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Dynamic average consensus on synchronous communication networks

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Abstract— **We propose a class of dynamic average consensus algorithms that allow a group of agents to track the average of their measured signals. The algorithms are implemented in discrete time and require a synchronous communication schedule. The convergence results rely on the input-to-output stability properties of consensus algorithms and require that the union of communication graphs over a bounded period of time be strongly connected. The only requirement on the set of signals is that the difference of the n^{th}-order derivatives of any two signals be bounded for some $n \geq 0$.**

I. Introduction

We consider the problem in which a set of autonomous agents aims to track the average of individually measured time-varying signals by local communication with neighbors. This problem is referred to as dynamic average consensus in opposition to the more studied static consensus. The dynamic average consensus problem arises in different contexts, such as formation control [7], sensor fusion [17], [18], [25] distributed estimation [12] and distributed tracking [20], [31]. These tasks require that all agents agree on the average of time-varying signals and thus the consensus on a static average value, e.g., the initial states of the agents, is insufficient.

Literature review: The distributed static consensus problem was introduced in the literature of parallel processors in [27] and has attracted significant attention in the controls community. A necessarily incomplete list of references includes [6], [16] for continuous-time consensus, [2], [9], [15], [24] for discrete-time consensus, [1], [13] discuss asynchronous consensus, and [10], [3], [26] treat quantized consensus, randomized consensus and consensus over random graphs, respectively. The convergence rate of consensus algorithms is e.g., discussed in [22], [28], consensus propagation is considered in [14], and conditions on consensus algorithms to achieve different consensus values is discussed in [4]. Consensus algorithms find application in a variety of areas such as load balancing [5], [30], formation control [6], [7], and, as we have mentioned, sensor fusion [12], [17], [18], [25], distributed tracking [20], [31] and consensus-based belief propagation in Bayesian networks [19].

The dynamic average consensus problem in continuous-time is studied in [8], [17], [23], [25]. By using standard frequency-domain techniques, the authors in [25] showed that their algorithm was able to track the average of ramp inputs with zero steady-state error. In the context of input-to-state stability, [8] showed that proportional dynamic average consensus algorithm could track with bounded steady-state error the average of bounded inputs with bounded derivatives. On the other hand, they showed that proportional-integral dynamic average consensus algorithm could track the average of constant inputs with sufficiently small steady-state error. The authors in [17] proposed a dynamic consensus algorithm and applied it to the design of consensus filters. The algorithm in [17] can track with some bounded steady-state error the average of a common input with a bounded derivative. The problem studied in [23] is similar to that in [17], and consensus of agents is over a common time-varying reference signal. However, the algorithm in [23] assumes that agents know the nonlinear model which generates the time-varying reference function. The problem studied in the present paper is close to those in [8] and [25] and includes those in [17] and [23] as special cases.

Statement of contributions. In this paper, we propose a class of discrete-time dynamic average consensus algorithms and analyze their convergence properties. This paper contributes to the problem of dynamic average consensus in the following aspects: The continuous-time communication assumption for dynamic average consensus in [8] and [25] is relaxed, and we consider more realistic discrete-time synchronous communication models. This allows us to obtain a direct relation between the frequency of inter-agent communication and the difference of input signals. Our dynamic average consensus algorithms are able to track the average of a larger class of time-varying inputs than [8] and [25] with zero or sufficiently small steady-state error. This includes polynomials, logarithmic-type functions, periodic functions and other functions whose n^{th}-order differences are bounded, for $n \geq 0$. We can also handle the case where the difference of the common part, that appears in all the individual inputs, explodes. Our analysis for the dynamic average consensus algorithms relies upon the input-to-output stability property of discrete-time static consensus algorithms in the presence of external disturbances. This result is the counterpart of continuous-time static consensus algorithms in [11] but more general in that we allow for unbounded disturbances.

Organization of the paper. We now outline the reminder of the paper. In Section II, we introduce general notation and the statement of the problem we study. In Section III, we focus on a first-order algorithm for dynamic average consensus. Section IV generalizes this to a class of n^{th}- order algorithms for dynamic average consensus and analyze their convergence properties. In Section V, we present some remarks on the extension of the results in Section III and IV. In Section VI, an example and its simulation results are given. Finally, Section VII includes some concluding remarks.

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II. PRELIMINARIES AND PROBLEM STATEMENT

In this section, we introduce the notation to be employed along the paper and state the problem of dynamic average consensus.

The positive real number $h \in (0, 1]$ is the time discretization unit and the update time instants $t \in \mathbb{R}$ (or s, τ) will be of the form $t = \kappa h$ (or $s = \kappa h$, $\tau = \kappa h$) for $\kappa \in \mathbb{Z}$.

We will consider a network of N nodes or agents, labeled by $i \in V = \{1, \ldots, N\}$, interacting over a communication network. The topology of the network at time t will be represented by a directed graph $\mathcal{G}(t) = (V,\ E(t))$ with an edge set $E(t) \subset V \times V$. We consider that $(i, j) \in E(t)$ if and only if node i communicates to node j at time t. The neighbors of node i at time t are denoted by $\mathcal{N}_i(t) = \{j \in V : (j, i) \in E(t) \text{ and } j \neq i\}$. The matrix $A(t) = [a_{ij}(t)] \in \mathbb{R}^{N\times N}$ represents the adjacency matrix of the graph $\mathcal{G}(t)$ where $a_{ij}(t) \neq 0$ if edge $(j, i) \in E(t)$. Finally, $\mathbf{1} \in \mathbb{R}^N$ will the vector which entries are all ones.

At each time instant t, every node synchronously will measure the local continuous physical process $u_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$, communicates with its neighbors and updates the state of its consensus algorithm. We ignore any delays induced by the communication and computation process. In the reminder of this paper, the sample $u_i(t)$ is referred to as the (external) input of node i at time t. Denote by $\bar{u}(t) = \frac{1}{N}\sum_{i=1}^{N} u_i(t)$ the average of the inputs of the network at time t. We denote $x_i^{[n]}(t)$ as the last component of the consensus state $x_i(t) = \text{col}(x_i^{[1]}(t), \cdots, x_i^{[n]}(t)) \in \mathbb{R}^n$ of node i where the order n will depend on the input $u(t)$ and its differences.

Our objective is to design an n^{th}-order dynamic average consensus algorithm that the nodes can utilize to asymptotically achieve the average of the input $u(t) = \text{col}(u_1(t), \cdots, u_N(t)) \in \mathbb{R}^N$, in the following sense. For all $i \in \{1, \ldots, N\}$, there must hold that:

$$\lim_{t\to\infty} |x_i^{[n]}(t) - \bar{u}(t-h)| = 0. \tag{1}$$

The quantity of $\lim_{t\to\infty} \max_{i\in V} |x_i^{[n]}(t) - \bar{u}(t-h)|$ is referred to as the steady-state error of the dynamic average consensus algorithm. This can be interpreted as a measurement of how far the components of the consensus state $x^{[n]}(t) = \text{col}(x_1^{[n]}(t), \cdots, x_N^{[n]}(t))$ are from achieving the dynamic average consensus. Our algorithms will accomplish (1) with either zero steady-state error or rendering the steady-state error smaller than (or equal to) any given bound.

III. FIRST-ORDER DYNAMIC AVERAGE CONSENSUS ALGORITHM

In this section, we present first-order algorithms to achieve dynamic average consensus. Our presentation mainly follows [1], [2], and [22]. We define:

$$\begin{aligned}
&M(t) = \max_{i\in V} x_i(t), && m(t) = \min_{i\in V} x_i(t),\\
&D(t) = M(t) - m(t), && \Delta u_i(t) = u_i(t) - u_i(t-h),\\
&\Delta u_{\max}(t) = \max_{i\in V} \Delta u_i(t), && \Delta u_{\min}(t) = \min_{i\in V} \Delta u_i(t).
\end{aligned}$$

By induction, the n^{th}-order difference of $u_i(t)$ is $\Delta^{(n)}u_i(t) = \Delta^{(n-1)}u_i(t) - \Delta^{(n-1)}u_i(t-h)$ for $n \geq 2$ where $\Delta^{(1)}u_i(t) = \Delta u_i(t)$. We will use the notations $\Delta^{(n)}u_{\max}(t) = \max_{i\in V} \Delta^{(n)}u_i(t)$ and $\Delta^{(n)}u_{\min}(t) = \min_{i\in V} \Delta^{(n)}u_i(t)$ for $n \geq 2$.

We make use of the following assumption that was proposed in [9] and also used in [2][22].

Assumption 3.1 (Periodical strong connectivity): There is some positive integer T such that, for all time instant $t \geq 0$, the directed graph $(V, E(t) \cup E(t+h) \cup \cdots \cup E(t + (T-1)h))$ is strongly connected.

Assumption 3.2 (Bounded first-order differences): For any $0 < h \leq 1$, there exists a time invariant constant $\theta \geq 0$ such that

$$\Delta U(t) = \Delta u_{\max}(t) - \Delta u_{\min}(t) \leq h\theta, \tag{2}$$

holds for all $t \geq 0$.

Remark 3.1: Inequality (2) becomes $\max_{i\in V} \dot{u}_i(t) - \min_{i\in V} \dot{u}_i(t) \leq \theta$ as $h \to 0$. Hence, Assumption 3.2 can be viewed as the discretized version of the property $\max_{i\in V} \dot{u}_i(t) - \min_{i\in V} \dot{u}_i(t) \leq \theta$ for some fixed $\theta \geq 0$ and all time instant $t \geq 0$. •

We propose $first-order\ dynamic\ average\ consensus\ algorithm$ below to reach dynamic average consensus:

$$x_i(t+h) = x_i(t) + \sum_{j\neq i} a_{ij}(t)(x_j(t) - x_i(t)) + \Delta u_i(t), \tag{3}$$

where the input $u(t)$ satisfies Assumption 3.2.

Remark 3.2: First-order dynamic average consensus algorithm can be rewritten as:

$$\begin{aligned}
[x_i(t+h) - x_i(t)]/h = \delta \sum_{j\neq i} a_{ij}(t)(x_j(t) - x_i(t))&\\
+ [u_i(t) - u_i(t-h)]/h\,,&
\end{aligned} \tag{4}$$

where the parameters δ and h satisfy $h\delta = 1$. Observe that (4) is close to the discretized version of the continuous-time dynamic consensus algorithm in [25] but is not exactly the same. If $h \to 0$, then $\delta = \frac{1}{h} \to \infty$, and thus the right-hand side of (4) is not well-defined. •

We will further suppose that the coefficients $a_{ij}(t)$ in first-order dynamic average consensus algorithm satisfy the following two assumptions.

Assumption 3.3 (Non-degeneracy): There exists a positive constant α such that $a_{ii}(t) = 1 - \sum_{j\neq i} a_{ij}(t) \geq \alpha$, and $a_{ij}(t)$ $(i \neq j)$ satisfies $a_{ij}(t) \in \{0\} \cup [\alpha,\ 1]$ for all $t \geq 0$.

Assumption 3.4 (Balanced communication): There holds that $\mathbf{1}^T A(t) = A(t)\mathbf{1} = \mathbf{1}$, for all $t \geq 0$.

Lemma 3.1: Consider first-order dynamic average consensus algorithm and suppose that Assumption 3.3 is satisfied. Then the following inequalities hold for all $s \geq 0$ and $t > 0$:

$$m(t+s) \geq m(s) + \sum_{\tau=s}^{t+s-h} \Delta u_{\min}(\tau)\,,$$

$$M(t+s) \leq M(s) + \sum_{\tau=s}^{t+s-h} \Delta u_{\max}(\tau)\,.$$

Proof: Due to the space limit, we omit the proof. ■

Let us fix $k \in V$ for every $t \geq s$ and define $\mathcal{D}_0 = \{k\}$. By Assumption 3.1, there is a non-empty set $\mathcal{D}_1 \subset V \setminus \{k\}$ of nodes such that for all $i \in \mathcal{D}_1$, node k communicates to node i at least once during the time frame $[s, s + (T-1)h]$. By induction, a set $\mathcal{D}_{\ell+1} \subset V \setminus (\mathcal{D}_0 \cup \cdots \cup \mathcal{D}_\ell)$ can be defined by considering those nodes j that communicate to some $i \in \mathcal{D}_0 \cup \cdots \cup \mathcal{D}_\ell$ at least once during the time frame $[s + \ell Th, s + ((\ell+1)T - 1)h]$. By Assumption 3.1, $\mathcal{D}_{\ell+1} \neq \emptyset$ as long as $V \setminus (\mathcal{D}_0 \cup \cdots \cup \mathcal{D}_\ell) \neq \emptyset$. Thus, there exists $\mathcal{L} \leq N-1$ such that $\mathcal{D}_0 \cup \cdots \cup \mathcal{D}_{\mathcal{L}} = V$.

Lemma 3.2: Consider first-order dynamic average consensus algorithm and suppose that Assumptions 3.1 and 3.3 hold. Let $s \geq 0$ and $k \in V$ be fixed and consider the associated $\mathcal{D}_0, \ldots, \mathcal{D}_{\mathcal{L}}$. Then for every $t \in [s + \ell Th, s + (\mathcal{L}T + T - 1)h]$ and $i \in \mathcal{D}_\ell$, with $\ell \in \{1, \ldots, \mathcal{L}\}$, there exists a constant η_ℓ (independent of s) such that for every $t \in [s + \ell Th,\ s + (\mathcal{L}T + T - 1)h]$, and $i \in \mathcal{D}_\ell$, we have that

$$x_i(t) \geq m(s) + \sum_{\tau=s}^{t-h} \Delta u_{\min}(\tau) + \eta_\ell (x_k(s) - m(s)), \quad (5)$$

$$x_i(t) \leq M(s) + \sum_{\tau=s}^{t-h} \Delta u_{\max}(\tau) - \eta_\ell (M(s) - x_k(s)). \quad (6)$$

Proof: Without loss of generality, we only consider the case where $s = 0$, being the proof for a general s identical. Since $\sum_{j=1}^{N} a_{kj}(t) = 1$ at every $t \geq 0$, we have that

$$\begin{aligned}
&x_k(t+h) - m(0) - \sum_{\tau=0}^{t} \Delta u_{\min}(\tau) \\
&= \sum_{j=1}^{N} a_{kj}(t)(x_j(t) - m(0) - \sum_{\tau=0}^{t-h} \Delta u_{\min}(\tau)) \\
&+ \Delta u_k(t) - \Delta u_{\min}(t) \\
&\geq a_{kk}(t)(x_k(t) - m(0) - \sum_{\tau=0}^{t-h} \Delta u_{\min}(\tau)) \\
&\geq \alpha (x_k(t) - m(0) - \sum_{\tau=0}^{t-h} \Delta u_{\min}(\tau)), \qquad (7)
\end{aligned}$$

where we are using the property that $x_k(t) - m(0) - \sum_{\tau=0}^{t-h} \Delta u_{\min}(\tau) \geq m(t) - m(0) - \sum_{\tau=0}^{t-h} \Delta u_{\min}(\tau) \geq 0$ from Lemma 3.1. Applying repeatedly (7) we have that, for all $t \in [0, (\mathcal{L}T + T - 1)h]$,

$$\begin{aligned}
&x_k(t) - m(0) - \sum_{\tau=0}^{t-h} \Delta u_{\min}(\tau) \\
&\geq \alpha^{\frac{t}{h} - 1}(x_k(h) - m(0) - \Delta u_{\min}(0)) \\
&\geq \alpha^{\frac{t}{h}}(x_k(0) - m(0)) \geq \eta_0 (x_k(0) - m(0)),
\end{aligned}$$

where $\eta_0 = \alpha^{NT-1}$ and we are using that $x_k(0) - m(0) \geq 0$. This proves inequality (5) for the nodes in $\mathcal{D}_0 = \{k\}$ and for any $t \in [0, (\mathcal{L}T + T - 1)h]$.

Now we proceed by induction on ℓ. Suppose that (5) holds for some $0 \leq \ell < \mathcal{L}$; then we should show (5) for $i \in \mathcal{D}_{\ell+1}$. It follows from the construction of the sets of $\{\mathcal{D}_0, \cdots, \mathcal{D}_{\mathcal{L}}\}$ that there exists some time $t' \in [\ell Th,\ (\ell T + T - 1)h]$ such that $a_{ij}(t') \neq 0$ for some $j \in \mathcal{D}_0 \cup \cdots \cup \mathcal{D}_\ell$ and $i \in \mathcal{D}_{\ell+1}$. By the induction hypothesis, we have that for all $t \in [\ell Th,\ (\mathcal{L}T + T - 1)h]$, there exists some $\eta_\ell > 0$ such that

$$x_j(t) - m(0) - \sum_{\tau=0}^{t-h} \Delta u_{\min}(\tau) \geq \eta_\ell (x_k(0) - m(0)).$$

Consequently, as in (7), we have

$$\begin{aligned}
&x_i(t' + h) - m(0) - \sum_{\tau=0}^{t'} \Delta u_{\min}(\tau) \\
&\geq a_{ij}(t')(x_j(t') - m(0) - \sum_{\tau=0}^{t'-h} \Delta u_{\min}(\tau)) \\
&\geq \alpha \eta_\ell (x_k(0) - m(0)).
\end{aligned}$$

Following along the same lines as in (7), we have that

$$x_i(t+h) - m(0) - \sum_{\tau=0}^{t} \Delta u_{\min}(\tau) = \eta_{\ell+1}(x_k(0) - m(0)),$$

holds for all $t \in [(\ell+1)Th,\ (\mathcal{L}T + T - 1)h]$, where $\eta_{\ell+1} = \alpha^{(N-\ell)T} \eta_\ell$. This establishes (5) for $i \in \mathcal{D}_{\ell+1}$. By induction, we have shown that (5) holds. The proof for (6) is analogous. ■

The following theorem shows the convergence properties of first-order dynamic average consensus algorithm.

Theorem 3.1: Let δ_1 be a positive constant and $h_1 = \sqrt{\frac{\delta_1}{2\theta(NT-1)\alpha^{-\frac{1}{2}N(N+1)T+1}}}$. Under Assumptions 3.1, 3.3, 3.4 and 3.2 with $\theta \neq 0$, the implementation of first-order dynamic average consensus algorithm with $h \leq h_1$ and initial conditions $x_i(0) = u_i(-h)$, $i \in \{1, \ldots, N\}$, achieves dynamic average consensus with nonzero steady-state error upper bounded by δ_1.

Proof: Let $\eta = \alpha^{\frac{1}{2}N(N+1)T-1}$, then $\eta \leq \eta_\ell$ for any $\ell \in \{1, \ldots, N-1\}$. By replacing s and t in (5) with t and $t_1 = t + (\mathcal{L}T + T - 1)h$ respectively, we have that for every $t \geq 0$, there holds that

$$\begin{aligned}
m(t_1) &= \min_{\ell \in \{0, \cdots, \mathcal{L}\}} \min_{i \in \mathcal{D}_\ell} x_i(t_1) \\
&\geq m(t) + \sum_{\tau=t}^{t_1-h} \Delta u_{\min}(\tau) + \min_{\ell} \eta_\ell (x_k(t) - m(t)) \\
&\geq m(t) + \sum_{\tau=t}^{t_1-h} \Delta u_{\min}(\tau) + \eta (x_k(t) - m(t)).
\end{aligned}$$

Similarly, we have

$$M(t_1) \leq M(t) + \sum_{\tau=t}^{t_1-h} \Delta u_{\max}(\tau) - \eta (M(t) - x_k(t)).$$

Combining the above two inequalities gives that

$$D(t_1) \leq (1-\eta) D(t) + \sum_{\tau=t}^{t_1-h} \Delta U(\tau). \quad (8)$$

Let us denote $T_k = k(\mathcal{L}T + T - 1)h$ for an integer $k \geq 1$. It follows from (8) that

$$D(T_n) \leq \max\{2(1-\eta)^n D(0), 2\Omega(n)\},$$

where

$$\Omega(n) = (1-\eta)^{n-1} \sum_{\tau=0}^{T_1-h} \Delta U(\tau) + \cdots + \sum_{\tau=T_{n-1}}^{T_n-h} \Delta U(\tau).$$

Since $\Delta U(t) = \Delta u_{\max}(t) - \Delta u_{\min}(t) \leq h\theta$, $D(t)$ is input-to-output stable with ultimate bound $\Xi = 2\Omega(\infty) \leq 2h^2\theta(\mathcal{L}T + T - 1)\frac{1}{\eta} \leq 2h^2\theta(NT-1)\alpha^{-\frac{1}{2}N(N+1)T+1}$, i.e., there exist $\Gamma > 0$ and $\lambda > 0$ such that

$$D(t) \leq \max\{\Gamma\lambda^t, \Xi\}, \qquad \forall\, t \geq 0. \tag{9}$$

Choose as initial state $x_i(0) = u_i(-h)$ for all $i \in \{1, \ldots, N\}$. By Assumption 3.4, the following conservation property of first-order dynamic average consensus algorithm is satisfied for all $t \geq 0$:

$$\begin{aligned}
&\sum_{i=1}^{N} x_i(t+h) = \sum_{i=1}^{N} x_i(t) + \sum_{i=1}^{N} \Delta u_i(t) \\
&= \sum_{i=1}^{N} x_i(0) + \sum_{i=1}^{N}\sum_{\tau=0}^{t} \Delta u_i(\tau) \\
&= \sum_{i=1}^{N} x_i(0) + \sum_{i=1}^{N}(u_i(t) - u_i(-h)) = \sum_{i=1}^{N} u_i(t), \qquad (10)
\end{aligned}$$

where we have used the induction in Line 2 of the above expressions.

It follows from (10) that $m(t+h) \leq \frac{1}{N}\sum_{i=1}^{N} u_i(t) \leq M(t+h)$ and thus

$$\lim_{t\to\infty} |x_i(t) - \frac{1}{N}\sum_{i=1}^{N} u_i(t-h)| \leq \lim_{t\to\infty} D(t) \leq \Xi.$$

Hence, for any given $\delta_1 > 0$, choosing $h \leq h_1$ gives an steady-state error $\Xi \leq \delta_1$. In other words, choosing a step of size h induces at least an error of order $h^2\theta(NT-1)\alpha^{-\frac{1}{2}N(N+1)T+1}$. ∎

Corollary 3.1: Under the Assumptions 3.1, 3.3, 3.4 and 3.2 with $\theta = 0$, the implementation of first-order dynamic average consensus algorithm with any $h > 0$ and initial state $x_i(0) = u_i(-h)$, $i \in \{1, \ldots, N\}$, achieves the dynamic average consensus at a geometric rate with zero steady-state error.

IV. Higher-order Algorithms for Dynamic Average Consensus

In this section, we present n^{th}-order algorithms for dynamic average consensus where $n \geq 2$. First of all, let us consider the case of $n = 2$. We will that the inputs satisfy the following condition weaker than Assumption 3.2.

Assumption 4.1 (Bounded second-order differences): For any $0 < h \leq 1$, there exists a time invariant constant $\theta_2 \geq 0$ such that

$$\Delta^{(2)} u_{\max}(t) - \Delta^{(2)} u_{\min}(t) \leq h\theta_2, \qquad \forall\, t \geq 0.$$

Correspondingly, we propose the following *second − order dynamic average consensus algorithm*

$$\begin{aligned}
x_i^{[2]}(t+h) &= x_i^{[2]}(t) + \sum_{j\neq i} a_{ij}(t)(x_j^{[2]}(t) - x_i^{[2]}(t)) \\
&+ x_i^{[1]}(t+h), \\
x_i^{[1]}(t+h) &= x_i^{[1]}(t) + \sum_{j\neq i} a_{ij}(t)(x_j^{[1]}(t) - x_i^{[1]}(t)) \\
&+ \Delta^{(2)} u_i(t).
\end{aligned} \tag{11}$$

Second-order dynamic average consensus algorithm can be written in the following vector form

$$\begin{aligned}
x^{[2]}(t+h) &= A(t)x^{[2]}(t) + x^{[1]}(t+h), \\
x^{[1]}(t+h) &= A(t)x^{[1]}(t) + \Delta^{(2)} u(t).
\end{aligned} \tag{12}$$

Theorem 4.1: Let δ_2 be a positive constant and $h_2 = \sqrt[3]{\frac{\delta_2}{8\theta_2(NT-1)^2\alpha^{-N(N+1)T+2}}}$. Under the Assumptions 3.1, 3.3, 3.4 and 4.1 with $\theta_2 \neq 0$, the implementation of second-order dynamic average consensus algorithm with $h \leq h_2$, and initial states $x_i^{[1]}(0) = \Delta u_i(-h)$, $x_i^{[2]}(0) = u_i(-h)$, $i \in \{1, \ldots, N\}$, achieves dynamic average consensus with nonzero steady-state error upper bounded by δ_2.

Proof: Note that the dynamic average consensus for $x^{[1]}$ in (12) has the same form as first-order dynamic average consensus algorithm, and can be obtained from this by replacing $\Delta u_i(t)$ with $\Delta^{(2)} u_i(t)$. Since Assumption 4.1 holds, it follows from Theorem 3.1 that by choosing the initial state as $x_i^{[1]}(0) = \Delta u_i(-h)$ we can find $\Gamma_1 > 0$ and $0 < \lambda_1 < 1$ such that for all $t \geq 0$ and all $i \in \{1, \ldots, N\}$, there holds that

$$|x_i^{[1]}(t) - \frac{1}{N}\sum_{i=1}^{N} \Delta u_i(t-h)| \leq D^{[1]}(t) \leq \max\{\Gamma_1\lambda_1^t, \Xi_1\},$$

where $D^{[1]}(t) = \max_{i\in V} x_i^{[1]}(t) - \min_{i\in V} x_i^{[1]}(t)$ and $\Xi_1 \leq 2h^2\theta_2(NT-1)\alpha^{-\frac{1}{2}N(N+1)T+1}$.

Hence, there exists a finite $\bar{t} \geq 0$ such that $\Gamma_1\lambda_1^t \leq \Xi_1$ for all $t \geq \bar{t}$. In this way, $x_i^{[2]}(t)$ in second-order dynamic average consensus algorithm can be written in the following way for $t \geq \bar{t}$

$$x_i^{[2]}(t+h) = x_i^{[2]}(t) + \sum_{j\neq i} a_{ij}(t)(x_j^{[2]}(t) - x_i^{[2]}(t)) + d_i(t), \tag{13}$$

with an input $d_i(t) = \frac{1}{N}\sum_{i=1}^{N} \Delta u_i(t) + \vartheta_i(t)$ and $|\vartheta_i(t)| \leq \Xi_1$. Note that for all $t \geq \bar{t}$, there holds that

$$\begin{aligned}
&\frac{\max_{i\in V} d_i(t) - \min_{i\in V} d_i(t)}{h} \leq \frac{2\Xi_1}{h} \\
&\leq 4h\theta_2(NT-1)\alpha^{-\frac{1}{2}N(N+1)T+1}.
\end{aligned}$$

Hence, (13) has the same form as first-order dynamic average consensus algorithm, and can be obtained from it by replacing $\Delta u_i(t)$ with $d_i(t)$ where $\theta = 4h\theta_2(NT-1)\alpha^{-\frac{1}{2}N(N+1)T+1}$ in Assumption 3.2.

Furthermore, consider as initial states $x_i^{[2]}(0) = u_i(-h)$ for all $i \in \{1,\dots,N\}$. Similarly to (10) with $\Delta u_i(t)$ instead of $u_i(t)$, we can obtain the following conservation property of second-order dynamic average consensus algorithm for every $t \geq 0$

$$\sum_{i=1}^{N} x_i^{[1]}(t+h) = \sum_{i=1}^{N} \Delta u_i(t),\ \sum_{i=1}^{N} x_i^{[2]}(t+h) = \sum_{i=1}^{N} u_i(t).$$

By using similar arguments to those employed in Theorem 3.1, we have that there exist $\Gamma_2 > 0$ and $0 < \lambda_2 < 1$ such that for all $t \geq \bar{t}$ and all $i \in V$, there holds

$$|x_i^{[2]}(t) - \frac{1}{N}\sum_{i=1}^{N} u_i(t-h)| \leq D^{[2]}(t) \leq \max\{\Gamma_2 \lambda_2^t, \Xi_2\}\,,$$

where $D^{[2]}(t) = \max_{i\in V} x_i^{[2]}(t) - \min_{i\in V} x_i^{[2]}(t)$ and $\Xi_2 = 2h^2\theta(NT-1)\alpha^{-\frac{1}{2}N(N+1)T+1} = 8h^3\theta_2(NT-1)^2\alpha^{-N(N+1)T+2}$. For any given $\delta_2 > 0$, choosing $h \leq h_2$ leads to $\Xi_2 \leq \delta$. ■

Corollary 4.1: Under Assumptions 3.1, 3.3, 3.4 and 4.1 with $\theta_2 = 0$, the implementation of second-order dynamic average consensus algorithm with any $h > 0$ and initial states $x_i^{[1]}(0) = \Delta u_i(-h)$, $x_i^{[2]}(0) = u_i(-h)$ for all $i \in \{1,\dots,N\}$ achieves dynamic average consensus at a geometric rate with zero steady-state error.

Now, let us consider the following $n^{th}-order\ dynamic\ average\ consensus\ algorithm$.

$$\begin{aligned} x_i^{[\ell]}(t+h) &= x_i^{[\ell]}(t) + \sum_{j\neq i} a_{ij}(t)(x_j^{[\ell]}(t) - x_i^{[\ell]}(t)) \\ &\quad + x_i^{[\ell-1]}(t+h)\,, \\ x_i^{[1]}(t+h) &= x_i^{[1]}(t) + \sum_{j\neq i} a_{ij}(t)(x_j^{[1]}(t) - x_i^{[1]}(t)) \\ &\quad + \Delta^{(n)} u_i(t), \qquad \ell \in \{2,\dots,n\}\,. \end{aligned} \tag{14}$$

The previous algorithm is the cascade of n first-order dynamic average consensus algorithms and can be compactly rewritten in the following vector form

$$\begin{aligned} x^{[\ell]}(t+h) &= A(t)x^{[\ell]}(t) + x^{[\ell-1]}(t+h)\,, \\ x^{[1]}(t+h) &= A(t)x^{[1]}(t) + \Delta^{(n)}u(t)\,, \end{aligned}$$

where $\ell \in \{2,\dots,n\}$. n^{th}-order dynamic average consensus algorithm is able to track the average of inputs which satisfy the following condition under which Theorem 4.2 holds.

Assumption 4.2 (Bounded n^{th}-order differences): For any $0 < h \leq 1$, there exists a time invariant constant $\theta_n \geq 0$ such that

$$\Delta^{(n)} u_{\max}(t) - \Delta^{(n)} u_{\min}(t) \leq h\theta_n\,, \qquad \forall\, t \geq 0.$$

Theorem 4.2: Let δ_n be a positive constant and $h_n = (\frac{\delta_n}{2^{2n-1}\theta_n(NT-1)^n\alpha^{n(-\frac{1}{2}N(N+1)T+1)}})^{\frac{1}{n+1}}$. Under the Assumptions 3.1, 3.3, 3.4 and 4.2 with $\theta_n \neq 0$, the implementation of n^{th}-order dynamic average consensus algorithm with $h \leq h_n$ and initial states $x_i^{[\ell]}(0) = \Delta^{(n-\ell)}u_i(-h)$ $(\ell = 1,\cdots,n-1)$, $x_i^{[n]}(0) = u_i(-h)$ for all $i \in \{1,\dots,N\}$, achieves the dynamic average consensus with a nonzero steady-state error upper bounded by δ_n.

Proof: The proof can be completed by applying Theorem 4.1 inductively. ■

Corollary 4.2: Under the Assumptions 3.1, 3.3, 3.4 and 4.2 with $\theta_n = 0$, the implementation of n^{th}-order dynamic average consensus algorithm with any $h > 0$ and initial states $x_i^{[\ell]}(0) = \Delta^{(n-\ell)}u_i(-h)$ $(\ell = 1,\cdots,n-1)$, $x_i^{[n]}(0) = u_i(-h)$ for all $i \in \{1,\dots,N\}$, achieves dynamic average consensus at a geometric rate with zero steady-state error.

V. Discussion

This section includes some remarks about the possible extension of the presented results. First, it can be shown that for any n^{th}-order polynomial $f(t) = \sum_{i=0}^{n} a_i t^i$, there holds that $\Delta^{(n)} f(t) = a_n n! h$. Hence, any set of n^{th}-order polynomials satisfies Assumption 4.2 for some bounded θ_n. If the leading coefficients of these polynomials are identical, then Assumption 4.2 is fulfilled for $\theta_n = 0$.

If the inputs $u_i(t)$ take the form of $u_i(t) = r(t) + \tilde{u}_i(t)$, $i \in V$, and the function $\tilde{u}_i(t)$ is a linear combination of polynomials, the logarithmic function, periodic functions and other functions whose n^{th}-order differences are bounded, then Assumption 4.2 also holds for any common $r(t)$ even when n^{th}-order difference of $r(t)$ explodes, e.g., like the exponential function. It is worth mentioning that it is unnecessary for Assumption 4.2 to hold that $\Delta^{(n)}u_i(t)$ be bounded for all i, $t \geq 0$ and any $h \in (0,1]$.

In the case that the communication is symmetric; i.e., when $(i,j) \in E(t)$ if and only if $(j,i) \in E(t)$, then Assumption 3.1 (periodical strong connectivity) in Corollary 4.2 can be weakened into:

Assumption 5.1 (Eventual strong connectivity): The directed graph $(V, \cup_{s\geq t} E(s))$ is strongly connected for all time instant $t \geq 0$.

Furthermore, Assumption 3.1 in Corollary 4.2 can also be replaced with the assumption in [15] that for any time instant $t \geq 0$, there is a leader in the directed graph $(V, \cup_{s\geq t} E(s))$. It is interesting to further think about the weaker assumption in [15] that there exists an integer $T \geq 1$ such that for any time instant $t \geq 0$, there is a leader in the directed graph $(V, E(t) \cup E(t+h) \cup \cdots \cup E(t+(T-1)h))$.

VI. Example

The section illustrates Theorem 4.1 with a simulation.

Let us consider a network consisting of four nodes, labeled 1 through 4. Suppose that graph $\mathcal{G}(t)$ satisfies Assumption 3.1 with $T = 4$. Assume the inputs $u_i(t)$ are:

$$\begin{aligned} u_1(t) &= 12\sin t + 0.2t^2, & u_2(t) &= 8\cos t - 0.3t^2, \\ u_3(t) &= 5\sin 0.5t + 0.1t^2, & u_4(t) &= \sin 2t. \end{aligned}$$

It can be verified that Assumption 4.1 is fulfilled with $\theta_2 = 21$. For the given $\delta_2 = 2$, we choose $\alpha = 0.495$ and can calculate $h < 0.0017$ for second-order dynamic average consensus algorithm. Figure 1 shows the state $x_i^{[2]}(t)$ for $i \in \{1, 2, 3, 4\}$ and the average of the inputs $\bar{u}(t-h)$.

VII. Conclusions

We have proposed a class of dynamic average consensus algorithms on synchronous communication networks and analyze their convergence properties. Due to slow convergence rates of the algorithms, tracking is shown at the expense of frequent communication and higher throughput. Future work will explore the algorithms application for sensor fusion.

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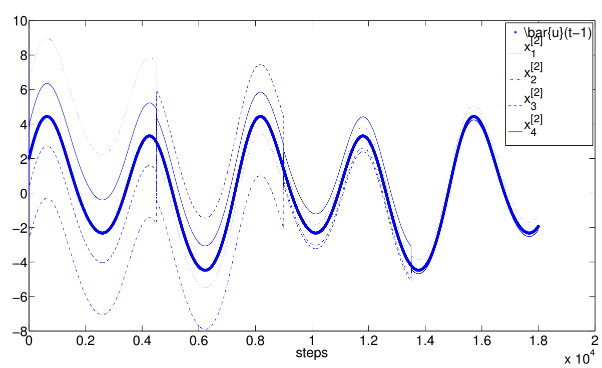

Fig. 1. Evolution of the states of the second order dynamic consensus algorithm in comparison with the average of the inputs