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Finite-Time Stability for Time-Varying Nonlinear Dynamical Systems

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Abstract— **Finite-time stability involves dynamical systems whose trajectories converge to an equilibrium state in finite time. Since finite-time convergence implies non-uniqueness of system solutions in backward time, such systems possess non-Lipschitzian dynamics. In this paper, we address finite-time and uniform finite-time stability of time-varying systems. Specifically, we provide Lyapunov and converse Lyapunov conditions for finite-time stability of a time-varying system. Furthermore, we show that finite-time stability leads to uniqueness of solutions in forward time. In addition, we establish necessary and sufficient conditions for continuity of the settling-time function of a nonlinear time-varying system.**

I. INTRODUCTION

The notions of asymptotic and exponential stability in dynamical systems theory imply convergence of the system trajectories to a Lyapunov stable equilibrium state over an infinite horizon. In many applications, however, it is desirable that a dynamical system possesses the property that trajectories that converge to a Lyapunov stable equilibrium state must do so in finite time rather than merely asymptotically. In the case when the dynamics of a time-varying system are Lipschitz continuous in the state and piecewise continuous in time, the dynamical system possesses a unique solution in forward and backward times for a given pair of initial time and state [1]. However, if a dynamical system possesses trajectories that converge to an equilibrium point in finite time, then clearly this system has multiple solutions starting at the equilibrium point with time running backwards. Hence, such systems cannot be Lipschitz continuous at the equilibrium point.

The absence of Lipschitz continuity is only a necessary condition for non-uniqueness of the system trajectories, and uniqueness of solutions in forward time can be preserved in the case of finite-time convergence. Sufficient conditions that ensure uniqueness of solutions in forward time in the absence of Lipschitz continuity are given in [2–5]. In addition, it is shown in [6, Theorem 4.3, p. 59] that uniqueness of solutions in forward time along with continuity of the system dynamics ensure that the system solutions are continuous functions of the system initial conditions even when the dynamics are not Lipschitz continuous.

In [7], a rigorous foundation for the theory of finite-time stability for autonomous nonlinear systems was developed. In this paper, we extend the results of [7] to address finite-time and uniform finite-time stability for time-varying systems. In addition, we establish necessary and sufficient conditions for continuity of the settling-time function, that is, the time at which a system trajectory reaches an equilibrium state. Finally, using the scalar comparison principle for time-varying systems [1], [2], we provide Lyapunov and converse Lyapunov results for finite-time stability of time-varying systems in terms of scalar differential inequalities.

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II. MATHEMATICAL PRELIMINARIES

In this section, we introduce notation, some definitions, and a key result that is necessary for developing finite-time stability for time-varying systems. Specifically, let $\mathbb{R}$ denote the set of real numbers, $\overline{\mathbb{R}}_+$ denote the set of nonnegative real numbers, $\overline{\mathbb{Z}}_+$ denote the set of nonnegative integers, $\mathbb{R}^n$ denote the set of $n \times 1$ column vectors, and $(\cdot)^{\mathrm{T}}$ denote transpose. Furthermore, we write $\|\cdot\|$ for an arbitrary spatial vector norm in $\mathbb{R}^n$, $\mathcal{B}_\varepsilon(\alpha)$, $\alpha \in \mathbb{R}^n, \varepsilon > 0$, for the open ball *centered* at α with *radius* ε, and $V'(x)$ for the Fréchet derivative of V at x.

In this paper, we consider nonlinear time-varying dynamical systems of the form

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad t \in \mathcal{I}_{x_0,t_0}, \tag{1}$$

where $x(t) \in \mathcal{D}$, $t \in \mathcal{I}_{x_0,t_0}$, is the system state vector, $\mathcal{I}_{x_0,t_0} \triangleq [t_0, \tau_{x_0,t_0})$, $t_0 < \tau_{x_0,t_0} \leq \infty$, is the maximal interval of existence of a solution $x(t)$ of (1), $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set with $0 \in \mathcal{D}$, $f : [0,\infty) \times \mathcal{D} \to \mathbb{R}^n$ is such that $f(\cdot,\cdot)$ is jointly continuous in t and x, and for every $t \in [0,\infty)$, $f(t, 0) = 0$. We assume that (1) possesses unique solutions in forward time for all initial conditions except possibly the origin in the following sense. For every $x \in \mathcal{D}\backslash\{0\}$ and $t_0 \in \mathbb{R}$, there exists $\tau_{x,t_0} > 0$ such that, if $x_1 : [t_0, \tau_1) \to \mathcal{D}$ and $x_2 : [t_0, \tau_2) \to \mathcal{D}$ are two solutions of (1) with $x_1(t_0) = x_2(t_0) = x$, then $\tau_{x,t_0} \leq \min\{\tau_1, \tau_2\}$ and $x_1(t) = x_2(t)$ for all $t \in [t_0, \tau_{x,t_0})$. Sufficient conditions for forward uniqueness in the absence of Lipschitz continuity can be found in [2–5]. Without loss of generality, we assume that for each x and $t_0 \in \mathbb{R}$, τ_{x,t_0} is chosen to be the largest such number in $\overline{\mathbb{R}}_+$. In this case, we denote the *trajectory* or *solution curve* of (1) on $[t_0, \tau_{x,t_0})$ satisfying the *consistency property* $s(t_0, t_0, x) = x$ and the *semi-group property* $s(t_2, t_1, s(t_1, t_0, x)) = s(t_2, t_0, x)$ for every $x \in \mathcal{D}$, $t_0 \in \mathbb{R}$, and $t_1 \leq t_2 \in [t_0, \tau_{x,t_0})$ by $s(\cdot, t_0, x)$ or $s^{t_0,x}(\cdot)$.

The next result presents the classical comparison principle for nonlinear time-varying dynamical systems.

Theorem 2.1 ([1]): Consider the nonlinear dynamical system (1) with $n = 1$ and let $x(t)$, $t \geq t_0$, be the solution to (1) with $x(t_0) = x_0$. Assume that there exists a continuously differentiable function $V : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x) \leq w(t, V(t,x)), \quad (t,x) \in [0,\infty) \times \mathcal{D}, \tag{2}$$

where $w(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ for all $t \in [t_0,\infty)$, $w(\cdot, x) : [t_0,\infty) \to \mathbb{R}$ is continuous on $[t_0,\infty)$ for all $x \in \mathbb{R}$, and

$$\dot{z}(t) = w(t, z(t)), \quad z(t_0) = z_0, \quad t \in \mathcal{I}_{z_0,t_0}, \tag{3}$$

has a unique solution $z(t)$, $t \in \mathcal{I}_{z_0,t_0}$. If $[t_0, t_0+\tau] \subseteq \mathcal{I}_{x_0,t_0} \cap \mathcal{I}_{z_0,t_0}$ and $V(t_0, x_0) \leq z_0$, $z_0 \in \mathbb{R}$, then $V(t, x(t)) \leq z(t)$, $t \in [t_0, t_0 + \tau]$.

III. Finite-Time Stability for Time-Varying Nonlinear Systems

In this section we develop the notion of finite-time stability for time-varying nonlinear dynamical systems. The following definition generalizes Definition 2.2 of [7] to time-varying systems.

Definition 3.1: Consider the nonlinear dynamical system (1). The zero solution $x(t) \equiv 0$ to (1) is *finite-time stable* if there exist an open neighborhood $\mathcal{N} \subseteq \mathcal{D}$ of the origin and a function $T : [0, \infty) \times \mathcal{N}\backslash\{0\} \to [0, \infty)$, called the *settling-time function*, such that the following statements hold:

i) *Finite-time convergence*. For every $t_0 \in [0, \infty)$ and $x_0 \in \mathcal{N}\backslash\{0\}$, $s(t, t_0, x_0)$ is defined on $[t_0, T(t_0, x_0))$, $s(t, t_0, x_0) \in \mathcal{N}\backslash\{0\}$ for all $t \in [t_0, T(t_0, x_0))$, and $\lim_{t \to T(t_0,x_0)} s(t, t_0, x_0) = 0$.

ii) *Lyapunov stability*. For every $\varepsilon > 0$ and $t_0 \in [0, \infty)$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{N}$ and for every $x_0 \in \mathcal{B}_\delta(0)\backslash\{0\}$, $s(t, t_0, x_0) \in \mathcal{B}_\varepsilon(0)$ for all $t \in [t_0, T(t_0, x_0))$.

The zero solution $x(t) \equiv 0$ to (1) is *uniformly finite-time stable* if *i)* holds and

iii) *Uniform Lyapunov stability*. For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{N}$ and for every $x_0 \in \mathcal{B}_\delta(0)\backslash\{0\}$, $s(t, t_0, x_0) \in \mathcal{B}_\varepsilon(0)$ for all $t \in [t_0, T(t_0, x_0))$ and every $t_0 \in [0, \infty)$.

Finally, the zero solution $x(t) \equiv 0$ to (1) is *globally finite-time stable* (respectively, *globally uniformly finite-time stable*) if it is finite-time stable (respectively, uniformly finite-time stable) with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$.

Remark 3.1: Note that the definition of uniform finite-time stability differs from that of finite-time stability only in that it requires Lyapunov stability to be uniform with respect to the initial time. Since the classical definition of uniform asymptotic stability requires uniform Lyapunov stability as well as uniform attractivity with respect to the initial time, a more mainsteam definition for uniform finite-time stability would involve uniform Lyapunov stability with uniform finite-time convergence.

Note that, for all $t_0 \in [0, \infty)$ and $x \in \mathcal{N}\backslash\{0\}$, $T(t_0, x) \geq t_0$ is the *absolute* time at which the trajectory starting at (t_0, x) reaches the origin. Hence, the time required for this trajectory to reach the origin is given by $T(t_0, x) - t_0$. Furthermore, if the zero solution $x(t) \equiv 0$ to (1) is (uniformly) finite-time stable, then it is (uniformly) asymptotically stable, and hence, (uniform) finite-time stability is a stronger notion than (uniform) asymptotic stability. Next, we show that if the zero solution $x(t) \equiv 0$ to (1) is finite-time stable, then (1) possesses a unique solution $s(\cdot, t_0, x_0)$ for every initial condition in an open neighborhood of the origin, including the origin, and $s(t, t_0, x_0) = 0$ for all $t \geq T(t_0, x_0), t_0 \in [0, \infty), x_0 \in \mathcal{N}$, where $T(t_0, 0) \triangleq t_0$.

Proposition 3.1: Consider the nonlinear dynamical system (1). Assume that the zero solution $x(t) \equiv 0$ to (1) is finite-time stable and let $\mathcal{N} \subseteq \mathcal{D}$ and $T : [0, \infty) \times \mathcal{N}\backslash\{0\} \to (0, \infty)$ be defined as in Definition 3.1. Then, for every $t_0 \in [0, \infty)$ and $x_0 \in \mathcal{N}$, there exists a unique solution $s(t, t_0, x_0)$, $t \geq t_0$, to (1) such that $s(t, t_0, x_0) \in \mathcal{N}$, $t \in [t_0, T(t_0, x_0))$ and $s(t, t_0, x_0) = 0$ for all $t \geq T(t_0, x_0)$, where $T(t_0, 0) \triangleq t_0$.

Proof. It follows from Lyapunov stability of the zero solution that $x(t) \equiv 0$ is the unique solution of (1) satisfying $x(t_0) = 0$ for all $t_0 \in [0, \infty)$. Thus, $s(t, t_0, 0) \equiv 0$ for all $t_0 \in [0, \infty)$. Next, let $t_0 \in [0, \infty)$ and $x_0 \in \mathcal{N}\backslash\{0\}$, and define

$$x(t) \triangleq \begin{cases} s(t, t_0, x_0), & 0 \leq t < T(t_0, x_0), \\ 0, & t \geq T(t_0, x_0). \end{cases} \tag{4}$$

Note that by construction, $x(\cdot)$ is continuously differentiable on $\mathbb{R}_+\backslash\{T(t_0, x_0)\}$ and satisfies (1) on $\mathbb{R}_+\backslash\{T(t_0, x_0)\}$. Furthermore, since $f(\cdot, \cdot)$ is jointly continuous,

$$\begin{aligned} \lim_{t \to T(t_0,x_0)^-} \dot{x}(t) &= \lim_{t \to T(t_0,x_0)^-} f(t, x(t)) \\ &= 0 \\ &= \lim_{t \to T(t_0,x_0)^+} \dot{x}(t), \end{aligned} \tag{5}$$

and hence, $x(\cdot)$ is continuously differentiable at $T(t_0, x_0)$ and $x(\cdot)$ satisfies (1). Hence, $x(\cdot)$ is a solution of (1) on $\mathbb{R}_+ \times \mathcal{N}$.

To show uniqueness, assume $y(\cdot)$ is a solution of (1) on $\mathbb{R}_+ \times \mathcal{N}$ satisfying $y(t_0) = x_0$. In this case, $x(t) = y(t)$ for all $t \in [t_0, T(t_0, x_0))$ by the uniqueness assumption in Section II. In addition, by continuity, $x(t) = y(t)$ at $t = T(t_0, x_0)$, and hence $x(t) = y(t)$, $t \in [t_0, T(t_0, x_0)]$, which implies that $y(T(t_0, x_0)) = 0$. Now, Lyapunov stability implies that $y(t) = 0$ for $t > T(t_0, x_0)$, which proves uniqueness of $x(\cdot)$. This proves the result. □

It follows from Proposition 3.1 that if the zero solution $x(t) \equiv 0$ to (1) is finite-time stable, then it defines a continuous global *semi-flow* on $\mathcal{N}$; that is, $s : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{N} \to \mathcal{N}$ is jointly continuous and satisfies the consistency property and the semi-group property. Hence,

$$s(t_0, t_0, x) = x, \tag{6}$$

$$s(t_2, t_1, s(t_1, t_0, x)) = s(t_2, t_0, x), \tag{7}$$

for every $x \in \mathcal{N}$, $t_0 \in [0, \infty)$, and $0 \leq t_1 \leq t_2 \in \mathbb{R}_+$. Furthermore, $s(\cdot, \cdot, \cdot)$ satisfies

$$s(T(t_0, x) + t_1, t_0, x) = 0, \tag{8}$$

for all $x \in \mathcal{N}, t_0 \in [0, \infty)$, and $t_1 \geq 0$. Finally, it is easy to see from Definition 3.1 that

$$T(t_0, x) = \inf\{t_1 \in \overline{\mathbb{R}}_+ : s(t_1, t_0, x) = 0\}, \tag{9}$$

for all $x \in \mathcal{N}, t_0 \in [0, \infty)$, and $t_1 \geq t_0$.

In general, finite-time stability of the zero solution $x(t) \equiv 0$ does not imply that the settling-time function $T(\cdot, \cdot)$ is jointly continuous [7]. However, continuity of $T(\cdot, \cdot)$ plays a key role in Lyapunov stability analysis as we will see in the next section. The following proposition shows that the settling-time function $T(\cdot, \cdot)$ of a finite-time stable system is jointly continuous on $\mathbb{R}_+ \times \mathcal{N}$ if and only if it is jointly continuous at $(\cdot, 0)$.

Proposition 3.2: Consider the nonlinear dynamical system (1). Assume that the zero solution $x(t) \equiv 0$ to (1) is finite-time stable. Let $\mathcal{N} \subseteq \mathcal{D}$ be as in Definition 3.1 and let $T : [0, \infty) \times \mathcal{N}\backslash\{0\} \to (0, \infty)$ be the settling-time function. Then the following statements hold:

i) If $t_0 \in [0, \infty), t_1 \geq t_0$, and $x \in \mathcal{N}$, then

$$T(t_1, s(t_1, t_0, x)) = \max\{T(t_0, x), t_1\}. \tag{10}$$

ii) $T(\cdot, \cdot)$ is jointly continuous on $\mathbb{R}_+ \times \mathcal{N}$ if and only if $T(\cdot, \cdot)$ is jointly continuous at $(t, 0), t \geq 0$.

Proof. *i)* It follows from (9) that $T(t_1, s(t_1, t_0, x)) = \inf\{t_2 \in \mathbb{R}_+ : s(t_2, t_1, s(t_1, t_0, x)) = 0\}$. Now, for $t_0 \le t_1 \le T(t_0, x)$, (7) and (9) imply that $T(t_1, s(t_1, t_0, x)) = \inf\{t_2 \in \mathbb{R}_+ : s(t_2, t_0, x) = 0\} = T(t_0, x)$. Alternatively, for $t_0 \le T(t_0, x) \le t_1$, $T(t_1, s(t_1, t_0, x)) = t_1$, which proves (10).

ii) Necessity is immediate. To prove sufficiency, suppose that $T(\cdot,\cdot)$ is jointly continuous at $(t, 0), t \ge 0$. Let $t \in [0,\infty)$ and $x \in \mathcal{N}$, and consider the sequences $\{t_n\}_{n=0}^{\infty} \subset [0,\infty)$ converging to t and $\{x_n\}_{n=0}^{\infty} \subset \mathcal{N}$ converging to x. Let $\tau^- = \liminf_{n\to\infty} T(t_n, x_n)$ and $\tau^+ = \limsup_{n\to\infty} T(t_n, x_n)$. Note that $\tau^-, \tau^+ \in \mathbb{R}_+$ and

$$\tau^- \le \tau^+. \tag{11}$$

Next, let $\{t_m^+\}_{m=0}^{\infty} \subset [0,\infty)$ be a subsequence of $\{t_n\}_{n=0}^{\infty}$ and $\{x_m^+\}_{m=0}^{\infty} \subset \mathcal{N}$ be a subsequence of $\{x_n\}_{n=0}^{\infty}$ such that $T(t_m^+, x_m^+) \to \tau^+$ as $m \to \infty$. The sequence $\{T(t,x), t_m^+, x_m^+\}_{m=0}^{\infty}$ converges in $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{N}$ to $(T(t,x), t, x)$. It follows from (8) and continuity of solutions on initial conditions that $s(T(t,x), t_m^+, x_m^+) \to s(T(t,x), t, x) = 0$ as $m \to \infty$. Thus, since $T(t, 0)$ is continuous for all $t \ge 0$, it follows that

$$T(T(t,x), s(T(t,x), t_m^+, x_m^+)) \to T(t,x), \quad m \to \infty. \tag{12}$$

Now, with $t_1 = T(t,x)$, $t_0 = t_m^+$, and $x = x_m^+$, it follows from (10) and (12) that $T(T(t,x), s(T(t,x), t_m^+, x_m^+)) = \max\{T(t_m^+, x_m^+), T(t,x)\} \to T(t,x), m \to \infty$. Thus, $\max\{\tau^+, T(t,x)\} = T(t,x)$, which implies

$$\tau^+ \le T(t,x). \tag{13}$$

Finally, let $\{t_m^-\}_{m=0}^{\infty} \subset [0,\infty)$ be a subsequence of $\{t_n\}_{n=0}^{\infty}$ and $\{x_m^-\}_{m=0}^{\infty} \subset \mathcal{N}$ be a subsequence of $\{x_n\}_{n=0}^{\infty}$ such that $T(t_m^-, x_m^-) \to \tau^-$ as $m \to \infty$. It follows from (11) and (13) that $\tau^- \in \mathbb{R}_+$, and hence, the sequence $\{T(t_m^-, x_m^-), t_m^-, x_m^-\}_{m=0}^{\infty}$ converges to $\{\tau^-, t, x\}$. Since $s(\cdot,\cdot,\cdot)$ is jointly continuous, it follows that $s(T(t_m^-, x_m^-), t_m^-, x_m^-) \to s(\tau^-, t, x)$ as $m \to \infty$. Now, (8) implies that $s(T(t_m^-, x_m^-), t_m^-, x_m^-) = 0$ for each m. Hence, $s(\tau^-, t, x) = 0$ and, by (9),

$$T(t,x) \le \tau^-. \tag{14}$$

Now, it follows from (11), (13), and (14) that $\tau^- = T(t,x) = \tau^+$, and hence, $T(t_n, x_n) \to T(t,x)$ as $n \to \infty$, which proves that $T(\cdot,\cdot)$ is jointly continuous on $\mathbb{R}_+ \times \mathcal{N}$. □

Note that it follows from Proposition 3.2 that $T(\cdot, x)$ is continuous in t on $\mathbb{R}_+$ for all $x \in \mathcal{N}$ and $T(t, \cdot)$ is continuous in x on $\mathcal{N}$ for all $t \in [0,\infty)$ if $T(\cdot,\cdot)$ is jointly continuous at $(t, 0), t \ge 0$.

Example 3.1: Consider the scalar nonlinear time-varying dynamical system

$$\dot{y}(t) = -k(t)\text{sign}(y(t))|y(t)|^{\lambda}, \quad y(t_0) = y_0, \quad t \ge t_0, \tag{15}$$

where $y_0 \in \mathbb{R}$, $\text{sign}(y) \triangleq \frac{y}{|y|}$, $y \ne 0$, $\text{sign}(0) \triangleq 0$, $k(\cdot)$ is continuous on $\mathbb{R}$ and $k(t) > 0$ for almost all $t \in [t_0,\infty)$, and $\lambda > 0$. The right-hand side of (15) is continuous everywhere and locally Lipschitz continuous everywhere except the origin. Hence, every initial condition $y_0 \in \mathbb{R}\backslash\{0\}$ has a unique solution in forward time on a sufficiently small time interval. Let $\lambda \in (0,1)$. In this case, for every $y_0 \in \mathbb{R}$, since $k(\cdot)$ is continuous on $\mathbb{R}$ and $k(t) > 0$ for almost all $t \in [t_0,\infty)$, there exists $t_1 \ge t_0$ such that

$$\int_{t_0}^{t_1} k(\tau)\mathrm{d}\tau = \frac{|y_0|^{1-\lambda}}{1-\lambda}, \tag{16}$$

and the solution to (15) is given by

$$s(t, t_0, y_0) = \begin{cases} \text{sign}(y_0)[|y_0|^{1-\alpha} - (1-\alpha)\int_{t_0}^{t} k(\tau)\mathrm{d}\tau]^{\frac{1}{1-\alpha}}, & \\ & t_0 \le t < t_1, y_0 \ne 0, \\ 0, & t \ge t_1, y_0 \ne 0, \\ 0, & t \ge t_0, y_0 = 0. \end{cases} \tag{17}$$

Note that if $K : [0,\infty) \to \mathbb{R}$ is a continuously differentiable function such that $\dot{K}(t) = k(t)$ for all $t \in [t_0, t_1]$, then (16) implies $K(t_1) - K(t_0) = \frac{|y_0|^{1-\lambda}}{1-\lambda}$, which further implies that $t_1 = K^{-1}\left(K(t_0) + \frac{|y_0|^{1-\lambda}}{1-\lambda}\right)$. Without loss of generality, we can choose $K(\cdot)$ such that $K(t_0) = 0$ so that

$$t_1 = K^{-1}\left(\frac{|y_0|^{1-\lambda}}{1-\lambda}\right). \tag{18}$$

Note that $t_1 = T(t_0, y_0)$ is the settling-time function and t_1 is unique since $k(t) > 0$ for almost all $t \in [t_0,\infty)$. Lyapunov stability of (15) follows by considering the Lyapunov function $V(y) = y^2$. △

IV. Lyapunov and Converse Lyapunov Theory for Finite-Time Stability

In this section we present necessary and sufficient conditions for finite-time stability using a Lyapunov function involving a scalar differential inequality. For the following result define

$$\dot{V}(t,x) \triangleq \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x), \tag{19}$$

for a given continuously differentiable function $V : [0,\infty) \times \mathcal{D} \to \mathbb{R}$.

Theorem 4.1: Consider the nonlinear dynamical system (1). Then the following statements hold:

i) If there exist a continuously differentiable function $V : [0,\infty) \times \mathcal{D} \to \mathbb{R}$, a class $\mathcal{K}$ function $\alpha(\cdot)$, a function $k : [0,\infty) \to \mathbb{R}_+$ such that $k(t) > 0$ for almost all $t \in [0,\infty)$, a real number $\lambda \in (0,1)$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that

$$V(t, 0) = 0, \quad t \in [0,\infty), \tag{20}$$

$$\alpha(\|x\|) \le V(t,x), \quad t \in [0,\infty), \quad x \in \mathcal{M}, \tag{21}$$

$$\dot{V}(t,x) \le -k(t)(V(t,x))^{\lambda}, \quad t \in [0,\infty), \quad x \in \mathcal{M}, \tag{22}$$

then the zero solution $x(t) \equiv 0$ to (1) is finite-time stable.

ii) If $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$ and there exist a continuously differentiable function $V : [0,\infty) \times \mathcal{D} \to \mathbb{R}$, a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$, a function $k : [0,\infty) \to \mathbb{R}_+$ such that $k(t) > 0$ for almost all $t \in [0,\infty)$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that (20)–(22) hold, then the zero solution $x(t) \equiv 0$ to (1) is globally finite-time stable.

iii) If there exist a continuously differentiable function $V : [0,\infty) \times \mathcal{D} \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a function $k : [0,\infty) \to \mathbb{R}_+$ such that $k(t) > 0$ for almost all $t \in [0,\infty)$, a real number $\lambda \in (0,1)$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that (21) and (22) hold, and

$$V(t,x) \le \beta(\|x\|), \quad t \in [0,\infty), \quad x \in \mathcal{M}, \tag{23}$$

then the zero solution $x(t) \equiv 0$ to (1) is uniformly finite-time stable.

vi) If $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$ and there exist a continuously differentiable function $V : [0,\infty) \times \mathcal{D} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a function $k : [0,\infty) \to \mathbb{R}_+$ such that $k(t) > 0$ for almost all $t \in [0,\infty)$, a real number $\lambda \in (0,1)$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that (21)–(23) hold, then the zero solution $x(t) \equiv 0$ to (1) is globally uniformly finite-time stable.

Proof. *i)* Let $t_0 \in [0,\infty)$, let $\varepsilon > 0$ be such that $\mathcal{B}_\varepsilon(0) = \{x \in \mathcal{D} : \|x\| < \varepsilon\} \subset \mathcal{M}$, define $\eta \triangleq \alpha(\varepsilon)$, and define $\mathcal{D}_\eta \triangleq \{x \in \mathcal{B}_\varepsilon(0) : V(t_0,x) < \eta\}$. Since $V(\cdot,\cdot)$ is continuous and $V(t_0,0) = 0$, it follows that $\mathcal{D}_\eta$ is nonempty and there exists $\delta = \delta(\varepsilon,t_0) > 0$ such that $V(t_0,x) < \eta$, $x \in \mathcal{B}_\delta(0)$. Hence, $\mathcal{B}_\delta(0) \subseteq \mathcal{D}_\eta$. Next, it follows from (22) that $V(t,x(t))$ is a nonincreasing function of time and, hence, for every $x_0 \in \mathcal{B}_\delta(0) \subseteq \mathcal{D}_\eta$, it follows that

$$\alpha(\|x(t)\|) \le V(t,x(t)) \le V(t_0,x_0) < \eta = \alpha(\varepsilon). \quad (24)$$

Thus, for every $x_0 \in \mathcal{B}_\delta(0) \subseteq \mathcal{D}_\eta, x(t) \in \mathcal{B}_\varepsilon(0), t \ge t_0$. Furthermore, it follows from (22) and Theorem 2.1 that

$$V(t,x(t)) \le s(t,t_0,V(t_0,x_0)), \quad x_0 \in \mathcal{B}_\delta(0), \quad t \in [t_0,\infty), \quad (25)$$

where $s(\cdot,\cdot,\cdot)$ is given by (17) with $y(t) = V(t,x(t))$. Now, it follows from (17), (21), and (25) that

$$x(t) = 0, \quad x_0 \in \mathcal{B}_\delta(0), \quad t \ge t_1, \quad (26)$$

where t_1 is given by (18). Note that (26) implies finite-time convergence of the trajectory of (1) for all $t_0 \in [0,\infty)$ and $x_0 \in \mathcal{B}_\delta(0)$. This along with (24) implies finite-time stability of the zero solution $x(t) \equiv 0$ to (1).

ii) Let $\delta > 0$, $t_0 \ge 0$, and $x_0 \in \mathcal{D}$ be such that $\|x_0\| < \delta$. Since $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function, it follows that there exists $\varepsilon > 0$ such that $V(t_0,x_0) \le \alpha(\varepsilon)$. Now, (22) implies that $V(t,x(t))$ is a nonincreasing function of time, and hence, it follows from (21) and (23) that, for all $t_0 \in [0,\infty)$,

$$\alpha(\|x(t)\|) \le V(t,x(t)) \le V(t_0,x_0) \le \alpha(\varepsilon), \quad t \ge t_0. \quad (27)$$

Hence, $x(t) \in \mathcal{B}_\varepsilon(0), t \ge t_0$, for all $x_0 \in \mathcal{B}_\delta(0)$ and $t_0 \in [0,\infty)$. Finite-time convergence follows as in the proof of *i)*, implying global finite-time stability of the zero solution $x(t) \equiv 0$ to (1).

iii) Let $\varepsilon > 0$ and $\mathcal{B}_\varepsilon(0)$ be given as in the proof of *i)*. Now, let $\delta = \delta(\varepsilon) > 0$ be such that $\beta(\delta) = \alpha(\varepsilon)$. Hence, it follows from (21) and (23) that, for all $t_0 \in [0,\infty)$ and $x_0 \in \mathcal{B}_\delta(0)$,

$$\alpha(\|x(t)\|) \le V(t,x(t)) \le V(t_0,x_0) < \beta(\delta) = \alpha(\varepsilon), \quad (28)$$

and hence, $x(t) \in \mathcal{B}_\varepsilon(0), t \ge t_0$, for all $t_0 \in [0,\infty)$. This along with (26) implies uniform finite-time stability of the zero solution $x(t) \equiv 0$ to (1).

iv) Let $\delta > 0$ and $x_0 \in \mathcal{D}$ be such that $\|x_0\| < \delta$. Since $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function, it follows that there exists $\varepsilon > 0$ such that $\beta(\delta) \le \alpha(\varepsilon)$. Now, (22) implies that $V(t,x(t))$ is a nonincreasing function of time, and hence, it follows from (21) and (23) that, for all $t_0 \in [0,\infty)$,

$$\alpha(\|x(t)\|) \le V(t,x(t)) \le V(t_0,x_0) < \beta(\delta) \le \alpha(\varepsilon), t \ge t_0. \quad (29)$$

Hence, $x(t) \in \mathcal{B}_\varepsilon(0), t \ge t_0$, for all $x_0 \in \mathcal{B}_\delta(0)$ and $t_0 \in [0,\infty)$. Finite-time convergence follows as in the proof of *i)*, implying global uniform finite-time stability of the zero solution $x(t) \equiv 0$ to (1). □

Example 4.1: Consider the nonlinear dynamical system given by

$$\dot{x}(t) = -t(x(t))^{\frac{1}{3}} - t(x(t))^{\frac{1}{5}}, \quad x(t_0) = x_0, \quad t \ge t_0. \quad (30)$$

For this system, we show that the zero solution $x(t) \equiv 0$ to (30) is globally uniformly finite-time stable. To see this, consider the Lyapunov function candidate $V(t,x) = x^{\frac{4}{3}}$ and let $\mathcal{D} = \mathbb{R}$. Then,

$$\begin{aligned} \dot{V}(t,x) &= \frac{\partial V}{\partial x}(t,x)f(t,x) \\ &= \frac{4}{3}x^{\frac{1}{3}}(-tx^{\frac{1}{3}} - tx^{\frac{1}{5}}) \\ &= -\frac{4}{3}t(x^{\frac{2}{3}} + x^{\frac{8}{15}}) \\ &\le -k(t)V(t,x)^{\frac{1}{2}}, \end{aligned}$$

where $k(t) = 2t > 0, t > 0$. Hence, it follows from *iv)* of Theorem 4.1 that the zero solution $x(t) \equiv 0$ to (30) is globally uniformly finite-time stable. Figure 4.1 shows the state trajectory versus time of (30) for various initial conditions. Finally, consider the case when $t_0 = 0$ and $x_0 = 1$. In this case, $K(t) = t^2$, where $K(t)$ is defined in Example 3.1, and $K(0) = 0$. Then, by (18), $t_1 = 1.414$ and $T(0,1) \le t_1$ as shown in Figure 4.1. △

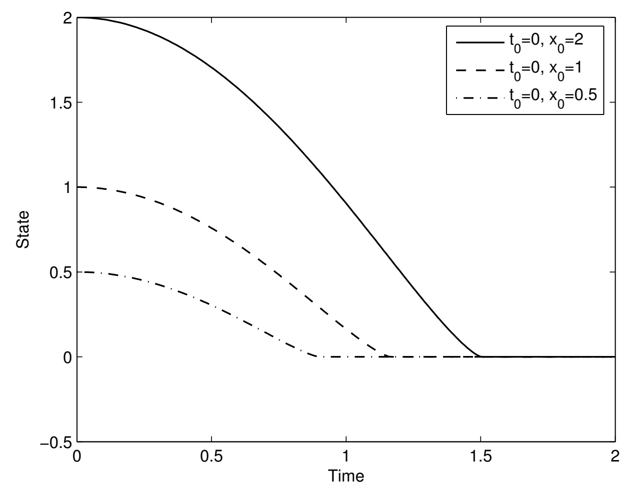

Fig. 1. State trajectory versus time for Example 4.1.

Finally, we present a partial converse theorem for finite-time stability in the case where the settling-time function is continuous. For the statement of this result, define

$$\dot{V}(t,s(t,t_0,x)) \triangleq \lim_{\tau \to t^-} \frac{1}{t-\tau}[V(t,s(t,t_0,x)) - V(\tau,s(\tau,t_0,x))], \quad (31)$$

for a given continuous function $V : [0,\infty) \times \mathcal{D} \to \mathbb{R}$ and for every $t_0 \in [0,\infty), t \ge t_0$, and $x \in \mathcal{D}$ such that the limit in (31) exists.

Theorem 4.2: Let $\lambda \in (0,1)$ and let $\mathcal{N}$ be as in Definition 3.1. Assume that there exists a class $\mathcal{K}$ function $\mu : [0,r] \to \mathbb{R}_+$, where $r > 0$, such that $\mathcal{B}_r(0) \subseteq \mathcal{N}$ and

$$\|f(t,x)\| \le \mu(\|x\|), \quad t \in [0,\infty), \quad x \in \mathcal{B}_r(0). \quad (32)$$

If the zero solution $x(t) \equiv 0$ to (1) is uniformly finite-time stable and the settling-time function $T(\cdot,\cdot)$ is jointly continuous at $(t,0)$, $t \geq 0$, then there exist a class $\mathcal{K}$ function $\alpha(\cdot)$, a positive constant $k > 0$, a continuous function $V : [0,\infty) \times \mathcal{N} \to \mathbb{R}$, and a neighborhood $\mathcal{M}$ of the origin such that $\dot{V}(t,x)$ is defined for $(t,x) \in [0,\infty) \times \mathcal{M}$ and

$$V(t,0) = 0, \quad t \in [0,\infty), \tag{33}$$

$$\alpha(\|x\|) \leq V(t,x), \quad t \in [0,\infty), \quad x \in \mathcal{M}, \tag{34}$$

$$\dot{V}(t,x) \leq -k(V(t,x))^{\lambda}, \quad t \in [0,\infty), \quad x \in \mathcal{M}. \tag{35}$$

Proof. First, it follows from Proposition 3.2 that the settling-time function $T : [0,\infty) \times \mathcal{N} \to \mathbb{R}_+$ is jointly continuous. Now, consider the Lyapunov function candidate $V : [0,\infty) \times \mathcal{N} \to \mathbb{R}$ given by $V(t,x) = [T(t,x) - t]^{\frac{1}{1-\lambda}}$. Note that $V(t,0) = [T(t,0) - t]^{\frac{1}{1-\lambda}} = [t-t]^{\frac{1}{1-\lambda}} = 0$, $t \in [0,\infty)$, which proves (33).

Next, since the zero solution $x(t) \equiv 0$ to (1) is uniformly finite-time stable, there exists $\delta = \delta(r) > 0$ such that $\|s(\tau,t,x)\| < r, \tau \geq t \geq 0, x \in \mathcal{B}_\delta(0)$. Now, since

$$s(\tau,t,x) = x + \int_t^\tau f(\sigma, x(\sigma))\mathrm{d}\sigma, \quad \tau \geq t, \tag{36}$$

it follows that, with $\tau = T(t,x)$,

$$\begin{aligned}\|x(t)\| &= \left\| - \int_t^{T(t,x)} f(\sigma,x(\sigma))\mathrm{d}\sigma \right\| \\ &\leq \int_t^{T(t,x)} \|f(\sigma,x(\sigma))\|\mathrm{d}\sigma \\ &\leq \int_t^{T(t,x)} \mu(\|x(\sigma)\|)\mathrm{d}\sigma \\ &\leq \int_t^{T(t,x)} \mu(r)\mathrm{d}\sigma \\ &= \mu(r)[T(t,x) - t],\ t \in [0,\infty),\ x \in \mathcal{B}_\delta(0). \end{aligned} \tag{37}$$

Let $\mathcal{M} = \mathcal{B}_\delta(0) \subset \mathcal{N}$ and note that, by (37),

$$\begin{aligned} V(t,x) &= [T(t,x) - t]^{\frac{1}{1-\lambda}} \\ &\geq \left(\frac{\|x\|}{\mu(r)}\right)^{\frac{1}{1-\lambda}} \\ &= \alpha(\|x\|), \quad t \in [0,\infty), \quad x \in \mathcal{M}, \end{aligned} \tag{38}$$

where $\alpha(\|x\|) \triangleq \left(\frac{\|x\|}{\mu(r)}\right)^{\frac{1}{1-\lambda}}, x \in \mathcal{M}$, is a class $\mathcal{K}$ function. This proves (34).

Finally, consider the Lyapunov derivative $\dot{V}(t,x(t))$ for some trajectory $x(t)$ starting at $t_0 \in [0,\infty)$ and $x_0 \in \mathcal{M}$. In this case,

$$\begin{aligned} \dot{T}(t,x(t)) &= \lim_{t' \to t} \frac{T(t',x(t')) - T(t,x(t))}{t'-t} \\ &= \lim_{t' \to t} \frac{T(t_0,x_0) - T(t_0,x_0)}{t'-t} \\ &= 0. \end{aligned} \tag{39}$$

Hence, it follows from (31) and (39) that

$$\begin{aligned} \dot{V}(t,x) &= \frac{1}{1-\lambda}[T(t,x(t)) - t]^{\frac{\lambda}{1-\lambda}}[\dot{T}(t,x(t)) - 1] \\ &= -\frac{1}{1-\lambda}[V(t,x)]^{\lambda}, \end{aligned} \tag{40}$$

which proves (35) with $k = \frac{1}{1-\lambda}$. Note that since, by Proposition 3.2, $T(\cdot,\cdot)$ is jointly continuous and $\lambda \in (0,1)$, it follows that $[T(t,x(t)) - t]^{\frac{\lambda}{1-\lambda}}$ is continuous. Hence, it follows from (40) that for all $t \in [0,\infty), x \in \mathcal{N}$, $V(\cdot,\cdot)$ is jointly continuous. □

V. CONCLUSION

This paper extends the notion of finite-time stability for autonomous systems to time-varying dynamical systems. Specifically, Lyapunov and converse Lyapunov results for finite-time stability involving finite-time scalar differential inequalities are established. In addition, necessary and sufficient conditions for continuity of the settling-time function are also presented.

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