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Robust H$^\infty$ Control of an Uncertain System via a Stable Positive Real Output Feedback Controller

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Abstract— **The paper presents a new approach to the robust H^∞ control of an uncertain system via an output feedback controller which is both stable and positive real. The uncertain systems under consideration contain structured uncertainty described by integral quadratic constraints. The controller is designed to achieve absolute stabilization with a specified level of disturbance attenuation. The main result involves solving a state feedback version of the problem by solving an algebraic Riccati equation dependent on a set of scaling parameters. Then two further algebraic Riccati equations are solved which depend on a further set of scaling parameters. The required controller is constructed from the Riccati solutions.**

I. INTRODUCTION

This paper considers the problem of robust H^∞ control via an output feedback controller which is both stable and positive real. This means that we aim to design a controller which is passive; e.g., see [1], [2]. The results of this paper build on the results of the previous conference papers [3], [4] which considered the problem of robust H^∞ control via a stable output feedback controller and H^∞ norm bounded controller respectively.

It is well known that the use of stable controllers is preferable to the use of unstable feedback controllers in many practical control problems; e.g., see [5], [6]. Indeed, the use of unstable controllers can lead to problems with actuator and sensor failure, sensitivity to plant uncertainties and nonlinearities and implementation problems. Furthermore, the use of a positive real controller has significant robustness advantages in the control of strictly passive systems due to the passivity theorem which states that the negative feedback interconnection of a passive and strictly passive system is stable; e.g., see [7]. In particular, in the control of flexible structures, the use of positive real (passive) controller can provide significant advantages in terms of robustness to spillover dynamics; e.g., see [8]. Thus, we are motivated in this paper to extend the results of [3]–[6], [9] to solve a robust H^∞ control problem via a positive real controller.

We consider a class of uncertain systems with structured uncertainty described by Integral Quadratic Constraints (IQCs); e.g., see [10], [11]. Indeed, our results build on the results of [10] which provide necessary and sufficient conditions for the absolute stabilization of such uncertain systems with a specified level of disturbance attenuation (but with no requirement that the output feedback controller is stable or positive real). The key idea behind our approach is to begin with an uncertain system of the type considered in [10] and then add an additional uncertainty to form a new uncertain system. This additional uncertainty has the property that for one specific value of the uncertainty, the new uncertain system reduces to the original uncertain system and thus any suitable controller for the new uncertain system will also solve the problem of absolute stabilization with a specified level of disturbance attenuation for the original system. Also, for a different value of the new uncertainty, the new uncertain system reduces to a certain open loop system in such a way that the controller is forced to be stable and positive real. Because our approach involves the addition of new uncertainties, our results provide only sufficient conditions rather than necessary and sufficient conditions for absolute stabilization with a specified level of disturbance attenuation. However, because the new uncertainty is explicitly constructed, this can give some indication about the degree of conservatism introduced.

Our main result is obtained applying the results of [10] to the new uncertain system. This gives us a procedure for constructing a strict bounded real output feedback controller solving a problem of absolute stabilization with a specified level of disturbance attenuation. This is achieved by solving a pair of algebraic Riccati equations dependent on a set of scaling parameters and a scaling matrix. The output feedback controller obtained is of the same order as the plant.

The remainder of the paper proceeds as follows: In Section II of the paper, we set up the problem of absolute stabilization with a specified level of disturbance attenuation via a stable positive real output feedback controller. Section III introduces the new uncertain system for which we will apply the results of [10] in order to obtain a positive real controller which guarantees absolute stabilization with a specified level of disturbance attenuation. The construction of this new uncertain system involves solving a state feedback version of the approach of [10] applied to the original uncertain system. This involves solving an algebraic Riccati equation of the H^∞ type which is dependent on a set of scaling parameters. This leads to our main result which is a procedure for constructing the required positive real controller. This procedure involves solving a pair algebraic Riccati equations of the H^∞ type which are dependent on an additional set of scaling parameters and a scaling matrix. The final controller is constructed from the solutions to these Riccati equations. Section IV presents an example which illustrates the theory presented in the paper.

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II. Problem Statement

We consider an output feedback H^∞ control problem for an uncertain system of the following form:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) + \sum_{s=1}^{k} D_s \xi_s(t);\\
z(t) &= C_1 x(t) + D_{12} u(t);\\
\zeta_1(t) &= K_1 x(t) + G_1 u(t);\\
&\vdots\\
\zeta_k(t) &= K_k x(t) + G_k u(t);\\
y(t) &= C_2 x(t) + D_{21} w(t)
\end{aligned} \quad (1)$$

where $x(t) \in \mathbf{R}^n$ is the *state*, $w(t) \in \mathbf{R}^p$ is the *disturbance input*, $u(t) \in \mathbf{R}^m$ is the *control input*, $z(t) \in \mathbf{R}^q$ is the *error output*, $\zeta_1(t) \in \mathbf{R}^{h_1}, \dots, \zeta_k(t) \in \mathbf{R}^{h_k}$ are the *uncertainty outputs*, $\xi_1(t) \in \mathbf{R}^{r_1}, \dots, \xi_k(t) \in \mathbf{R}^{r_k}$ are the *uncertainty inputs* and $y(t) \in \mathbf{R}^m$ is the *measured output*. The uncertainty in this system is described by a set of equations of the form

$$\begin{aligned}
\xi_1(t) &= \phi_1(t, \zeta_1(\cdot)|_0^t)\\
\xi_2(t) &= \phi_2(t, \zeta_2(\cdot)|_0^t)\\
&\vdots\\
\xi_k(t) &= \phi_k(t, \zeta_k(\cdot)|_0^t)
\end{aligned} \quad (2)$$

where the following Integral Quadratic Constraint is satisfied.

Definition 1: (Integral Quadratic Constraint; see [10], [11].) An uncertainty of the form (2) is an admissible uncertainty for the system (1) if the following conditions hold: Given any locally square integrable control input $u(\cdot)$ and locally square integrable disturbance input $w(\cdot)$, and any corresponding solution to the system (1), (2), let $(0, t_*)$ be the interval on which this solution exists. Then there exist constants $d_1 \geq 0, \dots, d_k \geq 0$ and a sequence $\{t_i\}_{i=1}^\infty$ such that $t_i \to t_*$, $t_i \geq 0$ and

$$\int_0^{t_i} \|\xi_s(t)\|^2 dt \leq \int_0^{t_i} \|\zeta_s(t)\|^2 dt + d_s \quad \forall i \quad \forall s = 1, \dots, k. \quad (3)$$

Here $\|\cdot\|$ denotes the standard Euclidean norm and $\mathbf{L}_2[0,\infty)$ denotes the Hilbert space of square integrable vector valued functions defined on $[0,\infty)$. Note that t_i and $t_\star$ may be equal to infinity. The class of all such admissible uncertainties $\xi(\cdot) = [\xi_1(\cdot), \dots, \xi_k(\cdot)]$ is denoted Ξ.

For the uncertain system (1), (3), we consider a problem of absolute stabilization with a specified level of disturbance attenuation. The class of controllers considered are output feedback controllers of the form

$$\begin{aligned}
\dot{x}_c(t) &= A_c x_c(t) + B_c y(t),\\
u(t) &= C_c x_c(t).
\end{aligned} \quad (4)$$

We will additionally require that the controller is stable and is positive real; i.e.,

$$A_c \text{ is Hurwitz and } G_c(-j\omega)' + G_c(j\omega) \geq 0 \quad \forall \omega \quad (5)$$

where $G_c(s) = C_c(sI - A_c)^{-1} B_c$. Such a controller is said to be *stable positive real.*

Definition 2: The uncertain system (1), (3) is said to be *absolutely stabilizable with disturbance attenuation* γ via a stable positive real controller if there exists such a controller (4) satisfying (5) and constants $c_1 > 0$ and $c_2 > 0$ such that the following conditions hold:

1) For any initial condition $[x(0), x_c(0)]$, any admissible uncertainty inputs $\xi(\cdot)$ and any disturbance input $w(\cdot) \in \mathbf{L}_2[0,\infty)$, then

$$[x(\cdot), x_c(\cdot), u(\cdot), \xi_1(\cdot), \dots, \xi_k(\cdot)] \in \mathbf{L}_2[0,\infty)$$

(hence, $t_* = \infty$) and

$$\begin{aligned}
&\|x(\cdot)\|_2^2 + \|x_c(\cdot)\|_2^2 + \|u(\cdot)\|_2^2 + \sum_{s=1}^{k} \|\xi_s(\cdot)\|_2^2\\
&\leq c_1[\|x(0)\|^2 + \|x_c(0)\|^2 + \|w(\cdot)\|_2^2 + \sum_{s=1}^{k} d_s].
\end{aligned} \quad (6)$$

2) The following H^∞ norm bound condition is satisfied: If $x(0) = 0$ and $x_c(0) = 0$, then

$$J \triangleq \sup_{w(\cdot)\in\mathbf{L}_2[0,\infty)} \sup_{\xi(\cdot)\in\Xi} \frac{\|z(\cdot)\|_2^2 - c_2 \sum_{s=1}^{k} d_s}{\|w(\cdot)\|_2^2} < \gamma^2. \quad (7)$$

Here, $\|q(\cdot)\|_2$ denotes the $\mathbf{L}_2[0,\infty)$ norm of a function $q(\cdot)$. That is, $\|q(\cdot)\|_2^2 \triangleq \int_0^\infty \|q(t)\|^2 dt$.

III. The Main Results

The key idea behind our main result is to introduce some extra uncertainty into the uncertain system (1), (3). This is done in a way so that the controller must achieve absolute stabilization with disturbance attenuation γ when applied to the original uncertain system (1), (3). Also, the controller must achieve stability with disturbance attenuation γ when applied to a system constructed so that the controller is disconnected from the plant and the controller itself must be stable and satisfy a frequency domain property which ensures that it is positive real. In other words, the controller itself must have the stable positive real property.

In order to define the required new uncertain system, we first consider a state feedback version of the problem considered in [10]. Indeed, using the results of [10], we can give a condition for the uncertain system (1), (3) to be absolutely stabilizable with a specified level of disturbance attenuation via a state feedback controller. This condition is given in terms of the existence of solutions to a parameter dependent algebraic Riccati equations. The Riccati equation under consideration is defined as follows: Let $\tau_1 > 0$, …, $\tau_k > 0$ be given constants and consider the algebraic Riccati equation

$$\begin{aligned}
&(A - B_2 E_1^{-1} \hat{D}_{12}' \hat{C}_1)' X + X(A - B_2 E_1^{-1} \hat{D}_{12}' \hat{C}_1)\\
&\quad + X(\hat{B}_1 \hat{B}_1' - B_2 E_1^{-1} B'_2) X\\
&\quad + \hat{C}_1'(I - \hat{D}_{12} E_1^{-1} \hat{D}_{12}') \hat{C}_1 = 0;
\end{aligned}$$

where

$$\begin{aligned}
\hat{C}_1 &= \begin{bmatrix} C_1 \\ \sqrt{\tau_1}K_1 \\ \vdots \\ \sqrt{\tau_k}K_k \end{bmatrix}; \hat{D}_{12} = \begin{bmatrix} D_{12} \\ \sqrt{\tau_1}G_1 \\ \vdots \\ \sqrt{\tau_k}G_k \end{bmatrix}; \\
E_1 &= \hat{D}_{12}'\hat{D}_{12}; \\
\hat{B}_1 &= \begin{bmatrix} \gamma^{-1}B_1 & \sqrt{\tau_1}^{-1}D_1 & \ldots & \sqrt{\tau_k}^{-1}D_k \end{bmatrix}.
\end{aligned} \tag{9}$$

Assumption 1: We will restrict attention to scaling parameters $\tau_1 > 0$, …, $\tau_k > 0$ such that $E_1 > 0$.

We now present a result which follows directly from the main result of [10]. This result gives a condition under which there exists a state feedback controller such that the resulting closed loop system is absolutely stable with disturbance attenuation γ. In fact for our purposes, we only require a state feedback controller controller such that the resulting closed loop system is absolutely stable. However, using the result of [10], we can obtain a state feedback controller such that the resulting closed loop system is absolutely stable with disturbance attenuation γ without any loss of generality and it will be most convenient to use this controller.

Lemma 1: Suppose the uncertain system (1), (3) is absolutely stabilizable with disturbance attenuation γ via a controller of the form (4) (but which does not necessarily satisfy condition (5)). Furthermore, suppose that Assumption 1 is satisfied for all constants $\tau_1 > 0$, …, $\tau_k > 0$. Then, there exist constants $\tau_1 > 0, \ldots, \tau_k > 0$ such that the Riccati equation (8) has a solution $X > 0$. Furthermore, the uncertain system (1), (3) is absolutely stabilizable with disturbance attenuation γ via the state feedback controller

$$u(t) = Kx(t) \tag{10}$$

where

$$K = -E_1^{-1}(B_2'X + \hat{D}_{12}'\hat{C}_1). \tag{11}$$

Proof. Suppose the uncertain system (1), (3) is absolutely stabilizable with disturbance attenuation γ via a controller of the form (4) (but which does not necessarily satisfy condition (5)). Furthermore, suppose that Assumption 1 is satisfied for all constants $\tau_1 > 0$, …, $\tau_k > 0$. Then, it follows from the proof of Theorem 4.1 of [10] that there exist constants $\tau_1 > 0, \ldots, \tau_k > 0$ such that the controller (4) solves the H^∞ control problem defined by the system

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + \hat{B}_1\hat{w}(t) + B_2u(t); \\
\hat{z}(t) &= \hat{C}_1x(t) + \hat{D}_{12}u(t); \\
y(t) &= C_2x(t) + \hat{D}_{21}\hat{w}(t)
\end{aligned} \tag{12}$$

and the H^∞ norm bound condition

$$\hat{J} \triangleq \sup_{\hat{w}(\cdot)\in\mathbf{L}_2[0,\infty),x(0)=0,x_c(0)=0} \frac{\|\hat{z}(\cdot)\|_2^2}{\|\hat{w}(\cdot)\|_2^2} < 1. \tag{13}$$

Here,

$$\begin{aligned}
\hat{w}(\cdot) &= \begin{bmatrix} \gamma w(\cdot)' & \sqrt{\tau_1}\xi_1(\cdot)' & \ldots & \sqrt{\tau_k}\xi_k(\cdot)' \end{bmatrix}', \\
\hat{z}(\cdot) &= \begin{bmatrix} z(\cdot)' & \sqrt{\tau_1}\zeta_1(\cdot)' & \ldots & \sqrt{\tau_k}\zeta_k(\cdot)' \end{bmatrix}'
\end{aligned}$$

and the matrix coefficients $\hat{B}_1, \hat{C}_1, \hat{D}_{12}$ are defined by (9) and

$$\hat{D}_{21} = \begin{bmatrix} \gamma^{-1}D_{21} & 0_{l\times r_1} & \ldots & 0_{l\times r_k} \end{bmatrix}. \tag{14}$$

Then, it follows from a standard result on H^∞ control (e.g., see Theorem 3.3 of [12]) that there exists a state feedback control law $u = Kx$ which stabilizes the system (12) and leads to the satisfaction of the H^∞ condition (13). Furthermore, it also follows from standard H^∞ control theory (e.g., see Corollary 3.1 of [12] or Theorem 4.8 and Section 4.5.1 of [13]) that the Riccati equation (8) has a solution $X > 0$ and that the corresponding state feedback controller (10), (11) stabilizes the system (12) and leads to the satisfaction of the H^∞ condition (13). It now follows using the same argument that is used in the proof of Theorem 4.1 of [10] that the state feedback controller (10), (11) absolutely stabilizes the uncertain system (1), (3) with disturbance attenuation γ. □

We now suppose that constants $\tau_1 > 0, \ldots, \tau_k > 0$ have been found such that the Riccati equation (8) has a solution $X > 0$ and we will use the corresponding state feedback gain matrix K defined in (11) to define a new uncertain system as follows:

$$\begin{aligned}
\dot{x}(t) &= \tilde{A}x(t) + B_1w_1(t) + \tilde{B}_2u(t) + \sum_{s=1}^{k} D_s\xi_s(t) \\
&\quad +\Delta B_2\left(\frac{K}{2}x(t) - \frac{u(t)}{2}\right); \\
z_1(t) &= \frac{C_1}{2}x(t) + \frac{D_{12}}{2}u(t) \\
&\quad -\Delta\left(\frac{C_1}{2}x(t) + \frac{D_{12}}{2}u(t)\right); \\
z_2(t) &= u - \frac{\gamma w_2(t)}{2} + \Delta\left(u(t) - \frac{\gamma w_2(t)}{2}\right); \\
\zeta_1(t) &= \tilde{K}_1x(t) + \tilde{G}_1u(t) + \Delta G_1\left(\frac{K}{2}x(t) - \frac{u(t)}{2}\right); \\
&\ \ \vdots \\
\zeta_k(t) &= \tilde{K}_kx(t) + \tilde{G}_ku(t) + \Delta G_k\left(\frac{K}{2}x(t) - \frac{u(t)}{2}\right); \\
y(t) &= \frac{1}{2}C_2x(t) + \frac{1}{2}D_{21}w_1(t) + \frac{\gamma w_2}{2} - \frac{u}{2} \\
&\quad -\Delta\left(\frac{C_2}{2}x(t) + \frac{D_{21}}{2}w_1(t) - \frac{\gamma w_2(t)}{2} + \frac{u}{2}\right)
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
\tilde{A} &= A + \frac{1}{2}B_2K; \quad \tilde{B}_2 = \frac{1}{2}B_2; \\
\tilde{K}_1 &= K_1 + \frac{1}{2}G_1K; \quad \tilde{G}_1 = \frac{1}{2}G_1; \\
&\ \ \vdots \\
\tilde{K}_k &= K_k + \frac{1}{2}G_kK; \quad \tilde{G}_k = \frac{1}{2}G_k;
\end{aligned} \tag{16}$$

Here Δ is assumed to be a scalar uncertain parameter satisfying the bound

$$|\Delta| \leq 1. \tag{17}$$

Case 1. $\Delta = 1$. In this case, it is straightforward to verify that with this value of Δ the state equations (15) become

$$\begin{aligned}
\dot{x}(t) &= (A + B_2 K)\, x(t) + B_1 w_1(t) + \sum_{s=1}^{k} D_s \xi_s(t);\\
z_1(t) &= 0;\\
z_2(t) &= 2u(t) - \gamma w_2(t);\\
\zeta_1(t) &= (K_1 + G_1 K)\, x(t);\\
&\vdots\\
\zeta_k(t) &= (K_k + G_k K)\, x(t);\\
y(t) &= \gamma w_2(t) - u(t)
\end{aligned} \tag{18}$$

where the IQC (3) is satisfied. However, the uncertain system (18), (3) is the closed loop uncertain system obtained when the state feedback control law (10), (11) is applied to the original uncertain system (1), (3). Thus, according to the construction of K and Lemma 1, this uncertain system will be absolutely stable. It should also be noted that for the system (18), the output of the controller $u(t)$ does not affect the plant but only affects the disturbance output $z_2(t)$. Also, the input to the controller $y(t)$ is not affected by the plant but is only affected by the disturbance input $w_2(t)$. This situation illustrated in Figure 1. In this block diagram the block (Σ_{cl})

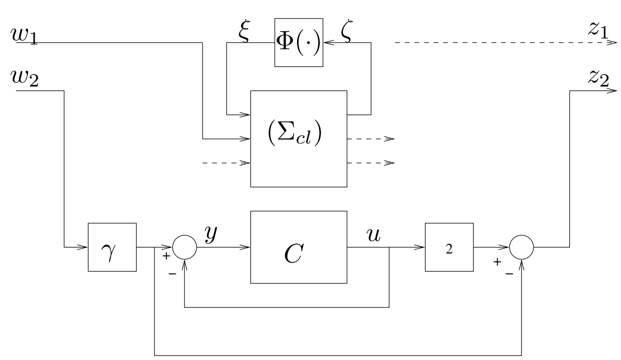

Fig. 1. Block diagram corresponding to Case 1.

refers to the closed loop uncertain system defined by (18), (3) and the block C refers to the output feedback controller of the form (4). Also, it follows the definition of absolute stabilizability with disturbance attenuation γ that we must have

$$\int_0^\infty z_2(t)' z_2(t) dt \leq \gamma^2 \int_0^\infty w_2(t)' w_2(t) dt \tag{19}$$

for all $w_2(\cdot) \in \mathbf{L}_2[0,\infty)$. However, it follows from (18) that

$$\gamma w_2 = y + u, \quad z_2 = u - y.$$

Hence, it follows from (19) that

$$\begin{aligned}
0 &\leq \int_0^\infty ((y+u)'(y+u) - (u-y)'(u-y))\, dt\\
&= 4 \int_0^\infty y' u dt
\end{aligned}$$

for all $y(\cdot) \in \mathbf{L}_2[0,\infty)$. From this, we can conclude that the output feedback controller must in fact be positive real; e.g., see [2].

Case 2. $\Delta = -1$. In this case, it is straightforward to verify that with this value of Δ the state equations (15) reduce to the state equations

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w_1(t) + B_2 u(t) + \sum_{s=1}^{k} D_s \xi_s(t);\\
z_1(t) &= C_1 x(t) + D_{12} u(t);\\
z_2(t) &= 0;\\
\zeta_1(t) &= K_1 x(t) + G_1 u(t);\\
&\vdots\\
\zeta_k(t) &= K_k x(t) + G_k u(t);\\
y(t) &= C_2 x(t) + D_{21} w_1(t)
\end{aligned} \tag{20}$$

which correspond to the original state equations (1). That is, when the controller (4) is applied to the uncertain system (15), (3), (17), this is equivalent to the situation shown in Figure 2. In this block diagram the block (Σ) refers to the

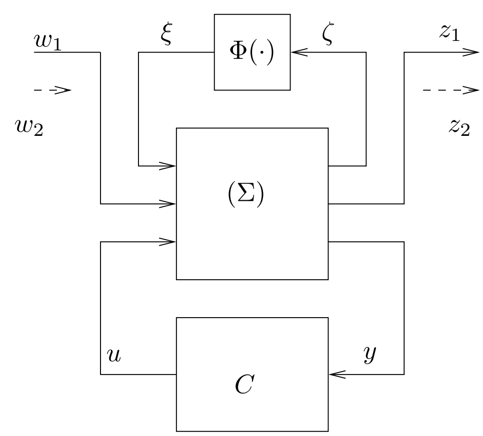

Fig. 2. Block diagram corresponding to Case 2.

original uncertain system defined by (1), (3) and the block C refers to the output feedback controller of the form (4). From this, we can conclude that the output feedback controller (4) solves the original problem of absolute stabilizability with disturbance attenuation γ.

Combining the conclusions from both cases, we can conclude that if there exists an output feedback controller of the form (4) which is absolutely stabilizing with disturbance attenuation γ for the uncertain system (15), (3), (17), then this controller is absolutely stabilizing with disturbance attenuation γ for the original uncertain system (1), (3) and furthermore, this controller is itself stable and has a transfer function which is positive real.

We now construct a new uncertain system in which the uncertainty overbounds the uncertainty in the uncertain system (15), (3), (17) and such that the results of [10] can be applied to construct the required output feedback controller. Indeed, we modify the state equations (15) to define a new uncertainty output vector ζ_{k+1} and a new uncertainty input vector ξ_{k+1} as follows:

$$\begin{array}{rcl} \dot{x}(t) &=& \tilde{A}x(t) + \tilde{B}_1\tilde{w}(t) + \tilde{B}_2u(t) + \sum_{s=1}^{k+1} D_s\xi_s(t) \\ \tilde{z}(t) &=& \tilde{C}_1x(t) + H\tilde{w}(t) + J\xi_{k+1} + \tilde{D}_{12}u(t); \\ \zeta_1(t) &=& \tilde{K}_1x(t) + F_1\xi_{k+1} + \tilde{G}_1u(t); \\ && \vdots \\ \zeta_k(t) &=& \tilde{K}_kx(t) + F_k\xi_{k+1} + \tilde{G}_ku(t); \\ \zeta_{k+1}(t) &=& \tilde{K}_{k+1}x(t) + L\tilde{w}(t) + \tilde{G}_{k+1}u(t); \\ y(t) &=& \tilde{C}_2x(t) + F_{k+1}\xi_{k+1} + \tilde{D}_{21}\tilde{w}(t) + D_{22}u \end{array} \tag{21}$$

where

$$\begin{array}{rcl} \tilde{z}(t) &=& \left[\begin{array}{c} z_1(t) \\ z_2(t) \end{array}\right]; \quad \tilde{w}(t) = \left[\begin{array}{c} w_1(t) \\ w_2(t) \end{array}\right]; \\ \tilde{B}_1 &=& \left[\begin{array}{cc} B_1 & 0 \end{array}\right]; \\ D_{k+1} &=& \left[\begin{array}{cccc} B_2 & 0 & 0 & 0 \end{array}\right] M^{-1}; \\ \tilde{C}_1 &=& \frac{1}{2}\left[\begin{array}{c} C_1 \\ 0 \end{array}\right]; \quad \tilde{D}_{12} = \left[\begin{array}{c} \frac{1}{2}D_{12} \\ \frac{1}{2}I \end{array}\right]; \\ J &=& \left[\begin{array}{cccc} 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 \end{array}\right] M^{-1}; \\ H &=& \frac{1}{2}\left[\begin{array}{cc} 0 & 0 \\ 0 & -\gamma I \end{array}\right]; \\ F_1 &=& \left[\begin{array}{cccc} G_1 & 0 & 0 & 0 \end{array}\right] M^{-1}; \\ && \vdots \\ F_k &=& \left[\begin{array}{cccc} G_k & 0 & 0 & 0 \end{array}\right] M^{-1}; \end{array} \tag{22}$$

$$\begin{array}{rcl} \tilde{K}_{k+1} &=& \frac{1}{2}M\left[\begin{array}{c} K \\ C_1 \\ 0 \\ C_2 \end{array}\right]; \quad L = \frac{1}{2}M\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & -\gamma I \\ D_{21} & -\gamma I \end{array}\right]; \\ \tilde{G}_{k+1} &=& \frac{1}{2}M\left[\begin{array}{c} -I \\ D_{12} \\ I \\ I \end{array}\right]; \quad \tilde{C}_2 = \frac{1}{2}C_2; \\ F_{k+1} &=& \left[\begin{array}{cccc} 0 & 0 & 0 & -I \end{array}\right] M^{-1}; \\ \tilde{D}_{21} &=& \frac{1}{2}\left[\begin{array}{cc} D_{21} & \gamma I \end{array}\right]; \quad D_{22} = -\frac{I}{2}; \end{array} \tag{23}$$

Here M is any nonsingular scaling matrix with appropriate dimensions. Now it is straightforward to verify that the state equations (21) reduce to the state equations (15) when we make the substitution

$$\xi_{k+1} = \left[\begin{array}{cccc} \Delta I & 0 & 0 & 0 \\ 0 & \Delta I & 0 & 0 \\ 0 & 0 & \Delta I & 0 \\ 0 & 0 & 0 & \Delta I \end{array}\right] \zeta_{k+1} = \Delta\zeta_{k+1}. \tag{24}$$

Also, it follows from (17) and (24) that uncertainty input ξ_{k+1} satisfies the following IQC of the form (3):

$$\int_0^{t_i} \|\xi_{k+1}(t)\|^2 dt \le \int_0^{t_i} \|\zeta_{k+1}(t)\|^2 dt + d_{k+1} \quad \forall i$$

for any constant $d_{k+1} > 0$. Hence, we can conclude that for any non-singular scaling matrix M, the uncertain system defined by the state equations (21) and the IQCs

$$\int_0^{t_i} \|\xi_s(t)\|^2 dt \le \int_0^{t_i} \|\zeta_s(t)\|^2 dt + d_s \ \ \forall i \ \ \forall s = 1, \ldots, k+1 \tag{25}$$

overbounds the uncertain system (15), (17). This leads us to the main result of this paper which is stated in terms of a pair of algebraic Riccati equations. The Riccati equations under consideration are defined as follows: Let $\tilde{\tau}_1 > 0, \ldots, \tilde{\tau}_k > 0$ be given constants and consider the algebraic Riccati equations

$$\begin{array}{l} (\check{A} - \check{B}_2\check{E}_1^{-1}\check{D}_{12}'\check{C}_1)'\check{X} + \check{X}(\check{A} - \check{B}_2\check{E}_1^{-1}\check{D}_{12}'\check{C}_1) \\ \quad +\check{X}(\check{B}_1\check{B}_1' - \check{B}_2\check{E}_1^{-1}\check{B}_2')\check{X} \\ \quad +\check{C}_1'(I - \check{D}_{12}\check{E}_1^{-1}\check{D}_{12}')\check{C}_1 = 0; \end{array} \tag{26}$$

$$\begin{array}{l} (\check{A} - \check{B}_1\check{D}_{21}'\check{E}_2^{-1}\check{C}_2)\check{Y} + \check{Y}(\check{A} - \check{B}_1\check{D}_{21}'\check{E}_2^{-1}\check{C}_2)' \\ \quad +\check{Y}(\check{C}_1'\check{C}_1 - \check{C}_2'\check{E}_2^{-1}\check{C}_2)\check{Y} \\ \quad +\check{B}_1(I - \check{D}_{21}'\check{E}_2^{-1}\check{D}_{21})\check{B}_1' = 0 \end{array} \tag{27}$$

where

$$\begin{array}{rcl} \check{A} &=& \tilde{A} + \bar{B}_1\bar{D}_{11}'\left(I - \bar{D}_{11}\bar{D}_{11}'\right)^{-1}\bar{C}_1; \\ \check{B}_2 &=& \tilde{B}_2 + \bar{B}_1\bar{D}_{11}'\left(I - \bar{D}_{11}\bar{D}_{11}'\right)^{-1}\bar{D}_{12}; \\ \check{C}_2 &=& \tilde{C}_2 + \bar{D}_{21}\bar{D}_{11}'\left(I - \bar{D}_{11}\bar{D}_{11}'\right)^{-1}\bar{C}_1; \\ \check{D}_{22} &=& D_{22} + \bar{D}_{21}\bar{D}_{11}'\left(I - \bar{D}_{11}\bar{D}_{11}'\right)^{-1}\bar{D}_{12}; \\ \check{B}_1 &=& \bar{B}_1\left(I - \bar{D}_{11}'\bar{D}_{11}\right)^{-\frac{1}{2}}; \\ \check{D}_{21} &=& \bar{D}_{21}\left(I - \bar{D}_{11}'\bar{D}_{11}\right)^{-\frac{1}{2}}; \\ \check{C}_1 &=& \left(I - \bar{D}_{11}\bar{D}_{11}'\right)^{-\frac{1}{2}}\bar{C}_1; \\ \check{D}_{12} &=& \left(I - \bar{D}_{11}\bar{D}_{11}'\right)^{-\frac{1}{2}}\bar{D}_{12}; \\ \check{E}_1 &=& \check{D}_{12}'\check{D}_{12}; \quad \check{E}_2 = \check{D}_{21}\check{D}_{21}'; \\ \bar{B}_1 &=& \left[\begin{array}{ccccc} \gamma^{-1}\tilde{B}_1 & \frac{1}{\sqrt{\tilde{\tau}_1}}D_1 & \ldots & \frac{1}{\sqrt{\tilde{\tau}_k}}D_k & D_{k+1} \end{array}\right]; \end{array} \tag{28}$$

$$\bar{C}_1 = \begin{bmatrix} \tilde{C}_1 \\ \sqrt{\tilde{\tau}_1}\tilde{K}_1 \\ \vdots \\ \sqrt{\tilde{\tau}_k}\tilde{K}_k \\ \tilde{K}_{k+1} \end{bmatrix};$$

$$\bar{D}_{11} = \begin{bmatrix} \gamma^{-1}H & 0 & \dots & 0 & J \\ 0 & 0 & \dots & 0 & \sqrt{\tilde{\tau}_1}F_1 \\ \vdots & & & & \\ 0 & 0 & \dots & 0 & \sqrt{\tilde{\tau}_k}F_k \\ \gamma^{-1}L & 0 & \dots & 0 & 0 \end{bmatrix};$$

$$\bar{D}_{12} = \begin{bmatrix} \tilde{D}_{12} \\ \sqrt{\tilde{\tau}_1}\tilde{G}_1 \\ \vdots \\ \sqrt{\tilde{\tau}_k}\tilde{G}_k \\ \tilde{G}_{k+1} \end{bmatrix};$$

$$\bar{D}_{21} = \begin{bmatrix} \gamma^{-1}\tilde{D}_{21} & 0 & \dots & 0 & F_{k+1} \end{bmatrix}. \tag{29}$$

Assumption 2: We will restrict attention to scaling parameters $\tilde{\tau}_1 > 0$, ..., $\tilde{\tau}_k > 0$ and a scaling matrix M such that the following conditions are satisfied:

(i) $\check{E}_1 > 0$.
(ii) $\check{E}_2 > 0$.
(iii) $\bar{D}_{11}\bar{D}_{11}' < I$.

Theorem 1: Suppose that the uncertain system (1), (3) is such that there exist constants $\tau_1 > 0, \dots, \tau_k > 0$ satisfying Assumption 1 such that the Riccati equation (8) has a solution $X > 0$ and let

$$K = -E_1^{-1}(B'_2X + \hat{D}_{12}'\hat{C}_1). \tag{30}$$

Furthermore, suppose there exist constants $\tilde{\tau}_1 > 0, \dots, \tilde{\tau}_{k+1} > 0$ and a scaling matrix M satisfying Assumption 2 such that the Riccati equations (26) and (27) have solutions $\check{X} > 0$ and $\check{Y} > 0$ and such that the spectral radius of their product satisfies $\rho(\check{X}\check{Y}) < 1$. Then the uncertain system (1), (3) is absolutely stabilizable with disturbance attenuation γ via a controller of the form (4) satisfying (5) where

$$\begin{aligned} A_c &= \check{A}_c - B_c\check{D}_{22}C_c \\ \check{A}_c &= \check{A} + \check{B}_2C_c - B_c\check{C}_2 + (\check{B}_1 - B_c\check{D}_{21})\check{B}_1'\check{X} \\ B_c &= (I - \check{Y}\check{X})^{-1}(\check{Y}\check{C}_2' + \check{B}_1\check{D}_{21}')\check{E}_2^{-1} \\ C_c &= -\check{E}_1^{-1}(\check{B}_2'\check{X} + \check{D}_{12}'\check{C}_1). \end{aligned} \tag{31}$$

Proof. It follows via a similar argument to the proof of Theorem 4.1 of [10] that the uncertain system (15), (25) is absolutely stabilizable with disturbance attenuation γ via a controller of the form (4) (not necessarily strict bounded real) if there exist constants $\tilde{\tau}_1 > 0, \dots, \tilde{\tau}_k > 0$ such that the controller (4) solves the H^∞ control problem defined by the system

$$\begin{aligned} \dot{x}(t) &= \tilde{A}x(t) + \bar{B}_1\bar{w}(t) + \tilde{B}_2u(t); \\ \bar{z}(t) &= \bar{C}_1x(t) + \bar{D}_{11}\bar{w}(t) + \bar{D}_{12}u(t); \\ y(t) &= C_2x(t) + \bar{D}_{21}\bar{w}(t) \end{aligned} \tag{32}$$

and the H^∞ norm bound condition

$$\bar{J} \triangleq \sup_{\bar{w}(\cdot)\in\mathbf{L}_2[0,\infty),x(0)=0,x_c(0)=0} \frac{\|\bar{z}(\cdot)\|_2^2}{\|\bar{w}(\cdot)\|_2^2} < 1. \tag{33}$$

Here,

$$\begin{aligned} \bar{w} &= \begin{bmatrix} \gamma w' & \sqrt{\tilde{\tau}_1}\xi_1' & \dots & \sqrt{\tilde{\tau}_k}\xi_k' & \xi_{k+1}' \end{bmatrix}'; \\ \bar{z} &= \begin{bmatrix} z' & \sqrt{\tilde{\tau}_1}\zeta_1' & \dots & \sqrt{\tilde{\tau}_k}\zeta_k' & \zeta_{k+1}' \end{bmatrix}' \end{aligned}$$

and the matrix coefficients $\bar{B}_1$, $\bar{C}_1$, $\bar{D}_{11}$, $\bar{D}_{12}$, $\bar{D}_{21}$ are defined by (28). Furthermore, it follows from standard loop shifting arguments in H^∞ control theory (e.g., see Sections 4.5.1 and 5.5.1 in [13] and Section 17.2 in [14]) that the H^∞ control problem (32), (33) has a solution if and only if the Riccati equations (26) and (27) have solutions $\check{X} > 0$ and $\check{Y} > 0$ and such that the spectral radius of their product satisfies $\rho(\check{X}\check{Y}) < 1$. Furthermore in this case, a controller of the form (4) (but not necessarily positive real) which solves the H^∞ control problem (32), (33) is defined by the equations (31).

We can now conclude that if the conditions of the theorem are satisfied, then the controller (4), (31) is absolutely stabilizing with disturbance attenuation γ for the uncertain system (15), (25). Then, using the arguments given above, it follows that the controller (4), (31) must in fact satisfy condition (5) and is absolutely stabilizing with disturbance attenuation γ for the uncertain system (1), (3). □

IV. ILLUSTRATIVE EXAMPLE

In this section, we consider a simple problem of absolute stabilization with a specified level disturbance attenuation in order to illustrate the theory developed above. We consider a system of the form (1) described by the following state equations

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -0.7 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} -0.1 \\ 0 \end{bmatrix} w(t) \\ &\quad + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t); \\ z(t) &= \begin{bmatrix} 0.1 & 0 \end{bmatrix} x(t) + 0.01u(t); \\ y(t) &= \begin{bmatrix} -0.5 & -1 \end{bmatrix} x(t) + 0.1w(t). \end{aligned} \tag{34}$$

Note that in this example, we are considering the special case in which the original uncertain system contains no uncertainty. The standard H^∞ central controller (e.g., see [14]) for this system (corresponding to $\gamma = 1$) is stable but is not positive real; see the Nyquist plot in Figure 3. Also, the corresponding state feedback gain matrix is $K = [4.2616 \quad -5.3407]$. We now apply the approach outlined in our main result Theorem 1 to this system. For the scaling matrix $M = 0.1I$, we find that the conditions of Theorem 1 are satisfied and we construct the corresponding controller which is described by the following state equations:

$$\begin{aligned} \dot{x}_c(t) &= \begin{bmatrix} -0.7041 & -1.0082 \\ 3.7451 & -2.7946 \end{bmatrix} x_c(t) \\ &\quad + \begin{bmatrix} -0.0166 \\ -0.1710 \end{bmatrix} y(t), \\ u(t) &= \begin{bmatrix} 1.7159 & -2.1182 \end{bmatrix} x_c(t). \end{aligned} \tag{35}$$

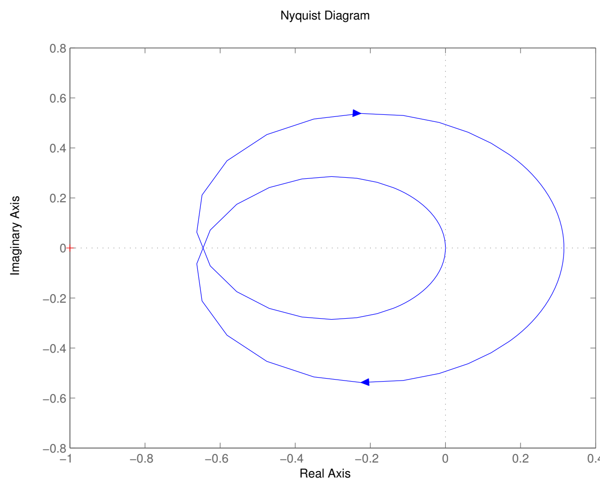

Fig. 3. Nyquist plot of H^∞ central controller.

This system is stable and positive real; see the Nyquist plot Figure 4. Furthermore, when the controller (35) is applied

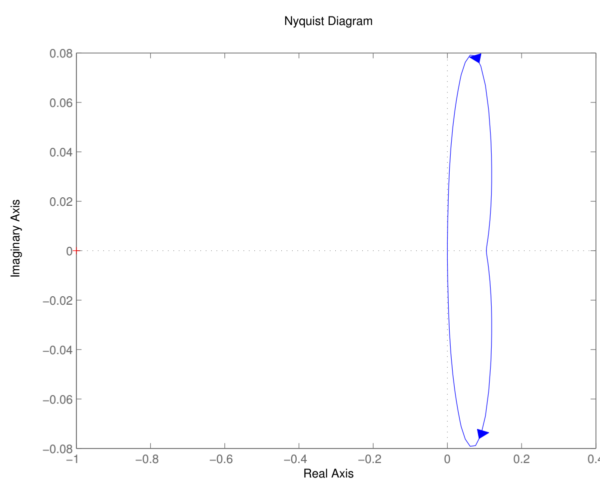

Fig. 4. Nyquist plot of positive real controller.

to the system (34), the resulting closed loop system has a magnitude bode plot shown in Figure 5. From this we can see that the stable positive real controller (35) does indeed solve the H^∞ control problem under consideration.

V. CONCLUSIONS

In this paper we have presented a new approach to the problem of absolute stabilization with a specified level of disturbance attenuation via the use of a stable positive real output feedback controller. The key idea of our approach is to add an additional uncertain parameter to the original uncertain system. For one value of this additional uncertain parameter, the new uncertain system reduces to the original uncertain system and for another value of the additional uncertain parameter, the system reduces to a system in which the controller itself is connected to the disturbance input and

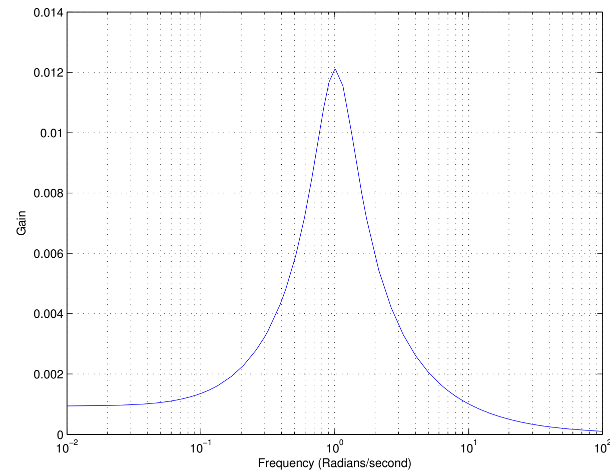

Fig. 5. Closed loop bode plot with stable positive real controller.

error output of the overall system in such a way that this forces the controller to be positive real.

A number of possible areas for future research are motivated by the results of this paper. One would be to reduce the conservatism of the approach by introducing dynamic multipliers to exploit the fact that the additional uncertain parameter is really only required to be constant but unknown. The use of such dynamic multipliers would result in the synthesis of a controller which was of higher order than the original plant. Another area for future investigation concerns the use of numerical optimization techniques to find suitable values for the scaling parameters and scaling matrix. In particular, it may be of interest to recast the problem in terms of LMIs instead of Riccati equations in order to be able to exploit LMI methods in solving these optimization problems.

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