2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Dissipative Analysis and Control of State-Space Symmetric Systems

Mona Meisami-Azad[†], Javad Mohammadpour[†], and Karolos M. Grigoriadis[†‡]

Abstract— **The paper addresses the problem of analysis and static output feedback control synthesis for strict quadratic dissipativity of linear time-invariant systems with state-space symmetry. As a particular case of dissipative systems, we consider the mixed H_∞ and positive real performance criterion and we develop an explicit expression for calculating the H_∞ norm of these systems. Subsequently, an explicit parametrization of the static output feedback control gains that solve the mixed H_∞ and positive real performance problem is obtained. Computational examples demonstrate the use and computational advantages of the proposed explicit solutions.**

I. Introduction

Since the introduction of the notion of dissipative systems in [12], and subsequently its generalization in [2], it has played a significant role in systems, circuits, and controls. Dissipativeness is a generalization of the concept of passivity in electrical networks and other dynamical systems that dissipate energy in some abstract sense [5]. In the past two decades, there has been an enormous interest in the problems of analysis and synthesis of H_∞ and positive real (or passivity-based) control systems. The H_∞ control design approach is based on the small gain theorem, while the positive real approach is based on the positivity theorem [15]. The paper [9] studies the problem of synthesizing a stabilizing controller for a LTI plant such that the closed-loop system is strictly positive real. The authors in [4] address the problem of finding an output feedback controller to make the closed-loop system strictly positive real using an LMI formulation. In an H_∞ control framework, the small gain theorem is used to ensure robust stability by requiring that the loop-gain be less than one over all frequencies. It is noted that that phase information is not used in checking stability. On the other hand, in the positivity theorem widely used in the analysis of passive control systems, phase information is taken into account. Based on the positivity theorem, the phase of a positive real system is less than 90 degrees, so that the closed-loop transfer function of a negative feedback connection of two positive real systems has a phase-lag less than 180 degrees. This guarantees stability regardless of the loop-gain. The small-gain and positivity theorems both deal with gain and phase performances separately, and therefore may lead to conservative results when used in applications [14]. The notion of dissipativeness provides not only a flexible trade-off between gain and phase but also an appropriate framework for less conservative robust control design, specially in applications where both gain and phase properties are important to consider.

Symmetric state-space system representations appear in many different engineering fields, such as electrical and power networks, structural systems, and chemical reactions. In particular, physical systems with only one type of energy storage capability, such as mechanical systems with only potential energy or only kinematic energy, and electrical systems with only electric energy or only magnetic energy (e.g., RL or RC circuits) provide models of such symmetric systems [1]. Moreover, systems with zeros interlacing the poles (ZIP) can be modeled as symmetric systems [8]. Stability criteria for state-space symmetric systems have been examined in [13]. The H_∞ control of symmetric systems has been addressed in [11].

In this paper we are concerned with the problem of quadratic dissipative control for linear time-invariant state-space symmetric systems. First, we establish necessary and sufficient conditions for quadratic dissipativeness of LTI state-space symmetric systems. Then, we focus on the mixed H_∞ and positive real performance analysis and control problem for such systems. This problem addresses a trade-off between passivity and H_∞ performance taking into account both gain and phase information resulting in less conservative results. The objective of the present work is to show that by exploiting the particular structure of symmetric systems, *explicit solutions* for the above problems can be achieved. For this purpose, a particular solution of the LMI formulation for quadratic dissipativity is obtained for stable symmetric systems. Then, an explicit expression for the mixed H_∞ and positive real performance for a symmetric system is developed that requires only the computation of the maximum eigenvalue of a matrix which contain the system data. Next, we derive explicit expressions for the optimally achievable closed-loop H_∞ performance level and the optimal control gains of the mixed H_∞ and positive real output feedback control synthesis problem. It is noticed that the static output control synthesis problem is solved in the above framework. In general, the solution of static output feedback synthesis problems are extremely cumbersome due to the lack of convexity of the corresponding synthesis formulations. However, by exploiting the symmetry in the proposed problems explicit analytical solutions of the static output feedback problem will be derived.

II. Preliminaries

We first present a formal definition of state-space symmetric systems, and we review the concept of dissipativity for dynamical systems. Also, some algebraic results that will be

[†] The authors are with the Department of Mechanical Engineering, University of Houston, Houston, TX 77204 USA
[‡]Corresponding author. Tel: 713-743-4387, Fax: 713-743-4503; Email: karolos@uh.edu

978-1-4244-2079-7/08/$25.00 ©2008 AACC.

useful later in the proofs of the main theorems of the paper are introduced.

A. Definition of Symmetric Systems

Consider the following state-space representation for a linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), \quad x(0) = 0 \\ z(t) &= Cx(t) + Dw(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^m$ is the vector of exogenous inputs, $z(t) \in \mathbb{R}^p$ is the vector of controlled outputs, and $\{A, B, C, D\}$ denote the state-space matrices.

We say that the state-space representation (1) is symmetric if the following conditions hold

$$A = A^T, \qquad B = C^T, \qquad D = D^T. \tag{2}$$

The above system property is often referred to as internal or state-space symmetry to make a distinction from external or system symmetry that requires $G(s) = G^T(s)$, where $G(s) = C(sI - A)^{-1}B + D$ is the transfer function representation of the system. Obviously, state-space symmetry (2) implies external symmetry, but the converse is not true, that is, there exist symmetric transfer matrices for which there is no symmetric realization.

B. Dissipativity of Dynamical Systems

In this section, we review the concept of dissipativity in dynamical system and we present the LMI formulation for quadratic dissipativity for linear time invariant (LTI) systems. Consider a dynamical system represented by

$$\begin{aligned} \dot{x}(t) &= f(x(t), w(t)), \qquad x(0) = x_0 \\ z(t) &= g(x(t), w(t)) \end{aligned} \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^m$ is the input, $z(t) \in \mathbb{R}^p$ is the output, and f and g are smooth real vector functions.

Let us introduce the following quadratic energy supply function E associated with the system (3).

$$E(w, z, \mathrm{T}) = \prec z, Qz \succ_{\mathrm{T}} + 2 \prec z, Sw \succ_{\mathrm{T}} + \prec w, Rw \succ_{\mathrm{T}} \tag{4}$$

where Q, S and R are real matrices of appropriate dimensions with matrices Q and R symmetric, and $\prec u, v \succ_{\mathrm{T}} = \int_0^{\mathrm{T}} u^T v dt$ for $u,v \in \mathrm{L}_{2e}^n$. Next, we recall the notion of quadratic dissipativeness [12].

Definition 1: Given matrices Q, S and R where Q and R are symmetric, the system (3) with energy supply function E is called (Q, S, R)-dissipative if for some real function $\beta(.)$ with $\beta(0) = 0$,

$$\mathrm{E}(w, z, \mathrm{T}) + \beta(x_0) \geq 0, \quad \forall w \in \mathrm{L}_{2e}^q, \quad \forall \mathrm{T} \geq 0 \tag{5}$$

Furthermore, if for some scalar $\alpha > 0$

$$\mathrm{E}(w, z, \mathrm{T}) + \beta(x_0) \geq \alpha \prec w, w \succ_{\mathrm{T}}, \quad \forall w \in \mathrm{L}_{2e}^q, \quad \forall \mathrm{T} \geq 0 \tag{6}$$

then (3) is called strictly (Q, S, R)-dissipative.

Given a system in input/state/output form and a supply rate, the question that arises is if there exists a storage such that the dissipation inequality is satisfied. If such a non-negative storage exists, we call the system dissipative with respect to the supply rate. The problem of constructing a nonnegative storage has been extensively studied for general systems and, in particular, for linear systems with a supply rate that is a quadratic function of the input and output variables. In the case of linear systems with quadratic supply rates, it has been shown that both the constructed storage and the required supply are quadratic. In this case, the dissipation inequality becomes a linear matrix inequality (LMI) as we will discuss later in this section.

Remark 1: The notion of strict (Q, S, R)-dissipativity includes H_∞ performance and passivity as special cases represented in one of the following categories:

1) When $Q = -\gamma^{-1}I$, $S = 0$ and $R = \gamma I$, strict (Q, S, R)-dissipativity reduces to the H_∞ norm constraint.
2) When $Q = 0$, $S = I$, $R = 0$, (6) reduces to the strict positive realness.
3) When $Q = -\gamma^{-1}\theta I$, $S = (1 - \theta)I$, $R = \gamma\theta I$, $\theta \in (0, 1)$, (6) represents a mixed H_∞ and positive real performance. In this case θ represents a weighting parameter that defines the trade-off between H_∞ and positive real performance.

We make the following assumptions concerning the system (1) and the weighting matrices Q, S and R.

$$R + D^T S + S^T D + D^T Q D > 0 \tag{7}$$

$$Q_- \triangleq -Q \geq 0 \tag{8}$$

The following result presents necessary and sufficient conditions for the state-space representation (1) to be strictly (Q, S, R)-dissipative [14].

Theorem 1: Let Q, S, R be given matrices where Q and R are symmetric, and consider the system (1) subject to assumptions (7) and (8). Then the system (1) is asymptotically stable and strictly (Q, S, R)-dissipative, if there exists a matrix $P > 0$ such that

$$\begin{bmatrix} A^T P + PA & PB - C^T S & C^T Q_-^{\frac{1}{2}} \\ \star & -(R + D^T S + S^T D) & D^T Q_-^{\frac{1}{2}} \\ \star & \star & -I \end{bmatrix} < 0 \tag{9}$$

where $Q_- = -Q > 0$.

The next lemmas will be useful in the proofs of the main results of the paper.

Lemma 1: [3] Consider matrices Γ and Λ such that Γ has full column rank, and Λ is symmetric positive definite. Then $\Lambda \geq \Gamma\Gamma^T$ if and only if $\lambda_{max}(\Gamma^T \Lambda^{-1} \Gamma) \leq 1$.

Lemma 2: (Finsler's Lemma) [7]. Consider matrices M and Z such that M has full column rank and $Z = Z^T$. Then the following statements are equivalent:

1) There exists a scalar μ such that

$$\mu M M^T - Z > 0. \tag{10}$$

2) The following condition holds

$$M^{\perp} Z M^{\perp T} < 0. \quad (11)$$

If the above statements hold, then all scalars μ satisfying (10) are given by

$$\mu > \lambda_{max}[M^{\dagger}(Z - ZM^{\perp T}(M^{\perp} Z M^{\perp T})^{-1} M^{\perp} Z) M^{\dagger T}]. \quad (12)$$

Lemma 3: Consider the following LMI with a matrix parameter P

$$\sum_{i=1}^{m} (A_i P B_i + B_i^T P A_i^T) < 0 \quad (13)$$

where $A_1, ..., A_m$ and $B_1, ..., B_m$ are given matrices. Suppose that for every $P > 0$ that is a positive definite solution of (13), P^{-1} is also a solution of (13). Then, the identity matrix is a solution of (13), that is

$$\sum_{i=1}^{m} (A_i B_i + B_i^T A_i^T) < 0$$

Proof. Consider the eigenvalue decomposition (EVD) of $P_0 > 0$ as follows

$$P_0 = U \Sigma_0 U^T, \quad U^T = U^{-1}, \quad \Sigma_0 = diag(\sigma_1, ..., \sigma_n) > 0.$$

Then $P_0^{-1} = U \Sigma_0^{-1} U^T$, where $\Sigma_0^{-1} = diag(\frac{1}{\sigma_1}, ..., \frac{1}{\sigma_n})$ and since $\sigma_1 > 0$, there exists $0 \le \alpha_1 \le 1$ such that $\alpha_1 \sigma_1 + (1 - \alpha_1)\sigma^{-1} = 1$. Define $P_1 = \alpha_1 P_0 + (1 - \alpha_1) P_0^{-1} = U \Sigma_1 U^T$, where $\Sigma_1 = diag(1, \bar{\sigma}_2, ..., \bar{\sigma}_n)$ with $\bar{\sigma}_i = \alpha_1 \sigma_i + (1 - \alpha_1)\sigma_i^{-1}$ for $i = 2, ..., n$. Then, because of convexity of (13), $P_1 > 0$ satisfies (13) and hence P_1^{-1} also satisfies (13). Since $\bar{\sigma}_2 > 0$, there exists $0 \le \alpha_2 \le 1$ such that $\alpha_2 \bar{\sigma}_2 + (1 - \alpha_2)\bar{\sigma}_2^{-1} = 1$. Defining $P_2 = \alpha_2 P_1 + (1 - \alpha_2) P_1^{-1} = U \Sigma_2 U^T$ where $\Sigma_2 = diag(1, 1, \bar{\bar{\sigma}}_3, ..., \bar{\bar{\sigma}}_n)$ with $\bar{\bar{\sigma}}_i = \alpha_2 \bar{\sigma}_i + (1 - \alpha_2)\bar{\sigma}_i^{-1}$ for $i = 3, ..., n$ we obtain that $P_2 > 0$ satisfies (13). By repeating this process we obtain that $P_n = U \Sigma_n U^T = I$ for $\Sigma_n = I$ is a solution of (13) and this concludes the proof. □

In the next section, we propose a simple formulation to determine whether or not the symmetric system (1)-(2) is asymptotically stable and strictly dissipative, without having to solve the above linear matrix inequality (9). Subsequently, we provide an explicit solution of the mixed H_∞ and positive real performance analysis problem for symmetric systems.

III. Stability and Dissipativity of Symmetric Systems

For simplicity of the formulations in this section, we assume $D = 0$ in (1). The results will be extended to the case where $D \neq 0$ later in this section.

Lemma 4: Consider the system (1) that satisfies the state-space symmetry conditions (2) with $D = 0$. Also, assume that $Q = Q^T$, S, and $R = R^T$ are given weighting matrices in (4) subject to assumptions (7) and (8). The system (1)-(2) is asymptotically stable and strictly (Q, S, R)-dissipative if and only if there exists a positive scalar α satisfying the following inequality

$$2\alpha A + B(Q_- + (\alpha I - S)R^{-1}(\alpha I - S^T))B^T < 0 \quad (14)$$

Proof. Let us consider the following Lyapunov matrix

$$P = \alpha I_n \quad (15)$$

where α is a positive scalar and I_n is the identity matrix. Taking (15) into account, the LMI (9) results in

$$\begin{bmatrix} 2\alpha A & \alpha B - BS & BQ_-^{\frac{1}{2}} \\ \star & -(R + DS + S^T D) & DQ_-^{\frac{1}{2}} \\ \star & \star & -I \end{bmatrix} < 0 \quad (16)$$

Then, (14) is obtained by applying Schur complement formula to the latter inequality assuming that $D = 0$. Proof of the necessity of (16) follows similar lines as the proof of Theorem 2, that will be presented next, and it is omitted here for brevity. □

Remark 2: It is noted that the problem of checking asymptotic stability and dissipativity of the system (1) requires solving the LMI problem (9). Lemma 4 states that for symmetric systems this condition collapses to an LMI that includes only one scalar decision variable α.

In the following, we study the special case of Lemma 4 for the mixed H_∞ and positive real performance problem stated in Remark 1. Note that the extreme values of θ are interpreted as $\theta \longrightarrow 0$ which corresponds to positive realness of the system, and $\theta \longrightarrow 1$ which corresponds to the H_∞ performance.

Theorem 2: Consider the following stable symmetric system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), \quad x(0) = 0 \\ z(t) &= Cx(t) \end{aligned} \quad (17)$$

where $A = A^T$ and $B = C^T$. Let $\theta \in (0, 1)$ be a given scalar weight representing a trade-off between H_∞ and positive real performances. The H_∞ norm of the system can be explicitly obtained from the following relation.

$$\bar{\gamma} = f(\theta) \times \lambda_{max}(-B^T A^{-1} B) \quad (18)$$

where $f(\theta)$ is given by

$$f(\theta) = 1 + \frac{\sqrt{\theta^2 + (\theta - 1)^2} - 1}{\theta} \quad (19)$$

Proof. Considering the symmetry conditions $A = A^T$, $B = C^T$, and substituting $Q = -\gamma^{-1}\theta I$, $S = (1 - \theta)I$, and $R = \gamma\theta I$ into (9) result in

$$\begin{bmatrix} AP + PA & PB - (1 - \theta)B & \sqrt{\frac{\theta}{\gamma}} B \\ B^T P - (1 - \theta)B^T & -\gamma\theta I & 0 \\ \sqrt{\frac{\theta}{\gamma}} B^T & 0 & -I \end{bmatrix} < 0 \quad (20)$$

Now, we consider the Lyapunov matrix $P > 0$ to be $P = \alpha P_1$, with α and P_1 being positive scalar and matrix, respectively. After substituting P in the above inequality and using Schur complement, (20) can be written as

$$\alpha A P_1 + \alpha P_1 A + \frac{\alpha^2}{\gamma\theta} P_1 B B^T P_1 - \frac{(1 - \theta)\alpha}{\gamma\theta} B B^T P_1 \\ - \frac{(1 - \theta)\alpha}{\gamma\theta} P_1 B B^T + \frac{(1 - \theta)^2 + \theta^2}{\gamma\theta} B B^T < 0 \quad (21)$$

Pre- and Post-multiplying (21) by P_1^{-1}, we obtain a similar condition with respect to P_1^{-1}, that is,

$$\alpha P_1^{-1}A + \alpha AP_1^{-1} + \frac{\alpha^2}{\gamma\theta}BB^T - \frac{(1-\theta)\alpha}{\gamma\theta}P_1^{-1}BB^T - \frac{(1-\theta)\alpha}{\gamma\theta}BB^TP_1^{-1} + \frac{(1-\theta)^2+\theta^2}{\gamma\theta}P_1^{-1}BB^TP_1^{-1} < 0 \tag{22}$$

Setting $\alpha^2 = \theta^2 + (\theta-1)^2$, (21) and (22) will result in the same inequality with respect to P_1 and P_1^{-1}, respectively. From Lemma 3 we obtain that $P_1 = I$ is a solution to the above inequalities. Substituting $P = \alpha I$ into (21), we obtain

$$f(\theta)BB^T \leq -\gamma A \tag{23}$$

where $f(\theta)$ is defined as in (19). Now, making use of Lemma 1, (23) can be written as

$$\gamma \geq f(\theta) \times \lambda_{max}(-B^TA^{-1}B) \tag{24}$$

and this concludes the proof. □

Remark 3: For the case of strict H_∞ performance (*i.e.,* for $\theta = 1$), the bound obtained from Theorem 2 is determined to be

$$\gamma = \lambda_{max}(-B^TA^{-1}B) \tag{25}$$

which is the same explicit formula as given in [11].

In the next theorem we consider the symmetric system in (1)-(2), where now D is a non-zero matrix.

Theorem 3: Let $\theta \in (0,1)$ be a given scalar weight representing a trade-off between H_∞ and positive real performances. The stable state-space symmetric system (1)-(2) has an H_∞ norm given by

$$\begin{aligned}\bar{\gamma} \quad = \quad & \max\{\lambda_{max}[-f(\theta)^{-1}D], \lambda_{max}[-f(\theta)B^TA^{-1}B \\ & +(2-\frac{2}{\theta}+f(\theta)^{-1})D]\}\end{aligned} \tag{26}$$

Proof. Similar to the proof of Theorem 2 we consider $P = \alpha I$. Substituting the system matrices, the corresponding weighting matrices, and $P = \alpha I$ into the matrix inequality (9) leads to

$$\begin{bmatrix} 2\alpha I & (\alpha+\theta-1)B^T & B \\ \star & -\gamma\theta I - 2(1-\theta)D & D \\ \star & \star & -\frac{\gamma}{\theta}I \end{bmatrix} < 0 \tag{27}$$

This inequality can be rewritten in the form

$$\gamma MM^T - Z > 0$$

where M and Z are defined as following

$$M = \begin{bmatrix} 0 & 0 \\ \sqrt{\theta}I & 0 \\ 0 & \frac{1}{\sqrt{\theta}}I \end{bmatrix}, \quad Z = \begin{bmatrix} 2\alpha A & (\alpha+\theta-1)B & B \\ \star & 2(\theta-1) & D \\ \star & \star & 0 \end{bmatrix}. \tag{28}$$

Note that

$$M^\perp = \begin{bmatrix} I & 0 & 0 \end{bmatrix}.$$

Hence, the solvability condition (11) in Finsler's Lemma 2 is satisfied since it results in $2\alpha A < 0$. It is also noted that the H_∞ norm of the symmetric system is given by (12). Since

$$M^\dagger = \begin{bmatrix} 0 & \frac{1}{\sqrt{\theta}}I & 0 \\ 0 & 0 & \sqrt{\theta}I \end{bmatrix}$$

the formula in (12) provides the following expression for the H_∞ norm of the symmetric system

$$\bar{\gamma} = \lambda_{max}\begin{bmatrix} \frac{(\alpha+\theta-1)^2}{2\alpha\theta}\Omega + \frac{2(\theta-1)}{\theta}D & \frac{\alpha+\theta-1}{2\alpha}\Omega \\ \frac{\alpha+\theta-1}{2\alpha}\Omega & \frac{\theta}{2\alpha}\Omega \end{bmatrix} \tag{29}$$

where $\Omega = -B^TA^{-1}B$. The right hand side of the above equality can be rewritten as

$$\lambda_{max}(\begin{bmatrix} \frac{\alpha+\theta-1}{\theta}I & 0 \\ I & I \end{bmatrix} \begin{bmatrix} \frac{\alpha+\theta-1}{2\alpha}I & 0 \\ \frac{\theta}{2\alpha}I & 0 \\ \frac{2(\theta-1)}{\alpha+\theta-1}I & I - \frac{2(\theta-1)}{\alpha+\theta-1}I \\ \frac{\theta}{\alpha+\theta-1}I & -\frac{\theta}{\alpha+\theta-1}I \end{bmatrix}^T \begin{bmatrix} \Omega & 0 \\ 0 & \Omega \\ D & 0 \\ 0 & D \end{bmatrix})$$

Taking into account the fact that $\lambda_i(AB) = \lambda_i(BA)$ for all nonzero eigenvalues and any pair of matrices A and B of compatible dimensions, we obtain that

$$\bar{\gamma} = \lambda_{max}\begin{bmatrix} -\frac{\alpha+\theta-1}{\theta}\Omega + (2-\frac{2}{\theta}+\frac{\theta}{\alpha+\theta-1})D & -\frac{\theta}{2\alpha}\Omega + \frac{\theta}{\alpha+\theta-1}D \\ (\frac{\alpha-\theta+1}{\theta} - \frac{\theta}{\alpha+\theta-1)})D & -\frac{\theta}{\alpha+\theta-1}D \end{bmatrix}. \tag{30}$$

Similar to the proof of the previous theorem, setting $\alpha^2 = \theta^2+(\theta-1)^2$ makes the latter matrix an upper triangular one, and consequently the maximum eigenvalue is determined from (26). □

IV. The Mixed H_∞ and Positive Real Control Synthesis Problem

Now consider the following state-space system representation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1w(t) + B_2u(t) \\ z(t) &= C_1x(t) + D_{11}w(t) \\ y(t) &= C_2x(t) \end{aligned} \tag{31}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^{m_1}$ is the vector of exogenous inputs, $u(t) \in \mathbb{R}^{m_2}$ is the vector of control inputs, $z(t) \in \mathbb{R}^{p_1}$ is the vector of controlled outputs, and $y(t) \in \mathbb{R}^{p_2}$ is the vector of measured outputs. We call this system state-space symmetric if the system state-space data satisfy the following symmetry conditions

$$A = A^T, \quad B_1 = C_1^T, \quad B_2 = C_2^T, \quad D_{11} = D_{11}^T. \tag{32}$$

The static symmetric output feedback mixed H_∞ and positive real control synthesis problem is to design a symmetric static output feedback gain K such that the control law

$$u(t) = -Ky(t) \tag{33}$$

renders the closed-loop system stable and guarantees a mixed H_∞ and positive real performance.

The closed-loop system of the open-loop system (31) and the controller (33) becomes

$$\dot{x}(t) = (A - B_2 K B_2^T)x(t) + B_1 w(t)$$
$$z(t) = C_1 x(t) + D_{11} w(t). \tag{34}$$

Note that the closed-loop system (34) is also symmetric. The following result provides explicit expressions for the H_∞ norm of the closed-loop system and the corresponding controller gain for the special case of mixed H_∞ and positive real performance described earlier.

Theorem 4: Consider the symmetric system represented by (31) where $D_{11} = 0$. The achievable level of H_∞ performance can be computed from

$$\gamma_{bound} = f(\theta) \times \lambda_{max}[B_1^T B_2^{\perp T}(-B_2^{\perp} A B_2^{\perp T})^{-1} B_2^{\perp} B_1] \tag{35}$$

where $\theta \in (0, 1)$ represents the trade-off between H_∞ performance and positive real performance and $f(\theta)$ is defined in (19). For any $\gamma \geq \gamma_{bound}$, a static symmetric output feedback H_∞ control gain which makes the closed-loop system stable with H_∞ norm less than γ can be selected as

$$K \geq B_2^{\dagger}[\Sigma - \Sigma B_2^{\perp T}(B_2^{\perp} \Sigma B_2^{\perp T})^{-1} B_2^{\perp} \Sigma] B_2^{\dagger T} \tag{36}$$

where Σ is defined by

$$\Sigma = A + \frac{f(\theta)}{\gamma} B_1 B_1^T. \tag{37}$$

Proof. Substituting the closed-loop system matrices (34) into (9), and taking $Q = -\gamma^{-1}\theta I$, $S = (1-\theta)I$, and $R = \gamma\theta I$ into account along with applying Schur complement while assuming $D_{11} = 0$, we obtain

$$B_2 K B_2^T > A + \frac{f(\theta)}{\gamma} B_1 B_1^T \tag{38}$$

Note that $f(\theta)$ is defined in (19), where the trade-off parameter θ is known. Now, applying Finsler's lemma on (38) results in the following solvability condition

$$B_2^{\perp}(A + \frac{f(\theta)}{\gamma} B_1 B_1^T) B_2^{\perp T} < 0 \tag{39}$$

Applying Lemma 1 on (39) results in (35) which provides a bound on the H_∞ norm γ. Inequality (36) is also obtained as the result of Finsler's lemma. □

Remark 4: It should be noted that the solution of a static output feedback control problem, in general, results in a non-convex formulation with no efficient solution available [10]; however, the results of Theorem 4 provides an explicit solution without having to solve a bilinear-matrix-inequality (BMI) problem.

Remark 5: The solution of the symmetric strictly positive real output feedback control problem is obtained when $\theta = 0$. For $\theta = 0$, we obtain $f(0) = 0$ and hence $\gamma_{bound} = 0$, and a static symmetric output feedback gain that renders the closed-loop system strictly positive real is given by

$$K \geq B_2^{\dagger}[A - A B_2^{\perp T}(B_2^{\perp} A B_2^{\perp T})^{-1} B_2^{\perp} A] B_2^{\dagger T}.$$

V. Numerical Examples

In this section, we validate our mixed H_∞ and positive real performance analysis and static output feedback synthesis results using numerical examples.

Example 1: We consider a state-space symmetric system with the randomly generated data given in [6]. Let us consider the mixed H_∞ and positive real performance analysis problem for the system represented by (A, B_1, C_1, D). We vary the weight θ in the interval $(0, 1)$ and plot the H_∞ norm of the open-loop system versus θ determined from Theorem 3 in comparison with the H_∞ norm of the system computed using MATLAB. This comparison is illustrated in Figure 1 where it is observed that the proposed formulation for the H_∞ norm computation matches the actual norm of the symmetric system.

Example 2: As a second example, we consider the RL circuit network shown in Figure 2. We choose the currents of the inductors L_1, L_2 and L_3 as state variables x_i, $i = 1, 2, 3$, the disturbance voltage V_d as the disturbance input w, and the current I as the output z. Assume that $L_1 = L_2 = L_3 = 1$, and $R_1 = 1, R_2 = 2, R_3 = 3, R_4 = 4$. The open-loop system we obtain is a symmetric system as in (1) with the following data

$$A = A^T = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -5 & 3 \\ 0 & 3 & -7 \end{bmatrix}, \quad B = C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad D = 1.$$

We plot the H_∞ norm of the open-loop system determined from Theorem 3 and the H_∞ norm of the system computed using MATLAB versus θ in Figure 3. It is shown that the two profiles are identical.

Example 3: In order to validate the synthesis condition results presented in Theorem 4, we consider the symmetric system in (31) with system matrices A and B_1 given in the Example 1 and the control input matrix given in [6]. We vary the trade-off parameter θ in the interval $(0, 1)$ and compute the best achievable level of the H_∞ performance of the closed-loop system γ_{opt} using (35). Then, we design the output feedback control (33) with the control gain computed using (36) which guarantees the closed-loop system H_∞ performance for any $\gamma > \gamma_{opt}$. For each $\theta \in (0, 1)$, we assume the desired level of the closed-loop system H_∞ norm to be $\gamma = 1.01\gamma_{opt}$. Comparison between the actual H_∞ norm of the closed-loop system and the desired γ is illustrated in Figure 4. It is observed that for any θ the closed-loop system H_∞ norm is exactly the desired value. It should be noted that the controller designed for each θ results in improved disturbance attenuation for the closed-loop system compared to the open-loop system. This comparison can be found in [6].

VI. Conclusion

We addressed the dissipative analysis and output feedback control synthesis problem for LTI state-space symmetric systems. We have derived explicit necessary and sufficient conditions for quadratic dissipativeness and we have developed an explicit expression for the H_∞ norm of such

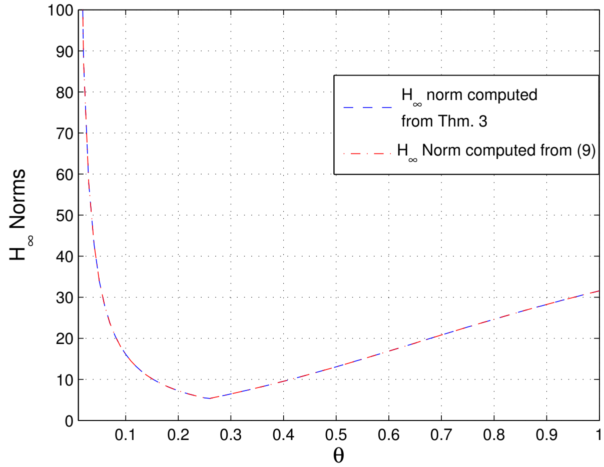

Fig. 1. Profiles of the H_∞ norms vs. θ for Example 1

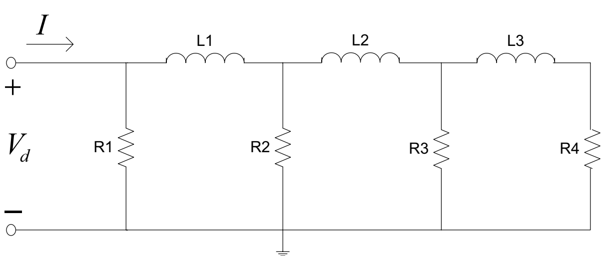

Fig. 2. RL circuit network

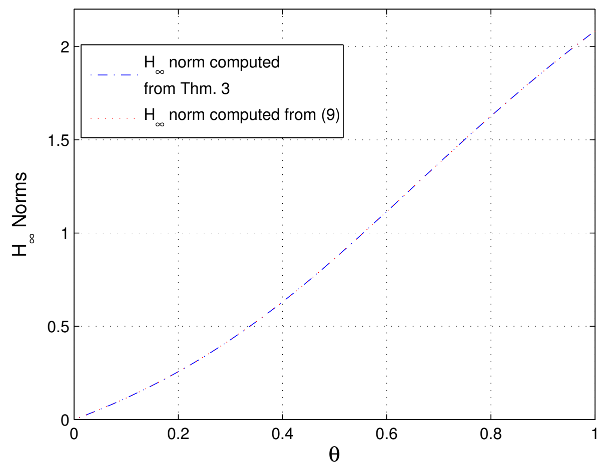

Fig. 3. Profile of the H_∞ norms vs. θ for Example 2

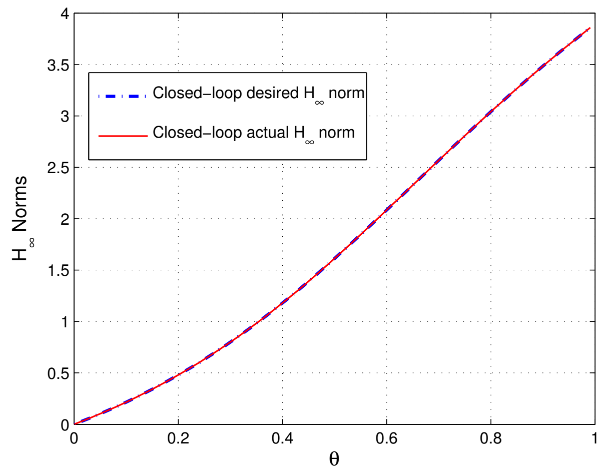

Fig. 4. Profiles of the closed-loop actual and desired H_∞ norms vs. θ for Example 3

systems for the special case of mixed H_∞ and positive real performance. Then, we developed explicit formulas for the mixed H_∞ and positive real static output feedback control law, along with an explicit expression for the achievable H_∞ norm of the closed-loop LTI symmetric system.

REFERENCES

[1] B. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis: A Modern Systems Theory*, Prentice-Hall, Englewood Cliffs, NJ, 1973.

[2] D.J. Hill and P.J. Moylan, "Dissipative Dynamical Systems: Basic Input-Output and State Properties," in *Journal Franklin Inst.*, 309: 327-357, 1980.

[3] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis,* Cambridge University Press, Cambridge, UK, 1991.

[4] C.-H. Huang, P.A. Ioannou, J. Maroulas, and M.G. Safonov, "Design of Strictly Positive Real Systems Using Constatnt Output Feedback," in *IEEE Trans. on Automatic Control*, 44(3): 569-573, 1999.

[5] R. Lozano, B. Brogliato, O. Egeland, and B. Maschke, *Dissipative Systems Analysis and Control: Theory and Applications*, Springer-Verlag, London, 2000.

[6] M. Meisami-Azad, *Control Design Methods for Large Scale Systems Using an Upper Bound Formulation*, Masters Thesis, University of Houston, TX, Aug. 2007.

[7] R.E. Skelton, T. Iwasaki, and K.M. Grigoriadis, *A Unified Algebraic Approach to Linear Control Design*, Taylor & Francis, 1998.

[8] B. Srinivasan and P. Myszkorowski, "Model Reduction of Systems with Zeros Interlacing the Poles," in *System and Control Letters*, 30(1): 19-24, 1997.

[9] W. Sun, P.P. Khargonekar, and D. Shim, "Solution to the Positive Real Control Problem for Linear Time-Invariant Systems," in *IEEE Trans. on Automatic Control*, 39(10): 2034-2046, 1994.

[10] V. Syrmos, C. Abdallah, P. Dorato, and K. M. Grigoriadis, "Static Output Feedback: A Survey," in *Automatica*, 33(2): 125-137, 1997.

[11] K. Tan and K.M. Grigoriadis, "Stabilization and H_∞ Control of Symmetric Systems: an Explicit Solution," in *Systems and Control Letters*, 44(1): 57-72, 2001.

[12] J.C. Willems, "Dissipative Dynamical Systems - Part 1: General Theory, Part 2: Linear Systems with Quadratic Supply Rates," *Archive for Rational Mechanics and Analysis*, no. 45, pp. 321-351 and 352-393, 1972.

[13] J.C. Willems, *Average Value Stability Criteria for Symmetric Systems*, Ricerche di Automatica, no. 4, pp. 87-108, 1973.

[14] S. Xie, L. Xie, and C.E. de Souza, "Robust Dissipative Control for Linear Systems with Dissipative Uncertainty," in *International Journal of Control*, 70(2): 169-191, 1998.

[15] K. Zhou, J.C. Doyle, and K. Glover, *Robust and Optimal Control,* Prentice-Hall, Inc., Upper Saddle River, NJ, 1996