frac{x_i}{\sum_{j=1}^N x_j}$$

represent the fraction of individuals in a population of animals playing pure strategy i, $i = 1, 2, \ldots, N$. Clearly $p_i \geq 0$ for $i = 1, 2, \ldots, N$, and $\sum_{i=1}^N p_i = 1$ for all $t \geq 0$. The vector $p = [p_1, p_2, \ldots, p_N]^\top$ is the "population state" which represents the strategy mix of the population [16]. Clearly, $p \in \Delta$ for all $t \geq 0$, where

$$\Delta = \left\{ p \in \mathbb{R}_+^N : \sum_{i=1}^N p_i = 1 \right\}$$

is the "constraint set" (simplex) that defines a subset of the state space. The vector $x = [x_1, x_2, \ldots, x_N]^\top$ lies in the simplex Δ_x, where $\Delta_x = \{x \in \mathbb{R}_+^N : \sum_{i=1}^N x_i = P\}$.

The replicator dynamics assume continuously mixed generations and are given by

$$\frac{\dot{p}_i}{p_i} = \beta \left[\{\text{fitness of animals that play } i \in H\} \right.$$
$$\left. \ldots - \{\text{average fitness in population}\} \right]$$

where $\beta > 0$ is a proportionality constant. The left-hand-side is the normalized rate of increase in the population share playing strategy i. The right-hand-side indicates that if i-strategists are more successful (less successful) than the average, their population share will increase (decrease, respectively).

In this specific case, we assume that the fitness of animals $i \in H$ at time t is given by Equation (6) i.e., we choose $f_i(q_i, x_i) = g_i(q_i, x_i)$. The average fitness is given by

$$\bar{f} = \sum_{j=1}^{N} p_j f_j(q_j, x_j)$$

Hence, the replicator dynamics are

$$\dot{p}_i = \beta p_i \left(f_i(q_i, x_i) - \bar{f} \right) \tag{9}$$

or,

$$\dot{x}_i = \beta x_i \left(\frac{q_i}{x_i} - \frac{1}{P} \sum_{j=1}^{N} x_j \frac{q_j}{x_j} \right) \tag{10}$$

Without loss of generality [11], if we assume $\beta = 1$, the set of differential equations for this case is

$$\begin{array}{rcl} \dot{x}_i & = & q_i - \frac{1}{P} x_i \sum_{j=1}^{N} q_j \\ \dot{q}_i & = & R_i - q_i \end{array} \tag{11}$$

Next, we show that the equilibrium point for (11)is GAS.

2) Stability Analysis: The equilibrium point for (11) is given by,

$$0 = q_i^* - \frac{1}{P} x_i^* \sum_{j=1}^{N} q_j^*$$

Since $q_i^* = R_i$,

$$x_i^* = \frac{P R_i}{\sum_{j=1}^{N} R_j}$$

Therefore the system in Equation (11) has (7) and (8) as its equilibrium points. Let us prove in the next theorem that the equilibrium point in Equation (8) is globally asymptotically stable.

Theorem 2.1: The equilibrium point given by Equation (8) for the replicator dynamics in Equation (11) is globally asymptotically stable (GAS), which region of asymptotic stability $\Delta_x - \partial \Delta_x$.

Proof: First, let $e_{x_i} = x_i - x_i^*$, and $e_{q_i} = q_i - q_i^*$ be the error coordinates for our system. Therefore,

$$\begin{array}{rcl} \dot{e}_{x_i} & = & e_{q_i} - \frac{1}{P} e_{x_i} \sum_{j=1}^{N} q_j^* - \frac{1}{P} e_{x_i} \sum_{j=1}^{N} e_{q_j} - \frac{1}{P} x_i^* \sum_{j=1}^{N} e_{q_j} \\ \dot{e}_{q_i} & = & -e_{q_i} \end{array} \tag{12}$$

We need to prove two things:

1) Stability in the sense of Lyapunov via a linear approximation.
2) Global attractivity.

The system in Equation (12) can be written as

$$\begin{array}{rcl} \dot{e}_{x_i} & = & -a_i^* e_{x_i} - \frac{1}{P} e_{x_i} \phi_i(t) + \psi_i(t) \\ \dot{e}_{q_i} & = & -e_{q_i} \end{array} \tag{13}$$

where $a_i^* = \frac{1}{P} \sum_{j=1}^{N} R_j > 0$, $\phi_i(t) = \sum_{j=1}^{N} e_{q_j}$, and $\psi_i(t) = e_{q_i} - \frac{R_i}{\sum_{j=1}^{N} R_j} \sum_{j=1}^{N} e_{q_j}$. The system can be seen as two interconnected systems, one driving the other in a open loop configuration (i.e., the system e_{x_i} is driven by e_{q_i}).

It is clear that if we do not have any kind of interconnections, both systems are globally exponentially stable (GES). That is

$$e_{q_i}(t) = e_{q_i}(0) e^{-t}$$

And the other system is 0-GES, because when $e_{q_i} = 0$, it can be written as

$$\dot{e}_{x_i} = -a_i^* e_{x_i}$$

Hence, we have an interconnection of one system that is GES, and another that is 0-GES, which implies that the whole interconnection is locally asymptotically stable (LAS). This can be seen from the first method of Lyapunov, i.e., by linearizing the system evaluated at the origin. For this case, we will have the Jacobian equal to

$$J(0,0) = \left[\begin{array}{cc} -a_i^* & \text{const} \\ 0 & -1 \end{array} \right]$$

Clearly, the Jacobian has eigenvalues $\lambda = -a_i^*$, and $\lambda = -1^*$, which implies that the system is stable in the sense of Lyapunov.

Now, we need to prove global attractivity. First of all, it is clear that the equation that has the interconnection is

$$\dot{e}_{x_i} = -a_i^* e_{x_i} - \frac{1}{P} e_{x_i} \phi_i(t) + \psi_i(t) \tag{14}$$

However, since $\phi_i(t)$ is a function of $e_{q_i}(t)$, and this function is bounded, we clearly have that

$$||\phi_i(t)|| \leq \alpha e^{-t} \leq M$$

where α and M are positive numbers. Therefore, we can see that the vector field described by Equation (14) is globally Lipschitz, i.e., for any x, y, we have that

$$\left| -a_i^* x - \frac{1}{P} x \phi_i(t) + \psi_i(t) + a_i^* y + \frac{1}{P} y \phi_i(t) - \psi_i(t) \right| =$$

$$\ldots \left| -(x - y) \left(a_i^* + \frac{1}{P} \phi_i(t) \right) \right| \leq L|x - y|$$

where L is the Lipschitz constant. Therefore, the vector field in Equation (14) is globally Lipschitz and uniformly with respect to time. This implies that the vector field is *complete*, which in turn implies that $e_{x_i}(t)$ is defined for all time t.

If we take the following Lyapunov candidate

$$V(e_{x_i}) = \frac{1}{2} e_{x_i}^2$$

Then, we obtain that the derivative across trajectories is given by

$$\dot{V} = e_{x_i} \dot{e}_{x_i}$$

$$\dot{V} = -a_i^* e_{x_i}^2 - \frac{1}{P} e_{x_i}^2 \phi_i(t) + e_{x_i} \psi_i(t) \tag{15}$$

What we want to prove is that for a finite time, we have that for every $\epsilon > 0$, there exists δ such that there is a level set

$\Omega_\delta = \{e_{x_i} : V(e_{x_i}) \leq \delta\} \subset B_\epsilon$, where B_ϵ is a ball of radius ϵ. We know that there exists a time t^* such that for every $\epsilon > 0$,

$$||e_q(t)|| \leq \epsilon$$

for all $t \geq t^*$. Hence, we need to prove that there exists a time T such that the trajectories e_{x_i} are inside the level set Ω_c, for all time $t \geq T$.

Recall that by Young's inequality we have,

$$xy \leq \bar{\epsilon}x^2 + \frac{1}{4\bar{\epsilon}}y^2$$

for all $\bar{\epsilon}, x, y > 0$. If we let $\bar{\epsilon} = \frac{1}{2}$, we will have that Equation (15) can be bounded as

$$\dot{V} \leq -a_i^* e_{x_i}^2 - \frac{1}{P} e_{x_i}^2 |\phi_i(t)| + |e_{x_i}||\psi_i(t)|$$

$$\dot{V} \leq -a_i^* e_{x_i}^2 - \frac{1}{P} e_{x_i}^2 |\phi_i(t)| + \frac{1}{2}|e_{x_i}|^2 + \frac{1}{2}|\psi_i(t)|^2$$

$$\dot{V} \leq -(a_i^* - \epsilon) e_{x_i}^2 - \left(\epsilon - \frac{1}{P}|\phi_i(t)| - \frac{1}{2}\right)|e_{x_i}|^2 + \frac{1}{2}|\psi_i(t)|^2$$

where $\epsilon < a_i^*$. It is clear that since $\phi_i(t)$ is exponentially decreasing (this term depends only on e_{q_i}), then, there exists a time t_1 such that the term $\epsilon - \frac{1}{P}|\phi_i(t)| - \frac{1}{2}$ is positive. Therefore, for all time $t \geq t_1$ we have

$$\dot{V} \leq -(a_i^* - \epsilon) e_{x_i}^2 + \frac{1}{2}|\psi_i(t)|^2$$

$$\dot{V} \leq -(a_i^* - \epsilon) 2V + \frac{1}{2}|\psi_i(t)|^2 \tag{16}$$

From Equation (16) is clear that if $V > \frac{1}{2}|\psi_i(t)|^2 \frac{1}{(a_i^* - \epsilon)}$, then $\dot{V} < 0$. If we let $\delta(t) = \frac{1}{2}|\psi_i(t)|^2 \frac{1}{(a_i^* - \epsilon)}$, it is clear that it exists a time t_2 such that $\delta(t) \geq \delta$, then the trajectories are globally attracted to the level set Ω_δ, which combined with the fact that the system is LAS, we can conclude that the whole interconnection is GAS.

III. Temperature Feedback Control Analogy

If we have a system that consists of multiple disjoint temperature zones (where a zone is composed by an actuator, e.g., a lamp, and a sensor), one simple way to model a zone with negligible accumulation of potential an kinetic energy is [22]:

$$\rho V_i C_v \dot{T}_i = -UA(T_i - T_a) + k_i x_i$$

where T_i is the temperature in the i^{th} zone; T_a is the ambient temperature; x_i is the control input (i.e., applied voltage to actuator); ρ is the density of the air; V_i is the volume of the zone; C_v is the heat capacity of the air; UA is the overall heat transfer coefficient and k_i is the proportionality constant for the applied voltage and the heating capacity of the lamp. Let $a_i = \frac{UA}{\rho V_i C_v} > 0$ and $b_i = \frac{k_i}{\rho V_i C_v} > 0$, we have

$$\dot{T}_i = -a_i T_i + b_i x_i + a_i T_a \tag{17}$$

Let $T_i(0) = T_a$, and assume that we have an upper bound temperature B such that $B >> T_a$. Let $q_i = B - T_i$ then, replacing in (17) we get that

$$\dot{q}_i = -a_i q_i - b_i x_i + a_i(B - T_a)$$

Hence,

$$\dot{q}_i = a_i(B - T_a) - x_i\left(b_i + \frac{a_i q_i}{x_i}\right) \tag{18}$$

It can be seen that (18) is equivalent to the resource loss in (1), where $R_i = a_i(B - T_a)$, and $g_i(x_i, q_i) = b_i + \frac{a_i q_i}{x_i}$.

In this case, the input rate (R_i), will correspond to a rate of dissipation of thermal energy to ambient, and the consumption rate ($g_i(x_i, q_i)$), will correspond to the heating contribution rate of the actuator to the i^{th} zone. This is due to the fact that an increment of T_i clearly represents a decrease in q_i. In this case we want to analyze the case of exploitation and interference competition, with no alternative mortality.

A. Control Dynamics

In Section II-B we assumed that $b_i = 0$, which in turn implies that there is not a clear influence of the x_i's in the dynamics of the standing crop. Now, if $b_i \neq 0$, and if we assume that the fitness $f(x_i, q_i)$ is equal to q_i, the controller dynamics are given by the replicator dynamics equation,

$$\dot{x}_i = x_i\left(q_i - \frac{1}{P}\sum_{j=1}^{N} x_j q_j\right)$$

Taking this equation and (18), we obtain the set of equations given by

$$\begin{aligned} \dot{x}_i &= x_i\left(q_i - \tfrac{1}{P}\textstyle\sum_{j=1}^{N} x_j q_j\right) \\ \dot{q}_i &= -a_i q_i - b_i x_i + a_i(B - T_a) \end{aligned} \tag{19}$$

Let us assume that $x_i > 0$, hence we want to live strictly inside the simplex Δ_x. Under this assumption, the equilibrium points of (19) are given by

$$\begin{aligned} x_i^* &= \frac{\frac{a_i}{b_i}P}{\sum_{j=1}^{N}\frac{a_j}{b_j}} \\ q_i^* &= (B - T_a) - \frac{P}{\sum_{j=1}^{N}\frac{a_j}{b_j}} \end{aligned} \tag{20}$$

Since $T_i = B - q_i$, we have that the final temperature is given by

$$T_i^* = T_a + \frac{P}{\sum_{j=1}^{N}\frac{a_j}{b_j}} \tag{21}$$

which clearly is a constant for all i. If we change coordinates, and we let $e_{q_i} = q_i - q_i^*$ and $e_{x_i} = x_i - x_i^*$, Equation (19) becomes

$$\begin{aligned} \dot{e}_{x_i} &= \left(e_{x_i} + x_i^*\right)\left(e_{q_i} - \tfrac{1}{P}\textstyle\sum_{j=1}^{N} e_{x_j}\left(e_{q_j} + q_j^*\right) - \tfrac{1}{P}\textstyle\sum_{j=1}^{N} e_{q_j} x_j^*\right) \\ \dot{e}_{q_i} &= -a_i e_{q_i} - b_i e_{x_i} \end{aligned} \tag{22}$$

For the $N = 2$ case, we obtain the system described by

$$\begin{aligned} \dot{e}_{x_1} &= \left(e_{x_1} + \tfrac{P a_1 b_2}{a_1 b_2 + a_2 b_1}\right)\left(e_{q_2} - e_{q_1}\right)\left(\tfrac{a_1 b_2}{a_1 b_2 + a_2 b_1} + \dots\right. \\ &\quad \left.\dots + \tfrac{e_{x_1} - P}{P}\right) \\ \dot{e}_{q_1} &= -a_1 e_{q_1} - b_1 e_{x_1} \\ \dot{e}_{q_2} &= -a_2 e_{q_2} + b_2 e_{x_1} \end{aligned} \tag{23}$$

where the origin is the unique equilibrium point.

B. Stability Analysis

Using SOS Techniques, the problem can be solved numerically with the construction of a set of constraints for the parameters and the application of extensions of Lyapunov's stability theorem. Using the ideas shown in [19], [20] first we define the main concepts for the SOS techniques, and then we show that the IFD is a locally asymptotically stable (LAS) equilibrium point for the $N = 2$ case. Consider the nonlinear system

$$\dot{x} = h(x, u)$$

with the constraints,

$$a_l(x, u) \leq 0 \qquad \text{for } l = 1, \dots, N_1$$

where $x \in \mathbb{R}^n$ is the state of the system, N_1 the number of constraints, and $u \in \mathbb{R}^m$ the set of uncertain parameters. We assume that a_l is a polynomial functions in (x, u), and $h(x, u)$ is a vector of polynomial or rational functions in (x, u) with no singularity in $\mathcal{D} \subset \mathbb{R}^{n \times m}$. $\mathcal{D}$ is defined as

$$\mathcal{D} = \{(x, u) \in \mathbb{R}^{n \times m} | a_l(x, u) \leq 0 \text{ for all } l\}$$

Without loss of generality, it is assumed that $h(x, u) = 0$ for $x = 0$, and $u \in \mathcal{D}_u^0$, where

$$\mathcal{D}_u^0 = \{u \in \mathbb{R}^m | (0, u) \in \mathcal{D}\}$$

The following theorem taken from [20], [19] gives us the conditions for local stability for the equilibrium point of the system.

Theorem 3.1: Suppose that for the system $\dot{x} = h(x, u)$ described above, there exist polynomial functions $V(x)$, $w(x, u)$ and $p_l(x, u)$ such that,

- $V(x)$ is positive definite in a neighborhood of the origin
- $w(x, u) > 0$ and $p_l(x, u) \geq 0$ in $\mathcal{D}$

Then,

$$-\frac{\partial V}{\partial x} h(x, u) + \sum p_l(x, u) a_l(x, u) \geq 0 \qquad (24)$$

or

$$Z(x, u) \geq 0 \qquad (25)$$

where

$$Z(x, u) = -w(x, u) \frac{\partial V}{\partial x} h(x, u) + \sum p_l(x, u) a_l(x, u) \quad (26)$$

guarantees that the origin of the state space is a stable equilibrium of the system.

An alternative proposition when we work with polynomial and rational functions is given next from [20], [19]. In this case, the positive definite conditions in the theorem above, can be expressed using the SOS definitions.

Proposition 3.2: For a polynomial function $\varphi(x) \geq 0$ (or SOS), the local stability can be guaranteed if:

- $V(x) - \varphi(x)$ is SOS, i.e., $V(x)$ positive definite.
- $Z(x, u)$ is SOS.
- $p_l(x, u)$ is SOS.

For the temperature system (23), the set of constraints of parameters is defined by the constant values a_i and b_i. These values where obtained through system identification [23] and their variability is estimated to be constrained by a sufficiently wide ranges

- $0 < a_i < 2$
- $0 < b_i < 0.01$

If we analyze (20), it is clear that in order to have always a positive q_i, we need another constraint

$$P < (B - T_a) \sum_{j=1}^{2} \frac{a_j}{b_j}$$

Using SOSTOOLS [24], we construct the SOS functions $\varphi(x)$ and $V(x)$ and the SOS polynomials $p_l(x, u)$ for $l = 1, \dots, 4$, such that the conditions in Theorem 3.1 are satisfied for a given P. The result was a fourth order Lyapunov's function that prove the local stability of the origin, which has 31 terms.

Figure 1 shows the response of the system for different initial conditions. We note that the equilibrium points for temperature and control signals are achieved at a finite time. Figure 2 shows the behavior of the numerically obtained

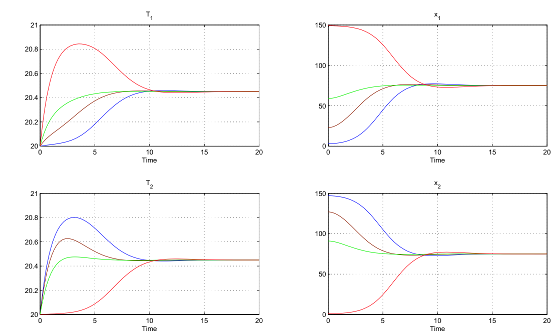

Fig. 1. Dynamics of system with $a_1, a_2 = 1, b_1, b_2 = 0.006$, and different IC's for $x_1 = (3, 23, 59, 149)$. In this case, $T_i^* = 20.45$ and $x_i^* = 75$ for $P = 150$.

$V(x)$ and $Z(x, u)$ along the trajectories of the system with different parameter variations. $V(x)$ and $Z(x, u)$ just need to

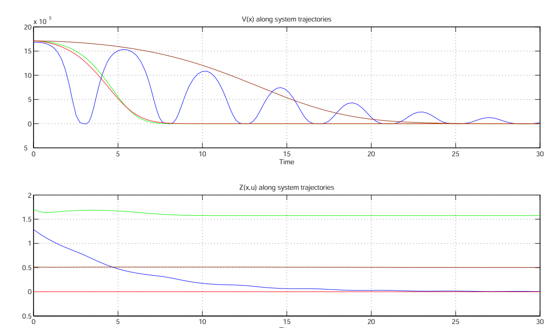

Fig. 2. $V(x)$ and $Z(x, u)$ along system trajectories with $x1_{init} = 149$ and parameters $a_1, a_2 = (0.1, 0.6, 0.9, 2)$ and $b_1, b_2 = (0.009, 0.001, 0.005, 0.01)$

be positive in order to satisfy the conditions of Theorem 3.1. This is shown in Figure 2. In addition, near the fixed limits of uncertainty, the functions are close to zero showing the activation of the constraints, and the need of finding another Lyapunov function for a wider range of uncertainty in the parameters.

IV. Conclusion

In this paper we analyze a model for a temperature control using ideas from theoretical ecology. This model is based on the Ideal Free Distribution (IFD), but without assuming continuous input. Instead, we use ideas from [5] in order to add dynamics to the resources. The temperature system is modeled using two approaches. First, the system can be seen as a open loop control model, we show that an IFD is reached, and also that the equilibrium point is globally asymptotically stable (GAS). Then, we change one of the parameters so that the system becomes a feedback control. In order to show stability of the system, we take a special case, and we use Sum of Squares (SOS) techniques (through the SOSTOOLS program) for bounded uncertainties in model parameters and changes in initial conditions. The future directions are the analysis of stability of the equilibrium point of feedback model without the dependence of numerical methods and the implementation of a test bed for the thermal system for real proves of the proposed control.

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