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Motion Planning and Tracking for Tip Displacement and Deflection Angle for Flexible Beams

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Abstract— **We present results for motion planning and tracking for flexible beams with Kelvin-Voigt damping, when the goal is to track sinusoidal reference signals for the displacement and deflection angle at the free-end of the beam using only actuation at the base. We present the solution to the motion planning problem for the string model, and a method of leveraging the string solution with PDE backstepping theory to solve the motion planning problem for the shear beam. We then present state-feedback boundary controllers that stabilize their respective systems around the motion planning solution.**

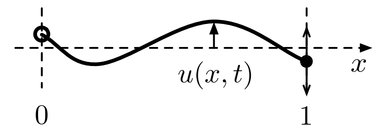

Fig. 1. Figure depicting a string/beam. The goal is to generate and track a reference trajectory at $x = 0$. The arrows at $x = 1$ represent actuation, and the circle at $x = 0$ represents the desired reference trajectory.

I. Introduction

Motion planning results for strings and flexible structures without internal/material damping have been presented in [1], [3], [12], [13], [14], [15]. We consider systems with Kelvin-Voigt damping since they are physically relevant, and note that the damping terms make the trajectory generation problem more difficult. The system models we consider are the wave equation (string and target system) and the shear beam model. The goal is to find motion planning solutions for sinusoidal free-end reference signals, using only actuation at the base. Figure 1 shows a graphical representation of that goal. A string is a single-input-single-output system, with the displacement at the base ($x = 1$) as the input, and the same quantity at the free-end ($x = 0$) as the output. A beam is a two-input-two-output system with the displacement and deflection angle at the base as the inputs, and the same quantities at the free-end as outputs. Motivation for this set-up comes from a particular shake table control problem where the table provides boundary actuation to a structure, modeled here as a flexible beam, in order to impart a desired reference trajectory at some point near its free-end.

The motion planning problem for the string is solved using a method based on postulating the solution as the infinite sum of the products of powers of the spatial variable and time dependent coefficients. This method has been discussed and implemented in [1], [2], [3], [4], [11], [12], [13], [14], [15]. A PDE backstepping approach is then used to obtain the shear beam motion planning solution, which is rather complicated, using the relatively simple solution for the string model.

The PDE backstepping approach also allows for the combination of the open-loop reference solutions with feedback to achieve exponential convergence to the reference trajectories. There are two main concepts in the control design. First a target system is chosen such that it has desirable performance characteristics, and such that it is analytically tractable (in particular, such that its motion planning and stabilization problems are easily solvable). The target system is the wave equation with a spring at one end and a damper at the other. Second a state transformation, mapping the plant state into the target state, is used to find conditions under which the application of boundary control produces a closed-loop system that emulates the target system. Details for the design of stabilizing boundary controllers for the wave equation, and shear beam can be found in [7], [8], and [5], [6], [9], [10] respectively.

Section II presents the string, target system, and shear beam models with Kelvin-Voigt damping. Section III presents the motion planning solutions for the string, target system, and shear beam. Section IV presents backstepping boundary controllers which stabilize the string and shear beam around their respective reference trajectories. Section V presents simulation results for string and shear beam.

II. Plant Models

A. String

The string model with Kelvin-Voigt (KV) damping is given by the wave equation

$$\varepsilon u_{tt} = (1 + d\partial_t)\, u_{xx} \tag{1}$$
$$u_x(0,t) = 0\,, \tag{2}$$

where $u(x,t)$ is the displacement along $0 \le x \le 1$ at time $0 \le t < \infty$, with initial conditions $u_0(x) = u(x,0)$ and $\dot{u}_0(x) = u_t(x,0)$, d is the KV damping coefficient, and ε is the inverse of the string stiffness. The boundary condition at $x = 0$ represents a free-end. The boundary input $u_x(1,t)$ will be used as a control input.

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B. Target System

The model for the target system is given by the wave equation

$$\begin{aligned} \varepsilon w_{tt} &= (1 + d\partial_t)\, w_{xx} && (3)\\ w_x(0,t) &= c_0 w(0,t) && (4)\\ w_x(1,t) &= -c_1 w_t(1,t)\,, && (5) \end{aligned}$$

where $w(x,t)$ is the displacement along $0 \le x \le 1$ at time $0 \le t < \infty$, with initial conditions $w_0(x) = w(x,0)$ and $\dot{w}_0(x) = w_t(x,0)$. The parameters c_0 and c_1 are design gains representing the spring stiffness and damping coefficient of the spring and damper located at opposite ends of the string. The spring stiffness c_0 should be large to emulate a pinned boundary condition at $x = 0$, and the damping coefficient c_1 should be chosen near $\sqrt{\varepsilon}$ to emulate a tuned damper at the end $x = 1$ [5], [6], [7], [8], [9], [10].

C. Shear Beam

The shear beam model with KV damping is given by a second-order-in-time, second-order-in-space PDE coupled with a second-order-in-space ODE

$$\begin{aligned} \varepsilon u_{tt} &= (1 + d\partial_t)\,(u_{xx} - \alpha_x) && (6)\\ 0 &= \varepsilon \alpha_{xx} + a\,(u_x - \alpha)\ . && (7) \end{aligned}$$

where $u(x,t)$ and $\alpha(x,t)$ denote the displacement the deflection angle along $0 \le x \le 1$ at time $0 \le t < \infty$, with initial conditions $u_0(x) = u(x,0)$, $\dot{u}_0(x) = u_t(x,0)$, $\alpha_0(x) = \alpha(x,0)$ and $\dot{\alpha}_0(x) = \alpha_t(x,0)$. The parameter a is proportional to the nondimensional cross-sectional area, and the parameter ε is inversely proportional to the nondimensional shear modulus. We consider a beam which is free at the end $x = 0$, i.e.,

$$\begin{aligned} u_x(0,t) &= \alpha(0,t) && (8)\\ \alpha_x(0,t) &= 0\,, && (9) \end{aligned}$$

and actuated at the end $x = 1$ through the boundary inputs $u_x(1,t)$ and $\alpha(1,t)$.

III. Motion Planning

Direct and inverse PDE backstepping transformations relating the string and shear beam to the target system are used to relate the motion planning solution of the string (easiest to find) to the motion planning solution for the target system and ultimately the shear beam.

Motion planning is done for sinusoidal tip reference trajectories since they are interesting functions in the context of shake table control (where reference signals tend to be periodic), and they form the basis for more complicated periodic reference trajectories. Note that the method used to find the string solution along with the PDE backstepping techniques used to find the target system and shear beam solutions can be used for reference trajectories other than sinusoids.

The motion planning solution for the string is found by postulating the reference solution as the infinite sum of time dependent coefficients and scaled powers of the spatial variable, i.e., $u^r(x,t) = \sum_{i=0}^{\infty} a_i(t) \frac{x^i}{i!}$. Examples of applications of this approach can be found in [1], [4], [11], [15].

Theorem 1: The string model (1), (2) is satisfied by the state reference trajectory

$$\begin{aligned} u^r(x,t) = \frac{A_u}{2}\Big[& e^{\hat{\beta}(\omega_u)x}\sin\left(\omega_u t + \beta(\omega_u)x\right)\\ & + e^{-\hat{\beta}(\omega_u)x}\sin\left(\omega_u t - \beta(\omega_u)x\right)\Big]\,, \quad (10) \end{aligned}$$

where the functions $\beta(n)$, and $\hat{\beta}(n)$ are defined as

$$\begin{aligned} \beta(n) &= n\sqrt{\varepsilon}\sqrt{\frac{\sqrt{1+n^2d^2}+1}{2\left(1+n^2d^2\right)}} && (11)\\ \hat{\beta}(n) &= n\sqrt{\varepsilon}\sqrt{\frac{\sqrt{1+n^2d^2}-1}{2\left(1+n^2d^2\right)}}\,. && (12) \end{aligned}$$

The output of the system satisfies the reference trajectory

$$u^r(0,t) = A_u \sin(\omega_u t)\,, \qquad (13)$$

where A_u and ω_u are the amplitude and frequency respectively. The open-loop Dirichlet control $u^r(1,t)$ can be found by evaluating (10) at $x = 1$. The expression for the open-loop Neumann control input $u_x^r(1,t)$ is given by the partial-derivative-with-respect-to-x of (10) evaluated at $x = 1$.

Proof: The motion planning solution (10) evaluated at $x = 0$ satisfies the desired reference trajectory (13). Equation (10) substituted into (1) and (2) satisfies the string PDE and free-end boundary condition. ■

The string model and the target system are related through the direct backstepping transformation

$$w(x,t) = u(x,t) + c_0 \int_0^x u(y,t)\,dy\,, \qquad (14)$$

which satisfies (3), (4). Therefore, the target system reference solution can be found by substituting (10) into the transformation

$$w^r(x,t) = u^r(x,t) + c_0 \int_0^x u^r(y,t)\,dy\,. \qquad (15)$$

Theorem 2: The target system (3), (4) is satisfied by the state reference trajectory

$$\begin{aligned} w^r(x,t) = & \frac{A_u}{2}\Big[e^{\hat{\beta}(\omega_u)x}\sin\left(\omega_u t + \beta(\omega_u)x\right)\\ & + e^{-\hat{\beta}(\omega_u)x}\sin\left(\omega_u t - \beta(\omega_u)x\right)\Big]\\ & - \frac{c_0 A_u}{2}\Big\{\gamma(\omega_u)\Big[e^{\hat{\beta}(\omega_u)x}\\ & \times \cos\left(\omega_u t + \beta(\omega_u)x\right)\\ & - e^{-\hat{\beta}(\omega_u)x}\cos\left(\omega_u t - \beta(\omega_u)x\right)\Big]\\ & - \hat{\gamma}(\omega_u)\Big[e^{\hat{\beta}(\omega_u)x}\sin\left(\omega_u t + \beta(\omega_u)x\right)\\ & - e^{-\hat{\beta}(\omega_u)x}\sin\left(\omega_u t - \beta(\omega_u)x\right)\Big]\Big\}\,, \quad (16) \end{aligned}$$

where the functions $\beta(n)$, and $\hat{\beta}(n)$ are defined in (11), (12), and $\gamma(n)$, and $\hat{\gamma}(n)$ are defined as

$$\gamma(n) = \frac{1}{n\sqrt{\varepsilon}}\sqrt{\frac{\sqrt{1+n^2d^2}+1}{2}} \quad (17)$$

$$\hat{\gamma}(n) = \frac{1}{n\sqrt{\varepsilon}}\sqrt{\frac{\sqrt{1+n^2d^2}-1}{2}}. \quad (18)$$

The output of the system satisfies the reference trajectory

$$w^r(0,t) = A_u \sin(\omega_u t). \quad (19)$$

An open-loop Dirichlet control input $w^r(1,t)$ can be found by evaluating (16) at $x=1$. The expression for the open-loop Neumann control input $w_x^r(1,t)$ is given by the partial-derivative-with-respect-to-x of (16) evaluated at $x=1$.

Proof: The motion planning solution (16) evaluated at $x=0$ satisfies the reference trajectory (19). Equation (16) substituted into (3) and (4) satisfies the target system PDE and $x=0$ boundary condition. ∎

The solution to the motion planning problem for the shear beam model (6)—(9) can be found by using a backstepping transformation from the target to plant state. The inverse transformation is

$$u(x,t) = w(x,t) - r(x,t) + \int_0^x l(x,y)\left(w(y,t)-r(y,t)\right)\,dy, \quad (20)$$

where $l(x,y)$ is the inverse transformation gain kernel, and $r(x,t)$ is the state of an auxiliary system required to satisfy the transformation from target to plant when the motion planning solution for the tip deflection angle is introduced into the design. The inverse transformation can be used to write the shear beam motion planning solution $u^r(x,t)$ in terms of the target system solution $w^r(x,t)$.

Theorem 3: The shear beam model (6)—(9) is satisfied by the state reference trajectory

$$u^r(x,t) = w^r(x,t) - r(x,t) + \int_0^x l(x,y)\left(w^r(y,t)-r(y,t)\right)\,dy, \quad (21)$$

where $w^r(x,t)$ is given in (16),

$$\begin{aligned} r(x,t) = {} & A_\alpha \left(f_1(x) - \int_0^x f_1(x-y)\Phi(y)\,dy \right)\sin(\omega_\alpha t) \\ & + A_\alpha \left(f_2(x) - \int_0^x f_2(x-y)\Phi(y)\,dy \right)\cos(\omega_\alpha t) \end{aligned} \quad (22)$$

with

$$\begin{aligned} f_1(x) &= \gamma(\omega_\alpha)\sin(\beta(\omega_\alpha)x)\cosh\left(\hat{\beta}(\omega_\alpha)x\right) \\ &\quad + \hat{\gamma}(\omega_\alpha)\cos(\beta(\omega_\alpha)x)\sinh\left(\hat{\beta}(\omega_\alpha)x\right) \end{aligned} \quad (23)$$

$$\begin{aligned} f_2(x) &= -\gamma(\omega_\alpha)\cos(\beta(\omega_\alpha)x)\sinh\left(\hat{\beta}(\omega_\alpha)x\right) \\ &\quad + \hat{\gamma}(\omega_\alpha)\sin(\beta(\omega_\alpha)x)\cosh\left(\hat{\beta}(\omega_\alpha)x\right) \end{aligned} \quad (24)$$

$$\Phi(x) = -b\sinh(bx) + b\int_0^x k(x,y)\sinh(by)dy, \quad (25)$$

where $\beta(\omega_\alpha)$, $\hat{\beta}(\omega_\alpha)$, $\gamma(\omega_\alpha)$, $\hat{\gamma}(\omega_\alpha)$ are given in (11), (12), (17), (18), and $l(x,y)$ satisfies

$$l_{xx} = l_{yy} - b^2 l - b^3\sinh b(x-y) - b^3\int_y^x \sinh b(x-\xi)l(\xi,y)\,d\xi \quad (26)$$

$$l(x,x) = -\frac{b^2}{2}x - c_0 \quad (27)$$

$$l_y(x,0) = c_0 l(x,0) - b^2\cosh bx, \quad (28)$$

with $b = \sqrt{\frac{a}{\varepsilon}}$. The outputs of the system satisfy the tip displacement and deflection angle reference trajectories

$$u^r(0,t) = A_u\sin(\omega_u t) \quad (29)$$

$$\alpha^r(0,t) = A_\alpha\sin(\omega_\alpha t), \quad (30)$$

where A_u, A_α, ω_u, ω_α are the amplitudes and frequencies respectively.

The open-loop control inputs are given by

$$\begin{aligned} u_x^r(1,t) = {} & w_x^r(1,t) - r_x(1,t) \\ & + l(1,1)\left(w^r(1,t)-r(1,t)\right) \\ & + \int_0^1 l_x(1,y)\left(w^r(y,t)-r(y,t)\right)dy \end{aligned} \quad (31)$$

$$\alpha^r(1,t) = \cosh(b)\alpha^r(0,t) + b\sinh(b)u^r(0,t) - b^2\int_0^1 \cosh\left(b(1-y)\right)u^r(y,t)\,dy, \quad (32)$$

where $w^r(1,t)$ is given by (16) evaluated at $x=1$, $w_x^r(1,t)$ is given by the partial-derivative-with-respect-to-x of (16) evaluated at $x=1$, $r(1,t)$ is given by (22) evaluated at $x=1$, and $r_x(1,t)$ is given by the partial-derivative-with-respect-to-x of (22) evaluated at $x=1$.

Proof: Substituting (21) into the "strict-feedback shear beam model for motion planning" [(6), (8) with $\alpha(x,t)$ substituted by $\alpha(x,t) = \cosh(bx)\alpha^r(0,t) - b\int_0^x \sinh\left(b(x-y)\right)u_y(y,t)\,dy$] satisfies the PDE and free-end boundary condition. The motion planning solution (21) evaluated at $x=0$ satisfies the reference trajectory (29). The expression for $\alpha(x,t)$ evaluated at $x=0$ satisfies the reference trajectory (30). ∎

IV. Reference Tracking

Definition 1: The reference trajectory $u^r(x,t)$ is said to be exponentially stable if there exist positive constants M

and m such that

$$\begin{aligned}&\left[\|u(t)-u^r(t)\|^2+\|u_t(t)-u_t^r(t)\|^2\right]^{1/2}\\ \leq\quad & Me^{-mt}\left[\|u_0-u_0^r\|^2+\|\dot{u}_0-\dot{u}_0^r\|^2\right]^{1/2}\,,\quad (33)\end{aligned}$$

where $\|\cdot\|$ denotes the norm of v, $\|v\|=\left(\int_0^1 v\left(x\right)^2\, dx\right)^{1/2}$, and $u_0(x)=u(x,0)$, $u_0^r(x)=u^r(x,0)$, $\dot{u}_0(x)=u_t(x,0)$, $\dot{u}_0^r(x)=u_t^r(x,0)$.

Theorem 4: The state feedback controller

$$\begin{aligned}u_x(1,t) &= -c_0u(1,t)-c_1u_t(1,t)-c_0c_1\int_0^1 u_t(y,t)\,dy\\ &\quad +w_x^r(1,t)+c_1w_t^r(1,t)\,, \qquad (34)\end{aligned}$$

exponentially stabilizes the string system (1), (2) about the state reference trajectory (10).

Proof: The expression for the boundary controller (34) is found by writing the stabilizing boundary controller in [7], [8] in terms of the reference tracking error $\tilde{u}(x,t)=u(x,t)-u^r(x,t)$, where $w_x^r(1,t)+c_1w_t^r(1,t)=u^r(1,t)+c_0u^r(1,t)+c_1u_t^r(1,t)+c_0c_1\int_0^1 u_t^r(y,t)\,dy$.

By Theorem 1 the string reference solution (10) satisfies the string model with boundary control (34), and is therefore governed by the same PDE and boundary conditions. The tracking error dynamics then resemble (1), (2), (34) [with $w_x^r(1,t)$ and $w_t^r(1,t)$ set to zero]. The direct and inverse backstepping transformations $\tilde{w}(x,t)=\tilde{u}(x,t)+c_0\int_0^x\tilde{u}(y,t)\,dy$, and $\tilde{u}(x,t)=\tilde{w}(x,t)-c_0\int_0^x e^{-c_0(x-y)}\tilde{w}(y,t)\,dy$ relate the tracking error dynamics and the exponentially stable tracking error target system

$$\begin{aligned}\varepsilon\tilde{w}_{tt} &= (1+d\partial_t)\,\tilde{w}_{xx} && (35)\\ \tilde{w}_x(0,t) &= c_0\tilde{w}(0,t) && (36)\\ \tilde{w}_x(1,t) &= -c_1\tilde{w}_t(1,t)\,. && (37)\end{aligned}$$

The state of the tracking error system $\tilde{u}(x,t)$ can be bounded by the state of the tracking error target system $\tilde{w}(x,t)$ by $\|\tilde{u}(t)\|\leq(1+c_0)\,\|\tilde{w}(t)\|$, and the same is true for the time derivatives, therefore the closed-loop system (1), (2), (34) is exponentially stable around the reference solution (10). ■

Theorem 5: The state feedback controllers

$$\begin{aligned}u_x(1,t) &= k(1,1)u(1,t)+\int_0^1 k_x(1,y)u(y,t)\,dy\\ &\quad -c_1u_t(1,t)+c_1\int_0^1 k(1,y)u_t(y,t)\,dy\\ &\quad +w_x^r(1,t)+c_1w_t^r(1,t)\\ &\quad -r_x(1,t)-c_1r_t(1,t) \qquad (38)\end{aligned}$$

$$\begin{aligned}\alpha(1,t) &= \cosh(b)\alpha^r(0,t)+b\sinh(b)u(0,t)\\ &\quad -b^2\int_0^1\cosh\left(b(1-y)\right)u(y,t)\,dy\,, \qquad (39)\end{aligned}$$

where $k(x,y)$ satisfies the partial-integro-differential equation

$$\begin{aligned}k_{xx} &= k_{yy}+b^2k-b^3\sinh\left(b(x-y)\right)\\ &\quad +b^3\int_y^x k(x,\xi)\sinh\left(b(\xi-y)\right)\,d\xi \qquad (40)\\ k(x,x) &= -\frac{b^2}{2}x-c_0 \qquad (41)\\ k_y(x,0) &= -b^2\cosh(bx)\\ &\quad +b^2\int_0^x k(x,y)\cosh(by)\,dy\,, \qquad (42)\end{aligned}$$

$b=\sqrt{\frac{a}{\varepsilon}}$, exponentially stabilize the shear beam (6)—(9) about the state reference trajectory (21).

Proof: The expression for the boundary controller (38) is found by expressing the target system boundary condition (5) in terms of the tracking error $[w_x(1,t)-w_x^r(1,t)]=-c_1\left[w_t(1,t)-w_t^r(1,t)\right]$, then expanding $w_x(1,t)$ and $w_t(1,t)$ using the direct backstepping transformation $w(x,t)=u(x,t)-\int_0^x k(x,y)u(y,t)\,dy$.

By Theorem 3 the shear beam reference solution (21) satisfies the "strict-feedback shear beam model for motion planning" and therefore has the same dynamics. The tracking error dynamics can then be written as

$$\begin{aligned}\varepsilon\tilde{u}_{tt} &= (1+d\partial_t)\left(\tilde{u}_{xx}+b^2\tilde{u}-b^2\cosh\left(bx\right)\tilde{u}(0,t)\right.\\ &\quad \left.+b^3\int_0^x\sinh\left(b\left(x-y\right)\right)\tilde{u}(y,t)\,dy\right) \qquad (43)\\ \tilde{u}_x(0,t) &= 0 \qquad (44)\\ \tilde{u}_x(1,t) &= k(1,1)\tilde{u}(1,t)+\int_0^1 k_x(1,y)\tilde{u}(y,t)\,dy\\ &\quad -c_1\tilde{u}_t(1,t)+c_1\int_0^1 k(1,y)\tilde{u}_t(y,t)\,dy\,, \qquad (45)\end{aligned}$$

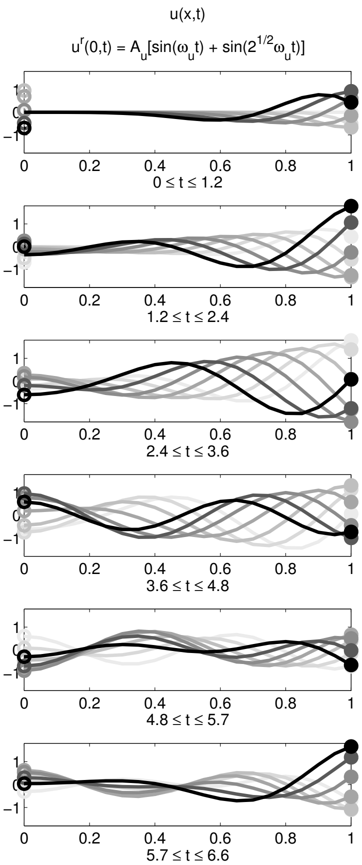
which resemble the closed-loop "strict-feedback shear beam model" dynamics. The direct and inverse backstepping transformations $\tilde{w}(x,t)=\tilde{u}(x,t)-\int_0^x k(x,y)\tilde{u}(y,t)\,dy$, and $\tilde{u}(x,t)=\tilde{w}(x,t)+\int_0^x l(x,y)\tilde{w}(y,t)\,dy$ relate the tracking error dynamics (43)–(45) to the exponentially stable tracking error target system (35)–(37). The state of the tracking error system $\tilde{u}(x,t)$ can be bounded by the state of the tracking error target system $\tilde{w}(x,t)$ by $\|\tilde{u}(t)\|\leq(1+\|l(1,y)\|_\infty)\,\|\tilde{w}(t)\|$, and the same is true for the time derivatives. Therefore the closed-loop system (6)—(9), (38), (39) is exponentially stable around the solution (21). ■

V. Simulation Results

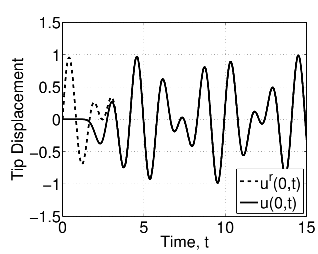
Simulations are computed in Matlab. Finite-differences are used to resolve partial derivatives in space, and the Crank-Nicolson method is used to march the equations in time.

A. String

Simulation results are shown for the string (1), (2) in closed-loop with the boundary controller (34), when the goal is to track the reference trajectory $u^r(0,t)=A_u\left[\sin\left(\omega_ut\right)+\sin\left(\sqrt{2}\omega_ut\right)\right]$. Tracking of two sinusoids is achieved by implementing the boundary controller as a function of the linear combination of the reference generation solutions for each frequency.



(a)

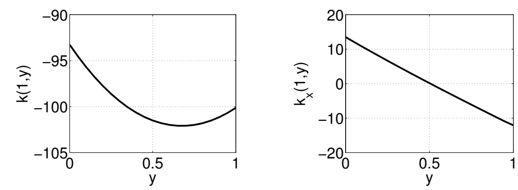


(b)

Fig. 2. String simulation results showing (a) the string state as snapshots in time, and (b) a comparison of the string tip displacement and reference trajectory.

The spatial and temporal step sizes used in simulation are $\Delta x = \frac{1}{100}$ and $\Delta t = \frac{1}{100}$. The string parameters are $d = 0.08$, and $\varepsilon = 5$. The controller parameters are chosen as $c_0 = 100$ and $c_1 = 0.99\sqrt{5}$. The reference trajectory parameters are $A_u = \frac{1}{2}$ and $\omega_u = \pi$. The string begins with zero initial displacement and velocity.

Figure 2(a) shows the evolution of the string state $u(x,t)$ on $0 \leq x \leq 1$ as a sequence of snapshots in time, with increasing darkness corresponding to increasing time in each sequence. The reference trajectory at the corresponding time is represented by a circle at $x = 0$ of the same shade. Figure 2(b) compares the tip displacement with the reference trajectory. The frequencies ω_u and $\sqrt{2}\omega_u$ are incommensurate, and therefore $u^r(0,t)$ never repeats itself.



(a) (b)

Fig. 4. Shear beam controller gains (a) $k(1,y)$, and (b) $k_x(1,y)$.

B. Shear Beam

Simulation results are shown for the shear beam model[1] (6)–(9) in closed-loop with the state feedback controllers (38) and (39), when the goal is to track the reference trajectories $u^r(0,t) = A_u \sin(\omega_u t)$ and $\alpha^r(0,t) = A_\alpha \sin(\omega_\alpha t)$.

The spatial and temporal step sizes used in simulation are $\Delta x = \frac{1}{100}$ and $\Delta t = \frac{1}{50}$. The beam parameters are $a = 5$, $d = 0.1$, $\varepsilon = 20$ and $\mu = 0.02$. The controller parameters are $c_0 = 100$ and $c_1 = 0.99\sqrt{20}$. The reference trajectory parameters are $A_u = \frac{1}{2}$, $\omega_u = \frac{\pi}{3}$, $A_\alpha = \frac{1}{4}$ and $\omega_u = \pi$. The beam is initialized with an initial displacement $u(x, t_0) = -\frac{1}{10}(1-x)^2$, initial deflection angle $\alpha(x, t_0) = \frac{1}{5}(1-x)$, and zero initial velocity.

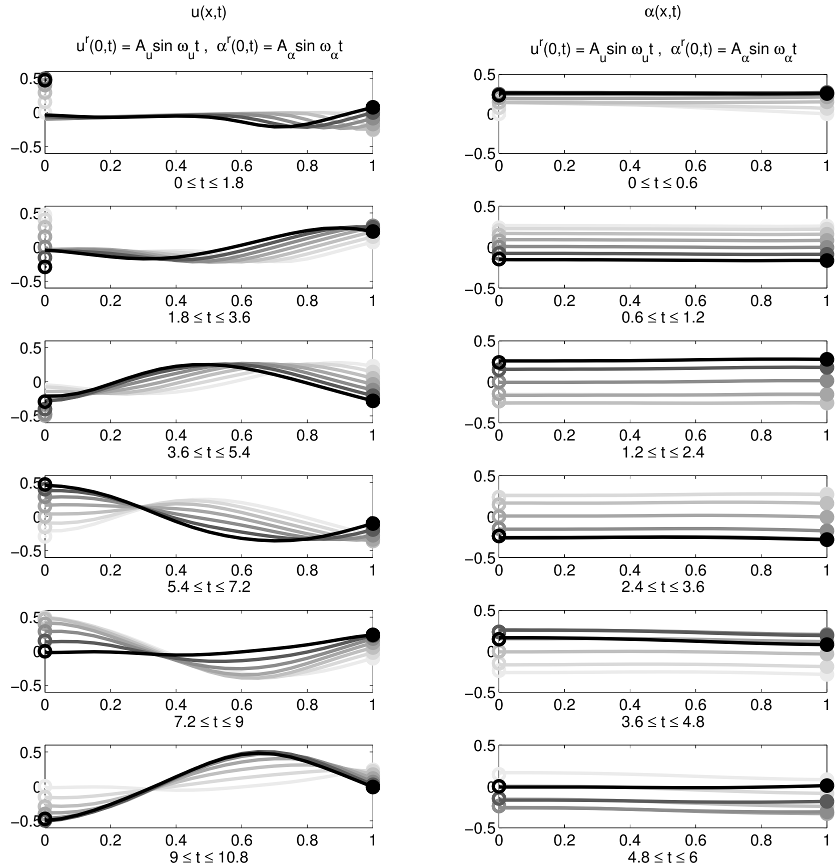
Figures 3(a) and (b) show the evolution of the beam states $u(x,t)$ and $\alpha(x,t)$ for the simultaneous tracking of the sinusoidal tip reference trajectories $u^r(0,t) = A_u \sin(\omega_u t)$ and $\alpha^r(0,t) = A_\alpha \sin(\omega_\alpha t)$. Figures 4(a) and (b) show the control gains $k(1,y)$ and $k_x(1,y)$ on the interval $0 \leq y \leq 1$. The curves are relatively simple and can be approximated by quadratic and a linear functions respectively.

VI. Conclusion

We have presented motion planning solutions for the string model, target system, and shear beam with Kelvin-Voigt damping, along with results for the combination of the motion planning solutions with exponentially stabilizing tracking controllers. The beam design combines PDE boundary backstepping methods with classical trajectory generation methods to simplify the problem from solving the motion problem for more complicated systems described by a PDE coupled with an ODE, to finding the motion planning solution for a target system with fewer spatial derivatives. Simulation results have been provided to highlight the performance of the tracking boundary controllers when applied to the string and beam.

While this work has focused on motion planning for periodic trajectories, our approach extends to a far broader class of temporal waveforms, that includes polynomials, exponentials, sinusoids, and products thereof as special cases. With a slight modification one can obtain motion planning

[1]Simulations are done for the more complicated Timoshenko beam model with KV damping, which reduces to the shear beam model via a singular perturbation (where μ, the small parameter, is proportional to the nondimensional moment of inertia of the beam). Shear beam results apply approximately—modulo an $O(\mu)$ tracking error—to the Timoshenko beam for small μ.



(a) (b)

Fig. 3. Shear beam simulation results showing snapshots of the beam states (a) $u(x,t)$, and (b) $\alpha(x,t)$ for simultaneous reference tracking.

solutions for all output reference trajectories that can be written in the form $u^r(0,t) = CX(t)$ where $X(t)$ is a solution of the autonomous linear 'exosystem' $\dot{X} = AX$ for a given initial condition $X(0)$. For example, if the reference output is $u^r(0,t) = t\mathrm{e}^{-t}\sin t$, we would choose the parameters of our exosystem as $C = [1\ 0\ 0\ 0]$, $A = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -8 & -8 & -4 \end{array}\right]$, $X(0) = [2\ 2\ 0\ 0]^{\mathrm{T}}$, and proceed to find the motion planning solution using the matrix exponentials of A.

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