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N Player Nash Cumulant Games

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***Abstract*— In stochastic game theory, the mean of a player's cost function has played a prominent role as a performance index. However, the mean is just one of many other cumulants. In fact, it is the first cumulant, with the second being the variance. The objective of this paper is to begin an N-player, higher order cumulant, stochastic differential Nash game. The problem is defined for a class of nonlinear systems with non-quadratic costs. Then sufficient conditions for the equilibrium solutions are developed. Lastly, for the case of linear systems with quadratic cost functions, the equilibrium solutions are determined with coupled Riccati equations.**

I. INTRODUCTION

Game theory has provided some interesting results for control theory. One such important application of game theory has been using a games approach to solving the H_∞ and H_2/H_∞ control problems [1] and [7]. These results can be used with a system with a stochastic input, where in this case the mean of the costs are used as the performance indices. However, the mean is the first cumulant. Using a game theoretic approach and higher order cumulants leads to a generalization of the H_2/H_∞ and H_∞ control techniques. Such generalizations are given for the variance in [2], [3], and [4]. In [9], k cumulant games were discussed for only the linear quadratic case. In fact, it was the quadratic nature of cumulants for the linear system and quadratic costs that was exploited. Here, the development will be done for a class of nonlinear systems with non-quadratic costs. This interaction between cumulants and games leads to this study of N-player, Nash cumulant games.

The mean is just one cumulant. Cumulants are a statistic that can be determined from the second characteristic function, which is the natural logarithm of the first characteristic function. As with moments and the first characteristic function, the kth cumulant is the kth derivative of the second characteristic function. Furthermore, if all of the cumulants are known, then the random variable can be completely described. In the case of structural control, adding additional cumulants beyond the mean was found to be worthwhile when the control was applied to a benchmark [8]. Each additional cumulant that was added reduced the various measures of the structural vibration. This helps motivate the study of using cumulants for N-player games.

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The discussion will begin by defining the problem, including the class of nonlinear systems and non-quadratic costs that will be used and the type of admissible player strategy. Sufficient conditions will be developed for both the k-th moment and k-th cumulant problems. These sufficient conditions will be for the nonlinear system and non-quadratic cost. Finally a linear system with quadratic costs will be examined. Equilibrium solutions for the players will be determined.

II. N PLAYER NASH GAME

Work on cumulant games started out with a two player, two cumulant game. The next logical step would be to consider an N-player, multiple-cumulant Nash game. That is what will be done here, in which a general sufficient condition will be determined for up to the fourth cumulant. Before this is presented, some preliminaries must be taken care of. For this game, consider the stochastic differential equation

$$\begin{aligned} dx(t) =& f(t, x(t), u_1(t), \cdots, u_N(t))dt \\ &+ \sigma(t, x(t))d\xi(t) \end{aligned} \tag{1}$$

where $x(t_0) = x_0$ is a random variable independent of ξ, $x \in \mathbb{R}^n$ is the state, $u_i \in U_i \subset \mathbb{R}^{p_i}$ is the i-th player, $i = 1, \cdots, N$, and ξ is a d-dimensional Brownian motion with variance W. The functions f, u_i will be assumed to satisfy both linear growth and Lipschitz conditions. That is, f and σ satisfy

(i) There exists a constant C such that

$$\|f(t, x, u_1, \cdots, u_N)\| \leq C\left(1 + \|x\| + \sum_{l=1}^{k} |u_l|\right)$$
$$\|\sigma(t, x)\| \leq C(1 + \|x\|)$$

for all $(t, x, u_1, \cdots, u_N) \in \bar{Q}_0 \times \mathcal{U}_1 \times \cdots \times \mathcal{U}_k$, $(t, x) \in \bar{Q}_0 = [t_0, t_f] \times \mathbb{R}^n$, and $\|\cdot\|$ is the Euclidean norm.

(ii) There is a constant K so that

$$\begin{aligned} &\|f(t, \tilde{x}, \tilde{u}_1, \cdots, \tilde{u}_N) - f(t, x, u_1, \cdots, u_N)\| \\ &\qquad \leq K\left(\|\tilde{x} - x\| + \sum_{l=1}^{k} \|\tilde{u}_l - u_l\|\right) \end{aligned}$$
$$\|\sigma(t, \tilde{x}) - \sigma(t, x)\| \leq K\|\tilde{x} - x\|$$

for all $t \in T$; $x, \tilde{x} \in \mathbb{R}^n$; $u_l, \tilde{u}_l \in \mathcal{U}_l$.

Furthermore, the players strategies $u_l(t) = \mu_l(t, x(t))$ satisfy

(i) for some constant $\tilde{C}$

$$\|\mu_l(t, x)\| \leq \tilde{C}(1 + \|x\|)$$

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(ii) there exists a constant $\tilde{K}$ such that

$$\|\mu_l(t,\tilde{x}) - \mu_l(t,x)\| \leq \tilde{K}(\|\tilde{x} - x\|)$$

where $t \in T$ and $x, \tilde{x} \in \mathbb{R}^n$. Often we will suppress the dependence on t and x and refer to the strategies as simply μ_l.

The cost function for the l-th player will be given as

$$\begin{aligned} J_l = &\int_{t_0}^{t_f} L_l(t, x(t), u_1(t), \cdots, u_N(t))dt \\ &+ \psi_l(t_f, x(t_f)) \end{aligned} \tag{2}$$

where L_l, ψ_l are continuous functions that satisfy polynomial growth conditions for $l = 1, \cdots, N$. If the strategies μ_l satisfy these conditions, then they are admissible strategies. We can rewrite the stochastic differential equation as

$$dx(t) = \tilde{f}(t, x(t))dt + \sigma(t, x(t))d\xi(t) \qquad x(t_0) = x_0 \tag{3}$$

where the strategy $(\mu_1, \cdots, \mu_k)$ has been substituted into f, called $\tilde{f}$. The conditions of Theorem V4.1 of [5] are now satisfied and we see that if $E\|x(t_0)\|^2 < \infty$, then a solution of (1) exists. Furthermore the solution $x(t)$ is unique in the sense that if there exists another solution $\tilde{x}(t)$ with $\tilde{x}(t_0) = x_0$, then the two solutions have the same sample paths with probability 1. The resulting process is a Markov diffusion process ([5] pg. 123) and the moments of $x(t)$ are bounded.

Also, the operator $\mathcal{O}^l = \mathcal{O}_1^l + \mathcal{O}_2$ will be given by

$$\begin{aligned} \mathcal{O}_1^l =& \frac{\partial}{\partial t} + f'(t, x, \mu_1^*, \cdots, \mu_{l-1}^*, \mu_l, \mu_{l+1}^*, \cdots, \mu_N^*)\frac{\partial}{\partial x} \\ \mathcal{O}_2 =& \frac{1}{2} tr\left(\sigma(t,x)W(t)\sigma'(t,x)\frac{\partial^2}{\partial x^2}\right). \end{aligned}$$

The performance index will be given as

$$\phi_i(t, x, u_1, \cdots, u_N) = \Lambda_k^i(t, x, u_1, \cdots, u_N) \tag{4}$$

where Λ_j^i is the k-th cumulant of the i-th player's cost function, J_i. Also, an admissible j-th moment cost function M_j^i is defined as in the two player,multiple cumulant game for $j = 1, \cdots, k-1$. Likewise, we have $\mathcal{U}_{Mi}$ and the j-th cumulant cost function $K_j^i(t,x)$.

Let $C^{1,2}(\bar{Q}_0)$ be the class of functions Φ that have continuous first partial derivatives with respect to t and continuous second partial derivatives with respect to x: $\Phi_t, \Phi_{x_i}, \Phi_{x_i x_j}$ for $i, j = 1, 2, \cdots, n$. Now let $C_p^{1,2}(\bar{Q}_0)$ be the class of functions $\Phi(t,x)$ that are of class $C^{1,2}(\bar{Q}_0)$ but where $\Phi, \Phi_t, \Phi_{x_i}, \Phi_{x_i,x_j}$ satisfy a polynomial growth condition. A polynomial growth condition for a function Φ is such that there exist constants c_1 and c_2 so that $\|\Phi(t,x)\| \leq c_1(1 + \|x\|^{c_2})$ for all $(t,x) \in \bar{Q}_0$, where $\bar{Q}_0 = [t_0, t_f] \times \mathbb{R}^n$. This yields the Dynkin formula

$$\begin{aligned} \Phi(t,x) =& E\left\{\int_t^{t_f} -\mathcal{O}^l \Phi(s, x(s))ds | x(t) = x\right\} \\ &+ E\left\{\Phi(t_f, x(t_f)) | x(t) = x\right\} \end{aligned} \tag{5}$$

where $\mathcal{O}^l$ is the backward evolution operator given by

$$\begin{aligned} \mathcal{O}^l =& \frac{\partial}{\partial t} + f'(t, x, u_1^*, \cdots, u_l, \cdots, u_N^*)\frac{\partial}{\partial x} \\ &+ \frac{1}{2} tr\left(\sigma(t,x)W(t)\sigma'(t,x)\frac{\partial^2}{\partial x^2}\right) \end{aligned} \tag{6}$$

with $E\{d\xi(t)d\xi'(t)\} = W(t)$. The expectation in (5) will now be referred to as E_{tx}.

III. Preliminaries

Consider the j-th moment of the cost function J_l, defined as

$$V_j^l(t, x; \mu_1, \cdots, \mu_N) = E_{tx}\{J_l^j(t, x; \mu_1, \cdots, \mu_N)\} \tag{7}$$

where E_{tx} is the expectation given $x(t) = x$. From [6], the $(j+1)$-st moment may be determined through

$$\begin{aligned} V_{j+1}^l(t, x; \mu_1, \cdots, \mu_N) = \sum_{i=0}^{j} \frac{j!}{i!(j-i)!}\Big[\\ V_{j-i}^l(t, x; \mu_1, \cdots, \mu_N) \cdot \Lambda_{i+1}^l(t, x; \mu_1, \cdots, \mu_N)\Big] \end{aligned} \tag{8}$$

where Λ_i^l is the i-th cumulant of the cost function J_l, where i, j are integers, $i \leq j$, and $0 \leq l \leq N$. This equation gives a much needed relationship between the moments and the cumulants.

A. Definitions

In this section we shall begin with some definitions for the problem of three cumulants. The approach taken here, in which the first to $k-1$th cumulants have admissible cumulant cost function is a growth from Won's work, [12].

Definition 1: A function $M_j^l : \bar{Q}_0 \to \mathbb{R}^+$ is an admissible j-th moment cost function for the lth player if there exists a strategy μ_l such that $M_j^l(t,x) = V_j^l(t, x; \mu_1, \cdots .\mu_N)$, where $M_j^l \in C^{1,2}(\bar{Q}_0)$ and $j = 1, 2, \cdots, k-1$. K_j^l is then the admissible jth cumulant cost function for the lth player that is defined through

$$\begin{aligned} K_{j+1}^l(t,x) =& M_{j+1}^l(t,x) \\ &- \sum_{i=0}^{j} \frac{(j)!}{i!(j-i)!} M_{j-i}^l(t,x) K_{i+1}^l(t,x). \end{aligned} \tag{9}$$

where K_i^l and M_i^l for $i = 1, \cdots, j$ are, respectively, admissible i-th cumulant and moment cost functions. Furthermore, for $\mu_l \in \mathcal{U}_{M_l}$, $K_j^l(t,x) = \Lambda_j^l(t, x; \mu_1^*, \cdots, \mu_l, \cdots \mu_N^*)$.

Definition 2: Let the class of admissible strategies $\mathcal{U}_{M_l}$ be such that if $\mu_l \in \mathcal{U}_{M_l}$ then μ_l is such that it satisfies the equality of Definition 1 from the definitions of $M_1^l, \cdots, M_j^l$, where $j = 1, 2, \cdots, k-1$.

Note that the first cumulant is the same as the first moment. The first and second cumulant cost functions $K_1^l, K_2^l \in C^{1,2}(\bar{Q}_0)$ are given by $K_1^l(t,x) = M_1^l(t,x)$ and $K_2^l(t,x) = M_2^l(t,x) - M_1^{l^2}(t,x)$ respectively.

Definition 3: Let $K_1^l, \cdots, K_j^l$ be admissible 1-st, $\cdots, j$-th cumulant cost functions. The control strategy μ^{l*} is the lth player's equilibrium solution if it is such that

$$M_{j+1}^l(t,x) = V_{j+1}^l(t,x;) \leq V_{j+1}(t, x; \mu_1^* \cdots, \mu_l, \cdots, \mu_N^*)$$

for all $\mu \in \mathcal{U}_M$. Furthermore the $j+1$-st cumulant cost function is given by

$$\begin{aligned}K_{j+1}(t,x) =&\Lambda_{j+1}(t,x;\mu_1^*,\cdots,\mu_N^*)\\ \leq&\Lambda_{j+1}(t,x;\mu_1^*,\cdots,\mu_l,\cdots,\mu_N^*).\end{aligned}$$

IV. The k-th Moment

It will be assumed that the other players have played their equilibrium strategies. The moment recursion formulae were first given in the paper by Sain 1967, [10]. This paper showed that, for the optimal control problem, the $(j+1)$-st moment of the cost function, $V_{j+1}^l(t,x;\mu_1^*,\cdots,\mu_l,\cdots,\nu^*)$, satisfies

$$\begin{aligned}&\mathcal{O}^l V_{j+1}(t,x;\mu_1^*,\cdots,\mu_l,\cdots,\mu_N^*)\\ &\quad+(j+1)V_j(t,x;\mu_1^*,\cdots,\mu_l,\cdots,\mu_N^*)\\ &\quad\cdot L_l(t,x;\mu_1^*,\cdots,\mu_l,\cdots,\mu_N^*)=0\end{aligned} \tag{10}$$

where $\mathcal{O}^l$ is the backward evolution operator. If the first j moment cost functions are admissible moment cost functions, then they satisfy

$$\begin{aligned}&\mathcal{O}^l M_1^l(t,x)+L_l(t,x,\mu_1^*,\cdots,\mu_l,\cdots,\mu_N^*)=0\\ &\mathcal{O}^l M_2^l(t,x)+2M_1^l(t,x)L_l(t,x,\mu_1^*,\cdots,\mu_l,\cdots,\mu_N^*)=0\\ &\qquad\vdots\\ &\mathcal{O}^l M_j^l(t,x)+jM_{j-1}^l(t,x)L_l(t,x,\mu_1^*,\cdots,\mu_l,\cdots,\mu_N^*)=0.\end{aligned} \tag{11}$$

Before moving further, the following useful lemma will be presented.

Lemma 1: Consider the running cost function $L_l(t,x,\mu_1^*,\cdots,\mu_l,\cdots,\mu_N^*)$, which is denoted by L_t. then the equality

$$(j+1)\int_t^{t_f} L_s\left[\int_s^{t_f} L_r dr\right]^j ds=\left[\int_t^{t_f} L_r dr\right]^{j+1} \tag{12}$$

holds.

Proof. First we should change the limits of integration:

$$\int_t^{t_f} L_s\left[\int_s^{t_f} L_r dr\right]^j ds=(-1)^j\int_{t_f}^{t} L_s\left[\int_{t_f}^{s} L_r dr\right]^j ds$$

Now recall that for two differential functions F and G we can integrate by parts

$$\int_{t_f}^{t} F(s)g(s)ds=F(t)G(t)-F(t_f)G(t_f)-\int_{t_f}^{t} f(s)G(s)ds$$

where $f(s)=\frac{dF(s)}{ds}, G(s)=\int_{t_f}^{s} g(r)dr$. Let $g(s)=L_s$ and

$$F(s)=\left[\int_{t_f}^{s} L_r dr\right]^j.$$

With these definitions we see that

$$f(s)=jL_s\left[\int_{t_f}^{s} L_r dr\right]^{j-1}$$

$$G(s)=\int_{t_f}^{s} L_r dr$$

which then yields

$$\begin{aligned}(-1)^j\int_{t_f}^{t} L_s\left[\int_{t_f}^{s} L_r dr\right]^j ds=&(-1)^j\left[\int_{t_f}^{t} L_s ds\right]^{(j+1)}\\ &-(-1)^j\int_{t_f}^{t} jL_s\left[\int_{t_f}^{s} L_r dr\right]^j ds\end{aligned}$$

which is

$$(j+1)\int_{t_f}^{t} L_s\left[\int_{t_f}^{s} L_r dr\right]^j ds=\left[\int_{t_f}^{t} L_s ds\right]^{(j+1)}$$

and the lemma is proved. □

Now consider the j-th moment equation. We can show that a function M_j^l that satisfies this equation is in fact the j-th moment.

Lemma 2: Consider a function $M_j^l \in C_p^{1,2}(Q)\cap C(\bar{Q})$ that satisfies

$$\mathcal{O}^k M_j^l(t,x)+jM_{j-1}(t,x)L_l(t,x,\mu_1^*,\cdots,\mu_l,\cdots,\mu_N^*)=0 \tag{13}$$

where M_{j-1}^l is an admissible $(j-1)$ moment cost function; then $M_j^l(t,x)=V_j^l(t,x;\mu_1^*,\cdots,\mu_l,\cdots,\mu_N^*)$.

Proof. Due to space constraints this proof is omitted. See [4] for details. □

Now consider the equation

$$\begin{aligned}\min_{\mu_l\in\mathcal{U}_{M_l}}&\left\{\mathcal{O}^l M_{j+1}^l(t,x)+(j+1)M_j^l(t,x)\right.\\ &\left.\cdot L_l(t,x,\mu_1^*,\cdots,\mu_l,\cdots,\mu_N^*)\right\}=0\end{aligned} \tag{14}$$

where $M_{j+1}^l(t,x)$ is a suitably smooth solution to (14) and $\mathcal{U}_{M_l}$. Suppose that the moment that is desired to be minimized is the $(j+1)$-st moment.

Theorem 1 (Verification Theorem): Let $M_j^l \in C_p^{1,2}(Q)\cap C(\bar{Q})$ be the j-th admissible moment cost function with an admissible class of control strategies, $\mathcal{U}_{M_l}$. If the function $M_{j+1}^l \in C_p^{1,2}(Q)\cap C(\bar{Q})$ satisfies

$$\begin{aligned}\min_{\mu_l\in\mathcal{U}_{M_l}}&\left\{\mathcal{O}^l M_{j+1}^l(t,x)+(j+1)M_j^l(t,x)\right.\\ &\left.\cdot L(t,x,\mu_1^*,\cdots,\mu_l,\cdots,\mu_N^*)\right\}=0,\end{aligned} \tag{15}$$

then $M_{j+1}^l(t,x)\leq V_{j+1}^l(t,x;\mu_1^*,\cdots,\mu_l,\cdots,\mu_N^*)$ for all $\mu_l\in\mathcal{U}_{M_l}$ and $(t,x)\in Q$. Furthermore if there is a μ_l^* such that

$$\begin{aligned}\mu_l^*=\arg\min_{\mu_l\in\mathcal{U}_{M_l}}&\left\{\mathcal{O}^l M_{j+1}^l(t,x)+(j+1)M_j^l(t,x)\right.\\ &\left.\cdot L_l(t,x,\mu_1^*,\cdots,\mu_l,\cdots,\mu_N^*)\right\}\end{aligned} \tag{16}$$

then $M_{j+1}^l(t,x)=V_{j+1}^l(t,x;\mu_1^*,\cdots,\mu_l,\cdots,\mu_N^*)$.

Proof. The proof is the same as that of Lemma 2 except that the equality sign is now an inequality

$$M_j^l(t,x)\leq E_{tx}\left\{\int_t^{t_f} jM_{j-1}^l(s,x(s))L_s ds+\psi_l^j(x(t_f))\right\}.$$

Beyond this, the proof is the same. For the case of $\mu_l=\mu_l^*$, the proof is exactly the same as the proof in Lemma 2. □

The problem, however, is not cast in terms of the moments of the cost function but rather the cumulants. For the case of one cumulant, the problem is the well known mean stochastic game. The two cumulant problem has already been given in this chapter. Now the case for three cumulants, and then four cumulants, will be examined. Furthermore a general r-cumulant HJB equation will be proposed.

V. kTH CUMULANT

With this proof, one can propose a general HJB equation for the k-th cumulant. Before doing so, we will consider several lemmas that will aid in the proof.

Lemma 3: Consider two functions $M_j^l(t,x), K_i^l(t,x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ where i and j are positive integers, then

$$\begin{aligned}\mathcal{O}^l[M_j^l(t,x)K_i^l(t,x)] =& \mathcal{O}^l[M_j^l(t,x)]K_i^l(t,x) \\ &+ \mathcal{O}^l[K_i^l(t,x)]M_j^l(t,x) \\ &+ \left(\frac{\partial M_j^l}{\partial x}(t,x)\right)' \sigma(t,x) \\ &\cdot W(t)\sigma'(t,x)\left(\frac{\partial K_i^l}{\partial x}(t,x)\right).\end{aligned} \tag{17}$$

Proof. For the sake of ease, we will suppress the dependence on time and state. From the definition of the operator $\mathcal{O}^l$, we have

$$\begin{aligned}\mathcal{O}^l[M_j^lK_i^l] &= \mathcal{O}_1^l[M_j^lK_i^l] + \mathcal{O}_2[M_j^lK_i^l] \\ &= \mathcal{O}_1^l[M_j^l]K_i^l + \mathcal{O}_1^l[K_i]M_j + \mathcal{O}_2[M_j^lK_i^l]\end{aligned}$$

where in the last line, the chain rule was used. But,

$$\begin{aligned}\mathcal{O}_2[M_j^lK_i^l] =& \frac{1}{2}tr\left(\sigma W\sigma'\frac{\partial}{\partial x}\left[\frac{\partial M_j^l}{\partial x}K_i^l + \frac{\partial K_i^l}{\partial x}M_j^l\right]\right) \\ =& \frac{1}{2}tr\Bigg(\sigma W\sigma'\Bigg[\frac{\partial^2 M_j^l}{\partial x^2}K_i^l + \left(\frac{\partial M_j^l}{\partial x}\right)\left(\frac{\partial K_i^l}{\partial x}\right)' \\ &+ \left(\frac{\partial K_i^l}{\partial x}\right)\left(\frac{\partial M_j^l}{\partial x}\right)' + \frac{\partial^2 K_i^l}{\partial x^2}M_j^l\Bigg]\Bigg) \\ =& K_i^l\mathcal{O}_2[M_j^l] + M_j\mathcal{O}_2[K_i^l] \\ &+ \left(\frac{\partial M_j^l}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial K_i^l}{\partial x}\right)\end{aligned}$$

and therefore

$$\begin{aligned}\mathcal{O}^l[M_j^lK_i^l] =& \mathcal{O}^l[M_j^l]K_i^l + \mathcal{O}^l[K_i^l]M_j^l \\ &+ \left(\frac{\partial M_j^l}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial K_i^l}{\partial x}\right).\end{aligned}$$

This is the desired result and completes the proof. □

Lemma 4: Let $K_1^l, \cdots, K_{k-1}^l \in C_p^{1,2}(Q) \cap C(\bar{Q})$, then

$$\begin{aligned}&\sum_{i=0}^{k-2}\frac{(k-1)!}{i!(k-i-1)!}\left(\frac{\partial K_{k-1-i}^l}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial K_{i+1}^l}{\partial x}\right) \\ &\quad = \frac{1}{2}\sum_{j=1}^{k-1}\frac{k!}{j!(k-j)!}\left(\frac{\partial K_j^l}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial K_{k-j}^l}{\partial x}\right)\end{aligned} \tag{18}$$

where the arguments of $K_1^l, \cdots, K_{k-1}^l$ have been suppressed and k, l are positive integers with $l \le k$.

Proof. Let $j = i+1$, then the first term in (18) becomes

$$\sum_{j=1}^{k-1}\frac{(k-1)!}{(j-1)!(k-j)!}\left(\frac{\partial K_{k-j}^l}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial K_j^l}{\partial x}\right),$$

by changing the limits of summation. Now consider $1 \le j \le k-1$. For odd k, the above summation is an even number of sums. So in pairs, we have

$$\begin{aligned}&\frac{(k-1)!}{(j-1)!(k-j)!}\left(\frac{\partial K_{k-j}^l}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial K_j^l}{\partial x}\right) \\ &+ \frac{(k-1)!}{(k-j-1)!(j)!}\left(\frac{\partial K_j^l}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial K_{k-j}^l}{\partial x}\right) \\ &= \left(\frac{j(k!)}{k(j!)(k-j)!} + \frac{(k-j)(k!)}{k(k-j)!(j!)}\right) \\ &\cdot\left(\frac{\partial K_j^l}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial K_{k-j}^l}{\partial x}\right) \\ &= \left(\frac{\partial K_j^l}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial K_{k-j}^l}{\partial x}\right) \\ &= \frac{1}{2}\left[\frac{k!}{j!(k-j)!}\left(\frac{\partial K_j^l}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial K_{k-j}^l}{\partial x}\right)\right. \\ &\left.+ \frac{k!}{(k-j)!j!}\left(\frac{\partial K_{k-j}^l}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial K_j^l}{\partial x}\right)\right].\end{aligned}$$

For the case of k even, it is much the same, with the exception of $j = k/2$. In this case we have

$$\frac{(k-1)!}{(k/2-1)!(k/2-j)!}\left(\frac{\partial K_{k/2}^l}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial K_{k/2}^l}{\partial x}\right)$$

but this is exactly the same as

$$\frac{1}{2}\frac{k!}{(k/2)!(k/2)!}\left(\frac{\partial K_{k/2}^l}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial K_{k/2}^l}{\partial x}\right)$$

and the lemma is proved. □

Theorem 2: Let $K_1^l, K_2^l, \cdots, K_{j-1}^l$ be admissible cumulant cost functions. If there exists a solution $K_j^l \in C_p^{1,2}(Q) \cap C(\bar{Q})$ that satisfies

$$\begin{aligned}&\mathcal{O}^lK_j^l(t,x) + \frac{1}{2}\sum_{s=1}^{j-1}\frac{k!}{s!(k-s)!}\left(\frac{\partial K_s^l}{\partial x}(t,x)\right)' \\ &\quad\cdot\sigma(t,x)W(t)\sigma'(t,x)\left(\frac{\partial K_{j-s}^l}{\partial x}(t,x)\right) = 0,\end{aligned} \tag{19}$$

then $K_j^l(t,x) = \Lambda_j^l(t,x,\mu_1^*,\cdots,\mu_l,\cdots,\mu_N^*)$, where μ_l is in $\mathcal{U}_{M_l}$.

Proof. The proof of this theorem follows very closely to that of the following theorem. □

Next, a general theorem will be given. This theorem has been shown to hold for k up to six, and it is conjectured that it holds further.

Theorem 3: Let $K_1^l, K_2^l, \cdots, K_{k-1}^l$ be admissible cumulant cost functions. If there exists a solution $K_k^l \in C_p^{1,2}(Q) \cap C(\bar{Q})$ that satisfies

$$\min_{\mu_l \in \mathcal{U}_{M_l}} \left\{ \mathcal{O}^l K_k^l(t,x) + \frac{1}{2} \sum_{s=1}^{k-1} \frac{k!}{s!(k-s)!} \left(\frac{\partial K_s^l}{\partial x}(t,x) \right)' \cdot \sigma(t,x) W(t) \sigma'(t,x) \left(\frac{\partial K_{k-s}^l}{\partial x}(t,x) \right) \right\} = 0, \tag{20}$$

then

$$\begin{aligned} K_k^l(t,x) =& \Lambda_k^l(t,x,\mu_1^*, \cdots, \mu_N^*) \\ \leq& \Lambda_k(t,x,\mu_1^*, \cdots, \mu_l, \mu_N^* \nu^*), \end{aligned}$$

where μ_l^* is the minimizing argument of (20) and the optimal strategy for the control.

Proof. Let K_k^l be of class $C_p^{1,2}(Q) \cap C(\bar{Q})$. Also, by definition,

$$\begin{aligned} K_k^l(t,x) =& M_k^l(t,x) \\ &- \sum_{i=0}^{k-2} \frac{(k-1)!}{i!(k-i-1)!} M_{k-1-i}^l(t,x) K_{i+1}^l(t,x) \end{aligned}$$

therefore $M_k^l(t,x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$. Now assume that M_k^l satisfies (14), that is M_k^l is a solution to

$$\begin{aligned} \mathcal{O}^l M_k^l(t,x) + & k M_{k-1}^l(t,x) \\ & \cdot L_l(t,x;\mu_1^*, \cdots, \mu_l, \cdots, \mu_N^*) = 0. \end{aligned}$$

Substituting for M_k yields

$$\mathcal{O}^l K_k^l + \sum_{i=0}^{k-2} \frac{(k-1)!}{i!(k-i-1)!} \mathcal{O}^l \left[M_{k-1-i}^l K_{i+1}^l \right] + k M_{k-1}^l L_l = 0$$

where the linearity of the operator $\mathcal{O}^l$ is used.

Using Lemma 3 and expanding the sum, we obtain

$$\begin{aligned} \mathcal{O}^l K_k^l &+ \sum_{j=0}^{k-3} \frac{(k-1)!}{j!(k-1-j)!} \mathcal{O}^l [M_{k-1-j}^l] K_{j+1}^l \\ &+ (k-1) \mathcal{O}^l [M_1^l] K_{k-1}^l \\ &+ \sum_{j=1}^{k-2} \frac{(k-1)!}{j!(k-1-j)!} \mathcal{O}^l [K_{i+1}^l] M_{k-1-i}^l \\ &+ \sum_{j=0}^{k-2} \frac{(k-1)!}{j!(k-1-j)!} \left(\frac{\partial M_{k-1-j}^l}{\partial x} \right)' \\ &\cdot \sigma W \sigma' \left(\frac{\partial K_{i+1}^l}{\partial x} \right) + k M_{k-1}^l L_l \\ &+ M_{k-1}^l \mathcal{O}_{\mu,\nu^*} [K_1^l] = 0. \end{aligned} \tag{21}$$

It can be shown that this reduces to

$$\begin{aligned} \mathcal{O}^l K_k^l &+ \sum_{j=1}^{k-2} \frac{(k-1)!}{j!(k-1-j)!} \mathcal{O}^l [K_{i+1}^l] M_{k-1-i}^l \\ &+ \sum_{j=0}^{k-2} \frac{(k-1)!}{j!(k-1-j)!} \left(\frac{\partial M_{k-1-j}^l}{\partial x} \right)' \\ &\cdot \sigma W \sigma' \left(\frac{\partial K_{i+1}^l}{\partial x} \right) = 0. \end{aligned} \tag{22}$$

By substitution for M_{k-1-i}^l and with the use of Lemma 4, we can obtain

$$\begin{aligned} \mathcal{O}^l K_k^l + & \frac{1}{2} \sum_{j=1}^{k-1} \frac{k!}{j!(k-j)!} \left(\frac{\partial K_j^l}{\partial x} \right)' \\ & \cdot \sigma W \sigma' \left(\frac{\partial K_{k-j}^l}{\partial x} \right) = 0 \end{aligned}$$

which is the desired equation, where it can be shown for various values of k that the following equation holds.

$$\begin{aligned} & \sum_{j=0}^{k-2} \frac{(k-1)!}{j!(k-1-j)!} \sum_{i=0}^{k-3-j} \frac{(k-2-j)!}{i!(k-2-j-i)!} \\ & \left[K_{i+1}^l \left(\frac{\partial M_{k-2-j-i}^l}{\partial x} \right) + M_{k-2-j-i}^l \left(\frac{\partial K_{i+1}^l}{\partial x} \right) \right] \\ & \qquad {}' \sigma W \sigma' \left(\frac{\partial K_{j+1}^l}{\partial x} \right) \\ & + \sum_{j=1}^{k-2} \frac{(k-1)!}{j!(k-1-j)!} \mathcal{O}^l [K_{j+1}^l] M_{k-1-i}^l = 0. \end{aligned} \tag{23}$$

□

VI. Case of Linear System with Quadratic Costs

This gives a sufficient condition for a class of nonlinear systems with non-quadratic costs. Now consider a linear system

$$dx(t) = \left[A(t)x(t) + \sum_{i=1}^{N} B_i(t) u_i(t) \right] dt + E(t) d\xi(t) \tag{24}$$

where $x(t_0) = x_0$. The $n \times n$ matrix A, $n \times m_i$ matrices B_i, and $n \times d$ matrix E all have continuous entries. The ith player's cost is assumed to be quadratic

$$J_i = \int_{t_0}^{t_f} \left[x' Q_i x + \sum_{j=1}^{N} u_j' R_{ij} u_j \right] dt + x_f' Q_i^f x_f \tag{25}$$

where $x(t_f) = x_f$, Q_i, Q_i^f, R_{ij} for $i \neq j$ are assumed to be positive semidefinite, and R_{ii} is positive definite for $i, j = 1, \cdots, N$.

Let K_j^i be quadratic with respect to the state, x, that is $K_i(t,x) = x' \mathcal{K}_j^i(t) x + k_j^i(t)$, where here $0 \leq i \leq N$ and $0 \leq j \leq k$.

Theorem 4: Let $\mathcal{K}_r^i \in C_p^{1,2}(\bar{Q}_0)$ be the r-th cumulant cost functions for $1, \cdots, k-1$. If $\mathcal{K}_r^i$ is a solution to the Riccati equations for $1 < r \leq k-1$,

$$\begin{aligned}
&\dot{\mathcal{K}}_r^i + A'\mathcal{K}_r^i + \mathcal{K}_r^i A - 2\gamma_r^i \mathcal{K}_r^i B_i R_{ii}^{-1} B_i' \mathcal{K}_r^i \\
&\quad - \sum_{s=1}^{r-1} \left[\gamma_s^i \mathcal{K}_s^i B_i R_{ii}^{-1} B_i' \mathcal{K}_r^i + \gamma_s^i \mathcal{K}_r^i B_i R_{ii}^{-1} B_i' \mathcal{K}_s^i \right] \\
&\quad - \sum_{s=r+1}^{k} \left[\gamma_s^i \mathcal{K}_s^i B_i R_{ii}^{-1} B_i' \mathcal{K}_r^i + \gamma_s^i \mathcal{K}_r^i B_i R_{ii}^{-1} B_i' \mathcal{K}_s^i \right] \\
&\quad - \sum_{s=1}^{r-1} \left[\gamma_s^i \mathcal{K}_s^i B_i R_{ii}^{-1} B_i' \mathcal{K}_r^i + \gamma_s^i \mathcal{K}_r^i B_i R_{ii}^{-1} B_i' \mathcal{K}_s^i \right] \\
&\quad - \sum_{j=1}^{i-1} \sum_{s=1}^{r} \left[\gamma_s^j \mathcal{K}_s^j B_j R_{jj}^{-1} B_j' \mathcal{K}_r^i + \gamma_s^j \mathcal{K}_r^i B_j R_{jj}^{-1} B_j' \mathcal{K}_s^j \right] \\
&\quad - \sum_{j=i+1}^{N} \sum_{s=1}^{r} \left[\gamma_s^j \mathcal{K}_s^j B_j R_{jj}^{-1} B_j' \mathcal{K}_r^i + \gamma_s^j \mathcal{K}_r^i B_j R_{jj}^{-1} B_j' \mathcal{K}_s^j \right] \\
&\quad + \sum_{s=1}^{r-1} \frac{k!}{s!(r-s)!} \left[\mathcal{K}_s^i EWE \mathcal{K}_{r-s}^i + \mathcal{K}_{r-s}^i EWE \mathcal{K}_s^i \right] = 0,
\end{aligned} \tag{26}$$

$$\dot{k}_r^i = -tr(E(t)W(t)E(t)\mathcal{K}_r^i(t)), \tag{27}$$

and for $r = 1$

$$\begin{aligned}
&\dot{\mathcal{K}}_1^i + A'\mathcal{K}_1^i + \mathcal{K}_1^i A - 2\gamma_1^i \mathcal{K}_1^i B_i R_{ii}^{-1} B_i' \mathcal{K}_1^i \\
&\quad - \sum_{s=1}^{r-1} \left[\gamma_s^i \mathcal{K}_s^i B_i R_{ii}^{-1} B_i' \mathcal{K}_1^i + \gamma_s^i \mathcal{K}_1^i B_i R_{ii}^{-1} B_i' \mathcal{K}_s^i \right] \\
&\quad - \sum_{s=r+1}^{k} \left[\gamma_s^i \mathcal{K}_s^i B_i R_{ii}^{-1} B_i' \mathcal{K}_1^i + \gamma_s^i \mathcal{K}_1^i B_i R_{ii}^{-1} B_i' \mathcal{K}_s^i \right] \\
&\quad - \sum_{s=1}^{r-1} \left[\gamma_s^i \mathcal{K}_s^i B_i R_{ii}^{-1} B_i' \mathcal{K}_1^i + \gamma_s^i \mathcal{K}_1^i B_i R_{ii}^{-1} B_i' \mathcal{K}_s^i \right] \\
&\quad - \sum_{j=1}^{i-1} \sum_{s=1}^{r} \left[\gamma_s^j \mathcal{K}_s^j B_j R_{jj}^{-1} B_j' \mathcal{K}_1^i + \gamma_s^j \mathcal{K}_1^i B_j R_{jj}^{-1} B_j' \mathcal{K}_s^j \right] \\
&\quad - \sum_{j=i+1}^{N} \sum_{s=1}^{r} \left[\gamma_s^j \mathcal{K}_s^j B_j R_{jj}^{-1} B_j' \mathcal{K}_1^i + \gamma_s^j \mathcal{K}_1^i B_j R_{jj}^{-1} B_j' \mathcal{K}_s^j \right] \\
&\quad + \sum_{j=1}^{N} \left(\sum_{s=1}^{k} \gamma_s^j \mathcal{K}_s^j \right) B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j' \left(\sum_{s=1}^{k} \gamma_s^j \mathcal{K}_s^j \right) \\
&\quad + Q_i = 0,
\end{aligned} \tag{28}$$

where $K_1^i(t_f) = Q_i^f$ and $K_r^i(t_f) = 0$. Then the Nash equilibrium solution is given as

$$u_i^*(t) = \mu_i^*(t, x(t)) = -R_{ii}^{-1}(t) B_i'(t) \left[\sum_{s=1}^{k} \gamma_s^i \mathcal{K}_s^i(t) \right] x(t). \tag{29}$$

Furthermore k_r^i is constructed with the use of $\mathcal{K}_r^i$, that is

$$\dot{k}_r^i = -tr(E(t)W(t)E(t)\mathcal{K}_r^i(t)) \tag{30}$$

for $r = 1, \cdots, k$ and $i = 1, \cdots, N$.

Proof. Due to space constraints, this proof it omitted. See [4]. □

VII. Conclusion

Sufficient conditions for equilibrium solutions of an N player stochastic game with a class of nonlinear systems with non-quadratic costs were developed. Using these sufficient conditions, equilibrium solutions were determined for the case of a linear system with quadratic costs. This involved the solution of coupled Riccati equations. The use of cumulants can be seen to generalize standard stochastic game theory, in which the mean (the first cumulant) is prominent.

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