2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Nonlinear control of dissipative PDE systems employing adaptive model reduction

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Abstract— **The problem of feedback control of distributed processes is considered. Typically this problem is addressed through model reduction where finite dimensional approximations to the original infinite dimensional system are derived. The key step in this approach is the computation of basis functions that are subsequently utilized to obtain finite dimensional ODE models using the method of weighted residuals. The most common approach for this task is the Karhunen-Loeve expansion combined with the method of snapshots. However, this approach requires *a priori* availability of a sufficiently large ensemble of PDE solution data, a requirement which is difficult to satisfy. In this work we focus on the recursive computation of eigenfunctions using a relatively small number of snapshots. The empirical eigenfunctions are continuously modified as additional data from the process becomes available. We use ideas from the recursive projection method to keep track of the dominant invariant eigenspace of the covariance matrix which is subsequently utilized to compute the empirical eigenfunctions required for model reduction. This dominant eigenspace is continuously modified with the addition of each snapshot with possible increase or decrease in its dimensionality, while simultaneously the computational burden is kept relatively small. The proposed approach is applied to control temperature in a jacketed tubular reactor where first order chemical reaction is taking place and the closed-loop system is successfully stabilized at an unstable steady-state.**

I. Introduction

Most of the processes relevant to the chemical process industry necessitate the consideration of transport phenomena (fluid flow, heat and mass transfer) often coupled with chemical reactions. Examples range from reactive distillation in petroleum processing to plasma enhanced chemical vapor deposition, etching and metallorganic vapor phase epitaxy (MOVPE) in semiconductor manufacturing. Mathematical descriptions of these transport-reaction processes can be derived from dynamic conservation equations and usually involve highly dissipative (typically parabolic) partial differential equations (PDEs). The problem of feedback control of such processes is nontrivial owing to the spatially distributed description of their dynamics.

The standard approach for feedback control of these systems involves the formulation of a finite dimensional approximation to the original infinite dimensional systems by spatial discretization using Galerkin's method [1]. This method yields a system of ordinary differential equations (ODEs) that describe the dominant dynamic behavior of the PDEs, which can be subsequently utilized to design feedback controllers. However, a drawback of this approach is that for nonlinear PDE systems typically a large number of eigenmodes are required to accurately capture the dynamics of the PDE when using Galerkin's method. Consequently, the dimensionality of the ODE approximation is large which leads to complex controller design and high dimensionality of the resulting controllers.

To overcome this issue, nonlinear model reduction techniques for highly dissipative PDEs were recently employed to formulate approximate low-order ODE models. In these investigations spatial discretization was carried out using the method of weighted residuals, using empirical eigenfunctions as basis functions. These eigenfunctions were generated by the application of Karhunen-Loève expansion (KLE, also known as proper orthogonal decomposition, principle component analysis and method of empirical eigenfunctions, [2], [3], [4]) on an ensemble of solution data of the PDEs for the span of process parameters. The motivation behind this approach was the presence of finite number of dominant spatial modes in the solution of highly dissipative PDEs which govern its long time dynamics, while the remaining infinite dimensional (stable) fast modes relax to these finite dimensional slow dynamics [5], [6]. The principle reason that allows model reduction is that the spatiotemporal behavior of the given PDE system is accounted for in the shape of the empirical eigenfunctions. Previous studies that focus on controller synthesis using KLE include [7]. Model reduction has also been extensively used for efficient solution of process optimization problems for spatially distributed processes [8], [9]. Alternatively, the frameworks of inertial and approximate inertial manifolds has also been employed towards control of distributed process systems [10].

Unfortunately, computation of eigenfunctions either through the direct solution of eigenvalue-eigenfunction problem of the spatial differential operator or empirically using the method of snapshots is a challenging task. Barring certain special cases, such as linear PDEs, analytical solutions to the eigenvalue-eigenvector problem of the spatial differential operator do not exist. Some studies have focussed on the linearization of the nonlinear operator around a steady-state [1], however, such eigenfunctions are applicable only locally in the neighborhood of the steady-state where the linearization takes place. Alternatively, method of snapshots has been extensively utilized to empirically compute the eigenfunctions of nonlinear PDEs. This approach relies on

Support by NSF CAREER Award # CBET 06-44519 and ACS award #PRF 44300-G9 is gratefully acknowledged

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the *a priori* availability of a large ensemble of PDE solution data (snapshots) which excites all of the possible spatial modes in the solution of the PDE system. Unfortunately, generating such an ensemble is not straightforward.

An alternative way is initially compute the eigenfunctions using the available ensemble of snapshots and continuously refine the eigenfunctions online as more snapshots of the process become available. The computations of the empirical eigenfunction requires the solution of the eigenvalue problem of the covariance matrix of the snapshots which might become expensive for online computations. In this article, we describe a procedure to recursively compute the empirical eigenfunctions of a given PDE system. The approach is motivated from the recursive projection method (RPM, [11]) and relies on the computation or approximation of the eigenspace of the covariance matrix corresponding to its significant eigenvalues. This dominant eigenspace is updated recursively as new snapshots from the process are added to the ensemble, simultaneously increasing or decreasing its dimensionality if required. We maintain that as long as the dimensionality of the dominant eigenspace remains small, the computational burden remains small and can be easily performed online. We present an application of the above approach to control temperature in a catalytic rod where an exothermic reaction is taking place. A finite dimensional adaptive controller is designed which stabilizes the process to an unstable steady-state.

II. Mathematical preliminaries

We focus on the problem of feedback control of spatially distributed processes described by highly dissipative PDEs of the following state-space description:

$$\begin{aligned} \frac{\partial x}{\partial t} &= \mathcal{A}(x) + f(x,t)u, \\ y &= h(x) \end{aligned} \tag{1}$$

subject to the mixed-type boundary conditions:

$$q(x, \frac{dx}{d\eta}, \ldots, \frac{d^{n_o-1}x}{d\eta^{n_o-1}}) = 0 \text{ on } \Gamma \tag{2}$$

and the following initial condition

$$x(z,0) = x_0(z). \tag{3}$$

In the above PDE system, $x(z,t) \in \mathbb{R}^n$ denotes the vector of state variables, y denotes the vector of controlled outputs, t is the time, $z = [z_1, z_2, z_3] \in \Omega \subset \mathbb{R}^3$ is the vector of spatial coordinates, Ω is the domain of definition of the process and Γ is its boundary. $\mathcal{A}(x)$ is a dissipative, possibly nonlinear, spatial differential operator which includes higher-order spatial derivatives, $f(t,x)$ and $h(x)$ are nonlinear, vector function which are assumed to be sufficiently smooth with respect to their arguments, $u(t) \in \mathbb{R}^p$ is the vector of design variables which are assumed to be piecewise continuous functions of time, $g(x, \frac{dx}{d\eta}, \ldots, \frac{d^{n_o-1}x}{d\eta^{n_o-1}})$ is a nonlinear vector function which is assumed to be sufficiently smooth (n_o, an even number, is the order of the PDE of Eq.1), $\left.\frac{dx}{d\eta}\right|_\Gamma$ denotes the derivative in the direction perpendicular to the boundary and $x_0(z)$ is a smooth vector function of z. We assume that for a given set of initial and boundary conditions and for each piecewise continuous vector function $d(t) \in \mathbb{R}^p$, the system of Eqs. 1-3 has a unique solution. We also define the inner product and norm in $L_2[\Omega]$, where $L_2[\Omega]$ is the space of square integrable functions defined in Ω, as follows:

$$(\phi_1, \phi_2) = \int_\Omega \phi_1(z)\phi_2(z)dz, \; ||\phi_1||_2 = (\phi_1, \phi_1)^{1/2} \tag{4}$$

where $\phi_1, \phi_2 \in L_2[\Omega]$.

III. Problem formulation and solution methodology

The control problem is formulated as deriving a feedback control law $d(t) = G(x(t))$ such that the closed-loop system is stabilized at a desired set point. The following methodology is adopted to achieve the above task:

1) Generate an ensemble of PDE solution data either experimentally or numerically.
2) Obtain a finite dimensional approximation to the infinite dimensional system of Eq. 1 using KLE and method of snapshots. Design a state feedback controller based on the finite dimensional approximation using the methodology described in section III-D.
3) Recursively modify the eigenfunctions and the finite dimensional approximation as new process measurements become available. This step may involve addition or deletion of eigenfunctions, if required. Adjust the feedback control law taking the modified finite dimensional approximation into account.

The following subsections are intended to describe each of the above steps in detail.

A. Derivation of finite dimensional approximations using method of weighted residuals

We derive finite-dimensional approximations of the infinite-dimensional PDE system of Eq.1 by using the method of weighted residuals. To simplify the notation, we consider the system of Eq.1 with $n = 1$. In principle, $x(z,t)$ can be represented as an infinite series in terms of a complete set of basis functions $\phi_k(z)$. We can obtain an approximation $x_N(z,t)$, by truncating the series expansion of $x(z,t)$ up to order N, as follows:

$$x_N(z,t) = \sum_{k=1}^{N} a_{kN}(t)\phi_k(z) \overset{N\to\infty}{\longrightarrow} x(z,t) = \sum_{k=1}^{\infty} a_k(t)\phi_k(z) \tag{5}$$

where $a_{kN}(t)$, $a_k(t)$ are time-varying coefficients.

Substituting the expansion of Eq.5 into Eq.1, multiplying the PDE with the weighting functions, $\psi_\nu(z)$, and integrating over the entire spatial domain, the following finite-dimensional ODE approximation to the system of Eq. 1 is

obtained,

$$-\sum_{k=1}^{N}\dot{a}_{kN}(\int_{\Omega}\psi_{\nu}(z)\phi_k(z)dz) + \int_{\Omega}\psi_{\nu}(z)\mathcal{A}(\sum_{k=1}^{N}a_{kN}(t)\phi_k(z))dz + \int_{\Omega}\psi_{\nu}(z)f(t,\sum_{k=1}^{N}a_{kN}(t)\phi_k(z))udz = 0 \quad (6)$$

where $a_{kN}(t)$ is the approximation of $a_k(t)$ obtained by an N-th order truncation.

B. Computation of empirical eigenfunctions using KLE

In this section, we use the solution data of the system of Eq.1 to construct global basis functions using KLE. The motivation for studying this approach is provided by the occurrence of dominant spatial patterns in the solution of several dissipative PDEs, which should be accounted for in the shape of the basis functions. This approach will be useful in the context of systems of dissipative PDEs that involve nonlinear spatial differential operators and spatially-varying coefficients that lead to non-symmetric solution profiles. KLE is a procedure used to compute an optimal set of empirical eigenfunctions from an appropriately constructed set of solutions of the PDE system of Eq.1, obtained from high-order discretizations (e.g., using standard packages or process data directly). In this work, the ensemble of solutions is constructed by computing the solutions of the PDE system of Eq.1 for different values of $d(t)$, and different initial conditions. Specifically, we construct a representative ensemble using the following procedure (see also [12], [13] for a detailed discussion on ensemble construction):

- First, we create a set of different initial conditions.
- We then discretize the interval in which each design variable d_m $(m = 1,\ldots,p)$ is constrained to be into m_{d_m} (not necessarily equispaced) subintervals. The discrete values of d_m are denoted by $d_{m,j}$, $j = 1,\ldots,m_{d_m}-1$.
- We also descritize the time-interval into n_{d_m} time subintervals (also not necessarily equispaced).
- Subsequently, we compute a set of time profiles for each of the design variables $d_m(t)$ by assigning values for $d_m(t)$ at different time instants t_j, $d_{m,j}$, and subsequently computing $d_m(t)$ for the entire time interval of process operation using linear interpolation.
- Finally, we compute an ensemble of PDE solution data for all possible combinations of initial conditions and profiles of $d(t)$.

Application of KLE to this ensemble of data provides an orthogonal set of basis functions (known as empirical eigenfunctions) for the representation of the ensemble, as well as a measure of the relative contribution of each basis function to the total energy (mean square fluctuation) of the ensemble. A truncated series representation of the ensemble data in terms of the dominant basis functions has a smaller mean square error than a representation by any other basis of the same dimension [14]. This implies that the projection on the subspace spanned by the empirical eigenfunctions will on average contain the most energy possible compared to all other linear decompositions, for a given number of modes. Therefore, the KLE yields the most efficient way for computing the basis functions (corresponding to the largest empirical eigenvalues) capturing the dominant patterns of the ensemble.

For simplicity of the presentation, we describe the KLE in the context of the system of Eq.1 with $n = 1$ and assume that there is available a sufficiently large set of solutions of this system for different values of d, $\{\bar{v}_\kappa\}$, consisting of K sampled states, $\bar{v}_\kappa(z)$, (which are typically called "snapshots"). The reader may refer to [15], [14], [3], [4] for a detailed presentation and analysis of the KLE. We define the ensemble average of snapshots as $< \bar{v}_\kappa >:= \frac{1}{K}\sum_{\kappa=1}^{K}\bar{v}_\kappa(z)$ (we note that non-uniform sampling of the snapshots and weighted ensemble average can be also considered; see, for example, [12]). Furthermore, the ensemble average of snapshots $< \bar{v}_\kappa >$ is subtracted out from the snapshots i.e.,:

$$v_\kappa = \bar{v}_\kappa - < \bar{v}_\kappa > \quad (7)$$

so that only fluctuations are analyzed. It is useful to analyze fluctuations in variables rather than the actual variables because usually fewer functions are required to fit them [3]. The issue is how to obtain the most typical or characteristic structure (in a sense that will become clear below) $\phi(z)$ among these snapshots $\{v_\kappa\}$. Mathematically, this problem can be posed as the one of obtaining a function $\phi(z)$ that maximizes the following objective function:

$$\text{Maximize } \frac{< (\phi, v_\kappa)^2 >}{(\phi,\phi)} \quad s.t.\ (\phi,\phi) = 1,\ \ \phi \in L^2([\Omega]) \quad (8)$$

which, other words, implies that the projection of $\bar{v}_k$ on the subspace spanned by $\phi(z)$ captures maximum energy. Here, (x, y) denotes complex inner-product defined as:

$$(x, y) = \int_{\Omega}\bar{x}(z)y(z)dz \quad (9)$$

The constraint $(\phi,\phi) = 1$ is imposed to ensure that the function, $\phi(z)$, computed as a solution of the above maximization problem, is unique. An alternative way to express the constrained optimization problem of Eq. 8 is to solve for ϕ such that:

$$\frac{d\bar{L}(\phi+\delta\psi)}{d\delta}(\delta = 0) = 0,\ (\phi,\phi) = 1 \quad (10)$$

where $\bar{L} =< (\phi, v_\kappa)^2 > -\lambda((\phi,\phi)-1)$ is the corresponding Lagrangian functional and δ is a real number.

Using the definitions of inner product and ensemble average, $\frac{d\bar{L}(\phi+\delta\psi)}{d\delta}(\delta = 0)$ can be computed from the

following expression:

$$\frac{d\bar{L}(\phi+\delta\psi)}{d\delta}(\delta=0)= \int_\Omega \left(\left\{ \int_\Omega < v_\kappa(z)v_\kappa(\bar{z}) > \phi(z)dz \right\} - \lambda\phi(\bar{z}) \right) \psi(\bar{z})d\bar{z} \tag{11}$$

Since $\psi(\bar{z})$ is an arbitrary function, the necessary conditions for optimality take the form:

$$\int_\Omega < v_\kappa(z)v_\kappa(\bar{z}) > \phi(z)dz = \lambda\phi(\bar{z}), \;\; (\phi,\phi)=1 \tag{12}$$

Introducing the two-point correlation function:

$$K(z,\bar{z}) =< v_\kappa(z)v_\kappa(\bar{z}) >= \frac{1}{K}\sum_{\kappa=1}^{K} v_\kappa(z)v_\kappa(\bar{z}) \tag{13}$$

and the linear operator:

$$R := \int_\Omega K(z,\bar{z})d\bar{z} \tag{14}$$

the optimality condition of Eq.12 reduces to the following eigenvalue-eigenfunction problem of the integral operator:

$$R\phi = \lambda\phi \Longrightarrow \int_\Omega K(z,\bar{z})\phi(\bar{z})d\bar{z} = \lambda\phi(z) \tag{15}$$

The computation of the solution of the above integral eigenvalue problem is, in general, a very expensive computational task. To circumvent this problem, Sirovich, in 1987, introduced the method of snapshots [3], [4]. The central idea of this technique is to assume that the requisite eigenfunction, $\phi(z)$, can be expressed as a linear combination of the snapshots i.e.:

$$\phi(z) = \sum_k c_k v_k(z) \tag{16}$$

Substituting the above expression for $\phi(z)$ on Eq.15, we obtain the following eigenvalue problem:

$$\int_\Omega \frac{1}{K}\sum_{\kappa=1}^{K} v_\kappa(z)v_\kappa(\bar{z})\sum_{k=1}^{K} c_k v_k(\bar{z})d\bar{z} = \lambda\sum_{k=1}^{K} c_k v_k(z) \tag{17}$$

Defining:

$$B^{\kappa k} := \frac{1}{K}\int_\Omega v_\kappa(\bar{z})v_k(\bar{z})d\bar{z} \tag{18}$$

the eigenvalue problem of Eq.17 can be equivalently written as:

$$Bc = \lambda c \tag{19}$$

The solution of the above eigenvalue problem (which can be obtained by utilizing standard methods from linear algebra) yields the eigenvectors $c = [c_1 \cdots c_K]$ which can be used in Eq.16 to construct the eigenfunction $\phi(z)$. From the structure of the matrix B, it follows that is symmetric and positive semi-definite, and thus, its eigenvalues, λ_κ, $\kappa = 1,\ldots,K$, are real and non-negative. The relative magnitude of the eigenvalues represents a measure of the fraction of the "energy" embedded in the ensemble captured by the corresponding eigenfunctions. Furthermore, the resulting eigenfunctions form an orthogonal set, i.e.:

$$\int_\Omega \phi_i(z)\phi_j(z)dz = 0, \;\; i \neq j \tag{20}$$

Remark 1: The value of m_{d_m} should be determined based on the effect of the design variable d_m on the solution of the system of Eq.1 (if, for example, the effect of the variable d_1 is larger that the effect of the variable d_2, then m_{d_1} should be larger than m_{d_2}).

Remark 2: It should be noted that the kernel in Eq. 12 is not symmetric for cylindrical or spherical geometries [16]. However, the reformulated problem given by Eq. 19 is symmetric irrespective of spatial geometry.

Remark 3: The basis that we compute using KLE is specific to the process under investigation and independent of the specific optimization problem we try to solve. Therefore, the same basis can be used to perform computationally efficient optimizations with respect to different objective functionals associated with the same underlying set of partial differential equations.

Remark 4: Even though it is expected that the use of more basis functions in the series expansion of Eq.5 would improve the accuracy of the computed approximate model of Eq.6, the use of empirical eigenfunctions corresponding to very small eigenvalues should be avoided because such eigenfunctions are contaminated with significant round-off errors.

Remark 5: Iterative methods, such as Krylov subspace methods can be used to reduce the computational cost associated with the computation of the system eigenvalues and eigenfunctions.

C. Recursive update of empirical eigenfunctions

The algorithm presented in the previous subsection requires *a priori* availability of a sufficiently large ensemble of PDE solution data to compute empirical eigenfunctions. However, in practice it is difficult to generate such an ensemble so that all possible spatial modes are contained within the corresponding snapshots. The resulting eigenfunctions, therefore, are representative of the corresponding ensemble only. During simulations situations may arise when the existing eigenfunctions fail to accurately represent the dynamics of the PDE system. One possible solution is to continue augmenting the ensemble of snapshots and subsequently recomputing the eigenfunctions as more information regarding the process becomes available. However, this would require the solution of the eigenvalue-eigenvector problem of Eq. 19, which may become computationally expensive and hence unsuitable for online computations. In this work we propose an algorithm that allows for recursive update of empirical eigenfunctions once new measurements from the process become available. The algorithm is based on the partition of the eigenspace of the covariance matrix into two subspaces; one containing the unstable or marginally stable modes

(denoted as $\mathbb{P}$) and the other containing the rest of highly stable modes (denoted as $\mathbb{Q}$). Such a partition is possible from the fact that the dynamics of highly dissipative PDEs is finite (typically small) dimensional. The orthonormal basis for the subspace $\mathbb{P}$ is recursively maintained upon the arrival of new snapshots, possibly by increasing or decreasing the size of the basis if required, while the orthonormal basis for $\mathbb{Q}$ can be computed from the fact that $\mathbb{Q}$ is the orthonormal complement of $\mathbb{P}$ in $\mathbb{R}^K$. We maintain that the extra work required for the above process is small as long as the dimension of $\mathbb{P}$ is small.

Let C_K denotes the covariance matrix obtained from K snapshots. We assume that out of K possible eigenvectors, m have the corresponding eigenvalues such that $\sum_{i=m+1}^{K} \lambda_i / \sum_{i=1}^{K} \lambda_i \leq \epsilon$, where ϵ is an arbitrarily chosen small parameter. An orthonormal basis for the subspace $\mathbb{P}$ can be obtained as:

$$Z = [\phi_1, \phi_2, \ldots, \phi_m],\ Z \in \mathbb{R}^{K \times m} \tag{21}$$

where $\phi_1, \phi_2, \ldots$ denote the eigenvectors of C_K corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots$. Note that the eigenvectors $\phi_1, \phi_2, \ldots, \phi_m$ define the dominant dynamics of the PDE system of Eq. 1. The projectors P and Q onto subspaces $\mathbb{P}$ and $\mathbb{Q}$ can be computed as

$$P = ZZ^T,\ Q = I - ZZ^T \tag{22}$$

where I denotes the identity matrix of dimension K. Now our task is to obtain the new the basis Z as new snapshots of the process become available. The algorithm outlined in the following computes an approximation to Z without requiring the solution of the eigenvalue-eigenvector problem of the covariance matrix. The algorithm requires the dimensionality of the covariance matrix to remain constant, which we achieve by discarding the oldest snapshot from the ensemble.

As a new snapshot from the process become available, the subspace $\mathbb{P}$ may change in the following three ways. The dimension of the subspace $\mathbb{P}$ may increase, that is eigenvalues corresponding to one or more highly stable eigenvectors from $\mathbb{Q}$ either become marginally stable or unstable. Another possibility is that the eigenvalues corresponding to eigenvectors in Z may decrease and join the cluster of small eigenvalues of C_K. In this case, the basis Z should be updated and its dimension should be simultaneously decreased. A third possibility is that the dimensionality of $\mathbb{P}$ remains unchanged, however the basis Z needs to be updated in order to account for the newly added snapshot. In the following, the above steps are explained in detail.

1) Increasing the size of the basis: In this section we consider the case when an isolated real eigenvalue λ_{m+1} leaves the cluster of highly stable eigenvalues when a new snapshot is added to C_K. We assume the during each step at most one eigenvalue-eigenvector pair joins the unstable subspace $\mathbb{P}$. Define $c_q = QC_KQ$, then the following power iteration produces iterates that asymptotically lie in the dominant eigenspace of c_q

$$q^{(v+1)} = (c_q)^v q^{(0)} \tag{23}$$

provided the initial iterate $q^{(0)}$ has a nonzero component in that direction.

2) Decreasing the size of the basis: As new snapshots are added and old snapshots are eliminated from the ensemble, the dominant eigenspace of C_K continuously changes. Power iteration in the previous section identifies scenarios when one of the eigenvalues become dominant. However, it is likely that during the process some of the dominant eigenvalues decrease in magnitude and join the remaining eigenvalues corresponding to $\mathbb{Q}$. In such cases, it is required to decrease the size of basis Z such that it spans the dominant eigenspace only. To test whether it is required to decrease the size of the basis we introduce the following $m \times m$ matrix

$$H = Z^T C_K Z \in \mathbb{R}^{m \times m}. \tag{24}$$

The eigenvalues of H are a subset of the eigenvalues of C_K and can be computed with little computational effort as long as m remains small. If only $\hat{m}$, with $\hat{m} < m$, eigenvalues of H are dominant, then $\mathrm{span}\{ZV\}$ provides a good approximation to the dominant eigenspace of C_K, where the basis $V \in \mathbb{R}m \times \hat{m}$ is obtained from the eigenvectors of H corresponding to its $\hat{m}$ dominant eigenvalues. Hence, the step

$$Z = \mathrm{orth}(ZV) \tag{25}$$

where $\mathrm{orth}(\cdot)$ denotes Gram-Schmidt orthonormalization, automatically reduces the size of the basis whenever required.

3) Maintaining the accuracy of the basis: The following one step orthogonal power iteration is performed in order to maintain the accuracy of the basis after each step

$$Z = \mathrm{orth}(C_K Z). \tag{26}$$

It can be shown that the orthogonal projections P and Q should satisfy $Q * C_K P = 0$. Hence, the accuracy of the basis can also evaluated by computing the matrix $\mathcal{E} = (I - ZZ^T)C_K(ZZ^T)$. If $\mathcal{E}$ becomes too large than a few additional steps of power iteration can be performed.

D. Finite dimensional controller design - Feedback linearization

In this section we employ feedback linearization to design state feedback controller for the system of Eq. 6. To simplify the development, we represent the system in the following compact form:

$$\begin{gathered} \dot{a} = f(a) + g(a)u = f(a) + \textstyle\sum_{i=1}^{m} g_m(a)u_m \\ y_m = \mathcal{S}a \\ y_{ci} = h_i(a),\ i = 1, \ldots, m \end{gathered} \tag{27}$$

where $f(a) = \int_\Omega \psi_\nu(z)\mathcal{A}(\sum_{k=1}^{N} a_{kN}(t)\phi_k(z))dz$, $g(a) = \int_\Omega \psi_\nu(z) f(t, \sum_{k=1}^{N} a_{kN}(t)\phi_k(z))dz$, y_m is the measured output

vector, $\mathcal{S}$ denotes the measurement sensor shape function, and y_{ci} is the i^{th} controlled output. We assume that the relative degree r_i of the system of Eq. 27 is well defined and less than p.

We use feedback linearization to design state feedback controllers which have the following general form:

$$u = p(a) + Q(a)v \tag{28}$$

where $p(x)$ is a smooth vector function, $Q(a)$ is a smooth matrix, and $v \in \mathbb{R}^m$ is the constant reference input vector. Based on relative degree of the system, we assign the following closed-loop behavior to the controlled outputs y_{ci}:

$$\sum_{i=1}^{m}\sum_{k=0}^{r_i}\beta_{ik}\frac{d^k y_{ci}}{dt^k} = v. \tag{29}$$

Combining Eq. 27 with Eq. 29 and assuming that the characteristic matrix $C_0(a)$ of the system of Eq. 27 is invertible, we can derive the state feedback controller of the following form that guarantee output behavior as described by Eq. 29:

$$u = \{[\beta_{1r_1}\dots\beta_{mr_m}]C_0(a)\}^{-1}\{v - \sum_{i=1}^{m}\sum_{k=0}^{r_i}\beta_{ik}L_f^k h_i(a)\}. \tag{30}$$

IV. Application to diffusion-reaction process

In this section we apply the proposed finite dimensional adaptive control methodology to a standard diffusion reaction process with nonlinearities [8]. Specifically, we consider an elementary exothermic reaction $\mathrm{A} \rightarrow \mathrm{B}$ taking place in a catalytic rod. The temperature of the rod is adjusted by means of an actuator located along the length of the rod. Assuming that the reactant A is present in excess, the spatial profile of the dimensionless temperature inside the rod is described by the following parabolic PDE:

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial z^2} + \beta_T(\mathrm{e}^{-\gamma/(1+x)} - \mathrm{e}^{-\gamma}) + \beta_U(b(z)u(t) - x) \tag{31}$$

subject to the following boundary conditions:

$$x(0,t) = 0,\ x(\pi,t) = 0,\ x(z,0) = x_0(z) \tag{32}$$

where x denotes the dimensionless rod temperature, z is the spatial coordinate along the axis of the rod, β_T denotes the dimensionless heat of reaction, γ denotes the dimensionless activation energy, β_U denotes the dimensionless heat transfer coefficient, $u(t)$ denotes the magnitude of actuation, and $b(z)$ accounts for the spatial profile of the actuator. Two different spatial distributions for the actuator were investigated. Initially $b(z) = H(z - 0.3\pi) - H(z - 0.7\pi)$, where $H(\cdot)$ denotes the standard Heaviside function, was considered which described a spatially distributed actuator centered at the midpoint of the rod. A point actuator located at the center of the rod was also considered such that $b(z) = \delta(z - 0.5\pi)$, where $\delta(\cdot)$ denotes the Dirac function. The nominal values of the parameters are: $\beta_T = 16$, $\gamma = 2$, and $\beta_U = 2$. Fig. 1 shows the evolution of the PDE for $u(t) = 0$ from an initial condition that is very close to the steady-state $x(z,t) = 0$. It can be seen that the system evolves away from the the above steady-state to another steady-state characterized by a non-uniform distribution of temperature across the rod with the maximum reaching at $z = \pi/2$. Hence, we conclude that the steady-state $x(z,t) = 0$ is an unstable steady-state. Our task is to design a feedback controller that stabilizes the rod temperature to the above open-loop unstable steady-state.

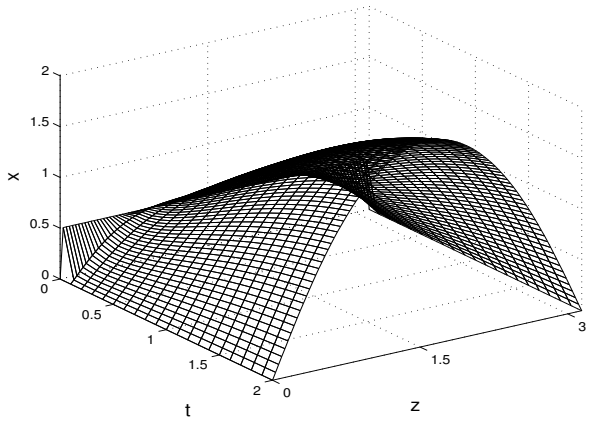

Fig. 1. Profile of the state of the PDE of Eq. 1 with $u(t) = 0$.

A. Simulation results

In order to obtain a finite dimensional approximation of the infinite dimensional system of Eq. 31, initially an ensemble of 100 snapshots was generated by keeping $u(t) = 0$. Note that it is not required to perform an exhaustive sampling of the state-space of the PDE by evolving the system from a number of different initial conditions and magnitudes of actuation during the ensemble generation phase. Application of KLE along with the method of snapshots resulted in a single dominant eigenfunction which captured more then 98% of the energy embedded in the ensemble. During closed-loop simulations, it was assumed that process measurements are available every $t_s = 0.1$ seconds. The finite dimensional process model and the control law were adjusted after each new process measurement.

1) Spatially distributed actuation: Fig. 2 shows the spatiotemporal profile of the rod temperature under closed loop operation with spatially distributed actuation $b(z) = H(z - 0.3\pi) - H(z - 0.7\pi)$. We observe that the controller is successfully able to stabilize the process at open-loop unstable steady-state. Fig. 3a shows the corresponding profile of the control action $u(t)$. Fig. 3b presents the variation in the number of empirical eigenfunctions employed to obtain the reduced-order process model. As mentioned before, originally a single eigenfunction captured most of the energy of the ensemble of snapshots. However, as more process measurements from the closed-loop operation were included in the ensemble while simultaneously old snapshots were removed, a new eigenfunction joined the dominant eigenspace at $t =$. Consequently, the dimensionality of the reduced-order ODE model was increased to $m = 2$ from $m = 1$. In addition, the eigenfunctions were being constantly modified to account for continuously changing ensemble of snapshots. This has been demonstrated in Fig. 4 and 5 which shows the temporal profiles of the two eigenfunctions.

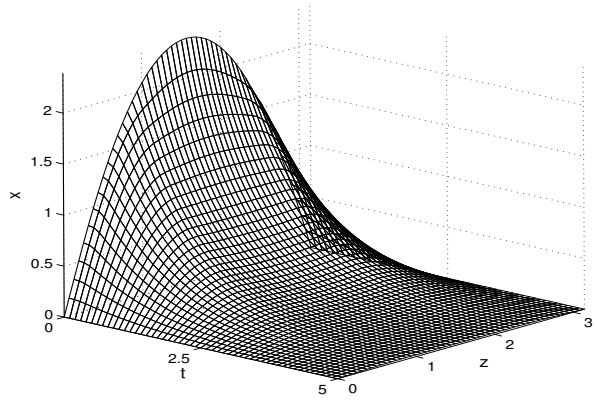

Fig. 2. Closed-loop temperature profile for distributed control action.

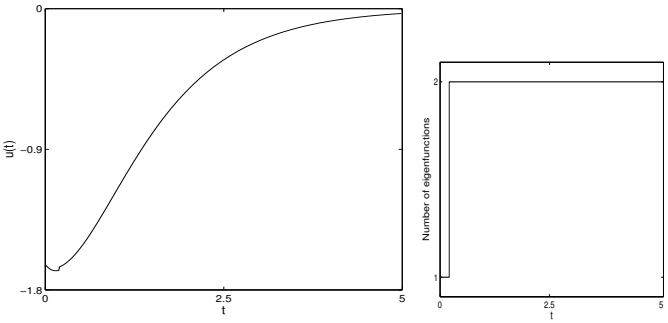

Fig. 3. (a) Profile of the magnitude of actuation for distributed control action, (b) Number of empirical eigenfunctions (also the order of the reduced ODE model) as a function of time for distributed control action.

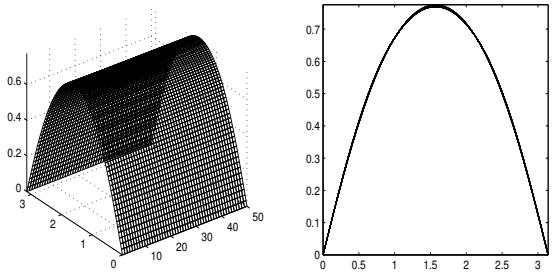

Fig. 4. Temporal profile of the dominant eigenfunction with spatially distributed control actuation.

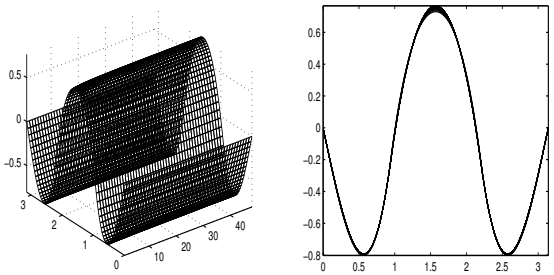

Fig. 5. Temporal profile of the second eigenfunction with spatially distributed control actuation.

2) Point actuation: Fig. 6 shows the spatiotemporal profile of the rod temperature under closed loop operation with point actuation $b(z) = \delta(z - 0.5\pi)$. We observe that the controller is successfully able to stabilize the process at open-loop unstable steady-state. Fig. 7a shows the corresponding profile of the control action $u(t)$. Fig. 7b presents the variation in the number of empirical eigenfunctions employed to obtain the reduced-order process model. Similarly to the previous case, as more process measurements from the closed-loop operation were included in the ensemble while simultaneously old snapshots were removed, a new eigenfunction joined the dominant eigenspace at $t =$. Consequently, the dimensionality of the reduced-order ODE model was increased to $m = 2$ from $m = 1$. In addition, the eigenfunctions were being constantly modified to account for continuously changing ensemble of snapshots. This has been demonstrated in Fig. 8 and 9 which show the temporal profiles of the two eigenfunctions.

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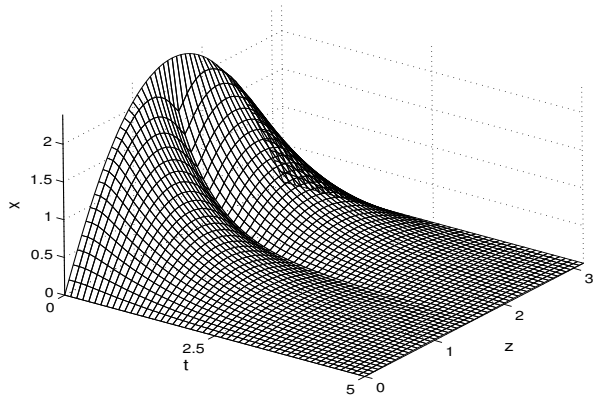

Fig. 6. Closed-loop temperature profile for point control action.

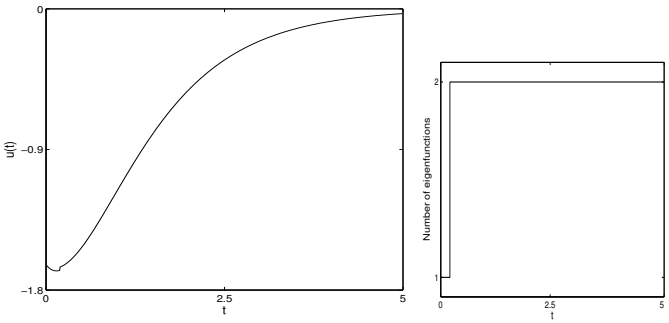

Fig. 7. (a) Profile of the magnitude of actuation for point control action, (b) Number of empirical eigenfunctions (also the order of the reduced ODE model) as a function of time for point control action.

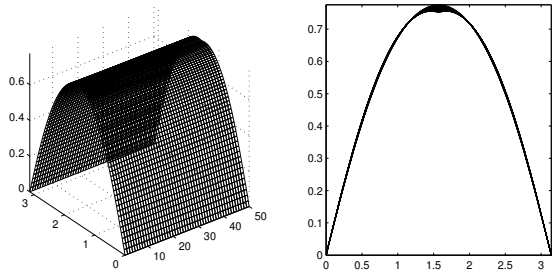

Fig. 8. Temporal profile of the dominant eigenfunction with localized control actuation.

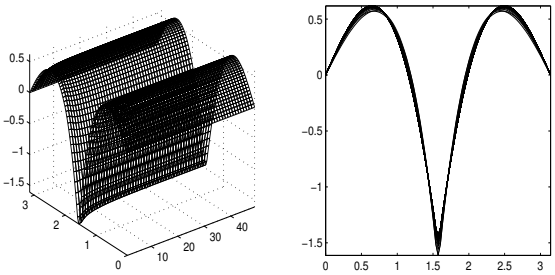

Fig. 9. Temporal profile of the second eigenfunction with localized control actuation.