2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Cooperative Kalman Filters for Cooperative Exploration

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Abstract— **Cooperative exploration requires multiple robotic sensor platforms to navigate in an unknown scalar field to reveal its global structure. Sensor readings from the platforms are combined into estimates to direct motion and reduce noise. We show that the combined estimates for the field value, the gradient and the Hessian satisfy an information dynamic model that does not depend on motion models of the platforms. Based on this model, we design cooperative Kalman filters that apply to general cooperative exploration missions. We rigorously justify a set of sufficient conditions that guarantee the convergence of the cooperative Kalman filters. These sufficient conditions provide guidelines on mission design issues such as the number of platforms to use, the shape of the platform formation, and the motion for each platforms.**

I. Introduction

Human activity has been one of the main factors leading to changes in the environment. Profound problems such as acid rain, red tides and global warming, result at least in part from chemicals and waste produced by society. Finding scientific evidence on specific factors most significant in affecting the atmosphere or the ocean, requires many sensors to gather comprehensive information from certain regions. When the area of interest is very large, mobile sensor networks are often the reasonable choice to reduce the number of sensor nodes that have to be deployed. Recent theoretical and experimental developments suggest that a balance between data collection and feasible motion of sensor platforms is key to mission success [1]–[3]. Finding an optimal strategy is a challenging task.

In this paper, we present a general Kalman filter design for mobile sensor networks to perform cooperative exploration missions. Exploration missions are frequently encountered in environmental applications where the mobile sensor platforms are commanded to measure an unknown scalar field corrupted by (correlated) noise. Since each platform can only take one measurement at a time, the platforms should move in a formation or a cluster to estimate local structures of the field. Our Kalman filter combines sensor readings from formation members to provide estimates for the field value and the gradient. A separate cooperative filter was developed to estimate the Hessian in our previous work [4], where a preliminary version of the Kalman filter is also derived. In this paper, significant new theoretical developments have been made to rigorously derive the cooperative Kalman filter. We prove a set of sufficient conditions that a formation and its motion need to satisfy to achieve the convergence of the Kalman filter. Derivation of these sufficient conditions is based on fundamental results connecting controllability and observability of a (time-varying) filtering system to its convergence in [5]–[7]. More recent developments in [8]–[10] have relaxed the conditions for convergence of Kalman filters to stabilizability and detectability, with even weaker conditions for some special cases. In this paper, we develop the sufficient conditions based on controllability and observability conditions because the resulting constraints on formation design are already mild enough, hence are acceptable in typical applications.

Kalman filtering for mobile sensor network applications have received recent attention in the literature. In [11], a distributed Kalman filter method was proposed to decompose a high order central Kalman filter into "micro" filters computable by each sensor node. The estimates made by each node are then combined using consensus filters [12]. A similar approach is taken in [13] to address the target tracking and coverage problems. Another type of Kalman filter design is proposed in [14] where the entire field is partitioned into cells and the movement of agents are controlled to maximize collected information. The above contributions assume that the (dynamic) model for a planar field is known to all nodes, hence each individual is able to compute a Kalman filter. Accordingly, the goal in these work is to implement a distributed algorithm on many sensor nodes to improve tracking or mapping precision.

For the cooperative exploration problem, on the other hand, the field is completely unknown; a Kalman filter can only be computed by combining readings across platforms. The interest here is to use the minimum number of sensor platforms to be able to navigate in the unknown scalar field to reveal its structures e.g. follow level curves or gradients. In [15], an adaptive scheme using the Kalman filter is developed for interpolating data to construct a scalar field. These contributions address different problems than this paper and are complementary to our results.

The organization of this paper is as follows. In Section II, we derive the information dynamics for a typical platform formation moving in a planar scalar field. In Section III, Kalman filtering techniques are applied to the information dynamics. We establish sufficient conditions for the cooperative Kalman filter to converge. A summary is presented in

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We want to thank the MIT and Harvard HOPS group (Pierre Lermusiaux, Allan Robinson, Pat Hayley, Wayne Leslie, in particular) for ocean modeling support. We thank Edward Fiorelli, Francois Lekien, and Pradeep Bhatta for collaborations, suggestions, and discussions. This work was supported in part by ONR grants N00014–02–1–0826, N00014–02–1–0861 and N00014–04–1–0534.

978-1-4244-2079-7/08/$25.00 ©2008 AACC.

Section IV.

II. Information Dynamics of Cooperative Exploration

In this section, we define the cooperative exploration problem and introduce the corresponding information dynamic model. Let $z(\mathbf{r})$ where $\mathbf{r} \in \mathbb{R}^2$ be a smooth scalar field in the plane that is unknown. The key idea for mobile sensor networks is to employ multiple moving sensor platforms to obtain the necessary estimates cooperatively and reduce noise. This requires the platforms to be in a formation, moving and collecting information simultaneously. In this paper, we focus on the information collection rather than the formation control aspects of the cooperative exploration problem.

In most applications, the sensor measurements are taken discretely over time. This is because the spatial range of the scalar field is usually very large. Hence very small scale fluctuations in the field should be filtered out as noise. Let the moment when new measurements are available be t_k where k is an integer index. To simplify the derivation, we do not consider the asynchronicity in the measurements; we assume that all platforms have new measurements at time t_k. In reality, when there exists asynchronicity, the technique we develop can still be applied with slight modifications.

Let the positions of the sensor platforms at time t_k be $\mathbf{r}_{i,k} \in \mathbb{R}^2$ where $i = 1,2,...,N$. We assume that the measurement taken by the ith platform is modeled as

$$p_{i,k} = z(\mathbf{r}_{i,k}) + w(\mathbf{r}_{i,k}) + n_{i,k} \tag{1}$$

where $z(\mathbf{r}_{i,k})$ is the value of the field at $\mathbf{r}_{i,k}$, $n_{i,k} \sim \mathcal{N}(0,\sigma_i^2)$ are i.i.d. Gaussian noise, and $w(\mathbf{r}_{i,k})$ are spatially correlated Gaussian noise. We define the following $N \times 1$ vectors:

$$\mathbf{p}_k = \left[p_{i,k}\right], \; \mathbf{z}_k = \left[z(\mathbf{r}_{i,k})\right], \; \mathbf{n}_k = \left[n_{i,k}\right], \; \mathbf{w}_k = \left[w(\mathbf{r}_{i,k})\right], \tag{2}$$

and assume that $\mathbf{n}_k$ and $\mathbf{w}_k$ are stationary, i.e., their statistics are time invariant. These assumptions are idealizations for physical scalar fields in the ocean or atmosphere.

We define the problem of *cooperative exploration* as follows:

Problem 2.1: Given measurements $\mathbf{p}_k$ for all time t_k and the statistics of the noise $\mathbf{n}_k$ and $\mathbf{w}_k$, find an estimate for the field $z(\mathbf{r})$ that minimizes an error metric J.

The choice of the error metric J depends on application. In this paper, J is chosen to be the mean square error over spatial domain.

The function $z(\mathbf{r}_{i,k})$ can be locally approximated by a Taylor series. Let $\mathbf{r}_{c,k}$ be the center of the platform formation at time t_k i.e. $\mathbf{r}_{c,k} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{r}_{i,k}$. If $\mathbf{r}_{i,k}$ is close to $\mathbf{r}_{c,k}$, then it is sufficient to use the Taylor series up to second order. Let $z_{i,k} = z(\mathbf{r}_{i,k})$, then

$$\begin{aligned} z_{i,k} \approx\; & z(\mathbf{r}_{c,k}) + (\mathbf{r}_{i,k} - \mathbf{r}_{c,k})^T \nabla z(\mathbf{r}_{c,k}) \\ & + \frac{1}{2}(\mathbf{r}_{i,k} - \mathbf{r}_{c,k}^T)\nabla^2 z(\mathbf{r}_{c,k})(\mathbf{r}_{i,k} - \mathbf{r}_{c,k}) \end{aligned} \tag{3}$$

for $i = 1,2,...,N$. We are interested in estimates of $z(\mathbf{r}_{c,k})$, $\nabla z(\mathbf{r}_{c,k})$, and $\nabla^2 z(\mathbf{r}_{c,k})$. In addition to providing insights on the structure of the scalar field, these estimates are also used in the steering control for the center of the formation, as shown in our previous work [4].

A. The Measurement Equations

Let $\mathbf{s}_k = [z(\mathbf{r}_{c,k}), \nabla z(\mathbf{r}_{c,k})^T]^T$. Let C_k be the $N \times 3$ matrix defined by

$$C_k = \begin{bmatrix} 1 & (\mathbf{r}_{1,k} - \mathbf{r}_{c,k})^T \\ \vdots & \vdots \\ 1 & (\mathbf{r}_{N,k} - \mathbf{r}_{c,k})^T \end{bmatrix}. \tag{4}$$

Let D_k be the $N \times 4$ matrix with its ith row vector defined by $\frac{1}{2}((\mathbf{r}_{i,k} - \mathbf{r}_{c,k}) \otimes (\mathbf{r}_{i,k} - \mathbf{r}_{c,k}))^T$ where $\otimes$ is the Kronecker product. For any 2×2 matrix H, we use the notation $\vec{H}$ to represent a column vector defined by rearranging the elements of H as follows

$$\vec{H} = [H_{11}, H_{21}, H_{12}, H_{22}]^T. \tag{5}$$

Then the Taylor expansions (3) for all sensor platforms near $\mathbf{r}_{c,k}$ can be re-written in a vector form as

$$\mathbf{z}_k = C_k \mathbf{s}_k + D_k \vec{\nabla}^2 z(\mathbf{r}_{c,k}) \tag{6}$$

where $\vec{\nabla}^2 z(\mathbf{r}_{c,k})$ is a 4×1 column vector obtained by re-arranging elements of the Hessian $\nabla^2 z(\mathbf{r}_c, k)$ as defined by (5).

Suppose that $\vec{H}_{c,k}$ is an estimate for the Hessian $\vec{\nabla}^2 z(\mathbf{r}_{c,k})$ in vector form. Equation (1) can now be written as

$$\mathbf{p}_k = C_k \mathbf{s}_k + D_k \vec{H}_{c,k} + \mathbf{w}_k + D_k \mathbf{e}_k + \mathbf{n}_k \tag{7}$$

where $\mathbf{e}_k$ represents the error in the estimate of the Hessian. Let $W_k = E[\mathbf{w}_k \mathbf{w}_k^T]$, $U_k = E[\mathbf{e}_k \mathbf{e}_k^T]$, and $R_k = E[\mathbf{n}_k \mathbf{n}_k^T]$. The noise $\mathbf{w}_k$ is "colored" because it originates from the spatial correlation of the field. Let $E[\mathbf{w}_k \mathbf{w}_{k-1}^T] = V_k$. We suppose that W_k, R_k and V_k are known once the positions of the platforms are known. This assumption is reasonable in ocean and meteorology applications since the statistical properties of ocean fields and atmospheric fields are usually known from accumulated observational data over a long period of time. We also assume that U_k determined by the accuracy of the Hessian estimation algorithm is known.

B. The State Dynamics

As the center of the formation moves, the states $\mathbf{s}_k = [z(\mathbf{r}_{c,k}), \nabla z(\mathbf{r}_{c,k})^T]^T$ evolve according to the following equations:

$$\begin{aligned} z(\mathbf{r}_{c,k}) &= z(\mathbf{r}_{c,k-1}) + (\mathbf{r}_{c,k} - \mathbf{r}_{c,k-1})^T \nabla z(\mathbf{r}_{c,k-1}) \\ \nabla z(\mathbf{r}_{c,k}) &= \nabla z(\mathbf{r}_{c,k-1}) + H_{c,k-1}(\mathbf{r}_{c,k} - \mathbf{r}_{c,k-1}). \end{aligned} \tag{8}$$

Let $\mathbf{h}_{k-1} = [0, E[H_{c,k-1}(\mathbf{r}_{c,k} - \mathbf{r}_{c,k-1})]^T]^T$ and $A_{k-1}^{s} = \begin{bmatrix} 1 & (\mathbf{r}_{c,k} - \mathbf{r}_{c,k-1})^T \\ 0 & I_{2\times 2} \end{bmatrix}$. We then rewrite (8) as

$$\mathbf{s}_k = A_{k-1}^{s} \mathbf{s}_{k-1} + \mathbf{h}_{k-1} + \boldsymbol{\epsilon}_{k-1} \tag{9}$$

where we have introduced the $N \times 1$ noise vector $\boldsymbol{\epsilon}_{k-1}$ which accounts for positioning errors, estimation errors for the

Hessians, and errors caused by higher order terms omitted from the Taylor expansion. We assume that ϵ_{k-1} are i.i.d Gaussian with zero mean and known covariance matrix M_{k-1} that is positive definite.

C. The Noise Dynamics

The noise $\mathbf{w}_k$ in the measurement equation (7) is colored. The standard technique (c.f. [16]) to handle this issue is to model $\mathbf{w}_k$ as

$$\mathbf{w}_k = A^{\mathrm{w}}_{k-1}\mathbf{w}_{k-1} + \boldsymbol{\eta}_{k-1} \tag{10}$$

where $\boldsymbol{\eta}_{k-1}$ is white noise with positive definite correlation matrix $Q_k = E[\boldsymbol{\eta}_k\boldsymbol{\eta}_k^T]$. Because

$$\begin{aligned} V_k &= E[\mathbf{w}_k\mathbf{w}_{k-1}^T] = A^{\mathrm{w}}_{k-1}E[\mathbf{w}_{k-1}\mathbf{w}_{k-1}^T] = A^{\mathrm{w}}_{k-1}W_{k-1} \\ W_k &= E[\mathbf{w}_k\mathbf{w}_k^T] = A^{\mathrm{w}}_{k-1}W_{k-1}(A^{\mathrm{w}}_{k-1})^T + Q_{k-1}, \end{aligned} \tag{11}$$

we have

$$\begin{aligned} A^{\mathrm{w}}_{k-1} &= V_kW_{k-1}^{-1} \\ Q_{k-1} &= W_k - A^{\mathrm{w}}_{k-1}W_{k-1}(A^{\mathrm{w}}_{k-1})^T. \end{aligned} \tag{12}$$

Remark 2.2: State equation (9) reveals the major difference between the cooperative exploration problem considered in this paper and the tracking/coverage problems considered in [11], [13], [14]. Equation (9), fundamental to the cooperative exploration problem, is only valid for the formation and does not make sense for each individual node, since A^{s}_{k-1} and $\mathbf{h}_{k-1}$ depend on the location of all platforms in the formation. Therefore, the distributed Kalman filter algorithms for tracking and coverage in [11], [13], and [14], which achieve consensus between nodes and increase computation efficiency, are not applicable here. The central problem here is to use the minimum number of platforms with coordinated motion to estimate the field. For this purpose, we design the cooperative Kalman filter in the next section.

III. The Cooperative Kalman Filter

We observe from the information dynamics modeled by (9), (10), and (7) that if the Hessian related term $\mathbf{h}_{k-1}$ is known for all k, then the system belongs to the category for which Kalman filters can be constructed. We have shown in [4] that $\mathbf{h}_{k-1}$ can be estimated. Thus standard procedures can be followed to obtain a Kalman filter, which will be called the cooperative Kalman filter in this section because it can only be computed by a formation and its performance depends on the configuration of the formation. As the new contribution, we establish sufficient conditions that a formation must satisfy for the cooperative Kalman filter to converge.

A. Cooperative Kalman Filter Equations

The equations for Kalman filters are obtained by canonical procedures, the formulas are derived following textbooks [16]–[18]. We will not repeat these equations due to space limit.

In order to design a Kalman filter with colored measurement noise $\mathbf{w}_k$, a well-known method devised in [19] can be applied by defining a new measurement $\widetilde{\mathbf{p}}_k$ as $\widetilde{\mathbf{p}}_k = \mathbf{p}_k - A^{w}_{k-1}\mathbf{p}_{k-1}$. This gives a new equation for measurements:

$$\begin{aligned} \widetilde{\mathbf{p}}_k =& (C_kA^{\mathrm{s}}_{k-1} - A^{\mathrm{w}}_{k-1}C_{k-1})\mathbf{s}_{k-1} + C_k\mathbf{h}_{k-1} \\ &+ (D_k\vec{H}_{c,k} - A^{\mathrm{w}}_{k-1}\vec{H}_{c,k-1}) \\ &+ C_k\boldsymbol{\epsilon}_{k-1} + D_k\mathbf{e}_k - A^{\mathrm{w}}_{k-1}D_{k-1}\mathbf{e}_{k-1} \\ &+ \mathbf{n}_k - A^{\mathrm{w}}_{k-1}\mathbf{n}_{k-1}. \end{aligned} \tag{13}$$

The equations (9), (10), and (13) are now the state and the measurement equations for the case when $\mathbf{w}_k \neq 0$. The states are $[\mathbf{s}_k^T, \mathbf{w}_k^T]^T$, the output is $\widetilde{\mathbf{p}}_k$, the state noise is $\boldsymbol{\epsilon}_{k-1}$, and the observation noise is $C_k\boldsymbol{\epsilon}_{k-1} + D_k\mathbf{e}_k - A^{\mathrm{w}}_{k-1}D_{k-1}\mathbf{e}_{k-1} + \mathbf{n}_k - A^{\mathrm{w}}_{k-1}\mathbf{n}_{k-1}$. The Kalman filter design procedure for this case can be found in most textbooks and will not be repeated here.

B. Convergence of the Cooperative Kalman Filter

Kalman filters converge if the time-varying system dynamics are uniformly completely controllable and uniformly completely observable [7]. In our case, these conditions are determined by the number of platforms employed, the geometric shape of the platform formation, and the speed of each platform. We develop a set of constraints for these factors so that the uniformly complete controllability and observability conditions are satisfied, which then guarantees convergence of the cooperative Kalman filter.

Let $\Phi(k,j)$ be the state transition matrix from time t_j to t_k where $k > j$, then one must have $\Phi(k,j) = A^{\mathrm{s}}_{k-1}A^{\mathrm{s}}_{k-2}\cdots A^{\mathrm{s}}_{j}$ and $\Phi(j,k) = \Phi^{-1}(k,j)$. The following lemma follows from direct calculation.

Lemma 3.1: For $\Phi(k,j)$ as defined above and C_k as defined in (4), we have, for $k \neq j$,

$$\Phi(k,j) = \begin{bmatrix} 1 & (\mathbf{r}_{\mathrm{c},k} - \mathbf{r}_{\mathrm{c},j})^T \\ 0 & I_{2\times 2} \end{bmatrix} \tag{14}$$

and

$$C_k\Phi(k,j) = \begin{bmatrix} 1 & (\mathbf{r}_{1,k} - \mathbf{r}_{\mathrm{c},j})^T \\ \vdots & \vdots \\ 1 & (\mathbf{r}_{N,k} - \mathbf{r}_{\mathrm{c},j})^T \end{bmatrix}. \tag{15}$$

Remark 3.2: Note that this lemma holds for both $k > j$ and $k < j$. It applies to formations with any shape and any motion.

For clarity, we restate the definitions for uniformly complete controllability and uniformly complete observability in [7] using notations in this paper.

Definition 3.3: The state dynamics (9) are *uniformly completely controllable* if there exist $\tau_1 > 0$, $\beta_1 > 0$, and $\beta_2 > 0$ (independent of k) such that the controllability Grammian

$$\mathscr{C}(k, k-\tau_1) = \sum_{j=k-\tau_1}^{k} \Phi(k,j)M_{j-1}\Phi^T(k,j) \tag{16}$$

satisfies

$$\beta_1 I_{3\times 3} \leq \mathscr{C}(k, k-\tau_1) \leq \beta_2 I_{3\times 3} \tag{17}$$

for all $k > \tau_1$. Here M_{j-1} is the covariance matrix for state noise $\boldsymbol{\epsilon}_{j-1}$.

Definition 3.4: Suppose $\mathbf{w}_k = 0$ for all k. The state dynamics (9) together with the measurement equation (7) is *uniformly completely observable* if there exist $\tau_2 > 0$, $\beta_3 > 0$, and $\beta_4 > 0$ (independent of k) such that the observability Grammian

$$\mathscr{J}(k,k-\tau_2) = \sum_{j=k-\tau_2}^{k} \Phi^T(j,k) C_j^T [D_j U_j D_j^T + R_j]^{-1} C_j \Phi(j,k) \tag{18}$$

satisfies

$$\beta_3 I_{3\times 3} \le \mathscr{J}(k,k-\tau_2) \le \beta_4 I_{3\times 3} \tag{19}$$

for all $k > \tau_2$. Here U_j and R_j are covariance matrices for noises $\mathbf{e}_j$ and $\mathbf{n}_j$ respectively.

If $\mathbf{w}_k \neq 0$, the measurement equation is (13) instead of (7). Then the observability Grammian is

$$\mathscr{J}^{\mathrm{w}}(k,k-\tau_2) = \sum_{j=k-\tau_2}^{k} \Phi^T(j,k) \widetilde{C}_j^T \widetilde{R}_j^{-1} \widetilde{C}_j \Phi(j,k) \tag{20}$$

where $\widetilde{C}_j = C_j A_{j-1}^{\mathrm{s}} - A_{j-1}^{\mathrm{w}} C_{j-1}$ and

$$\begin{aligned}\widetilde{R}_j = {} & C_j M_{j-1} C_j^T + D_j U_j D_j^T + A_{j-1}^{\mathrm{w}} D_{j-1} U_{j-1} D_{j-1} A_{j-1}^{\mathrm{w}T} \\ & + R_j + A_{j-1}^{\mathrm{w}} R_{j-1} A_{j-1}^{\mathrm{w}T}.\end{aligned} \tag{21}$$

The condition for uniformly complete observability is

$$\beta_3 I_{3\times 3} \le \mathscr{J}^{\mathrm{w}}(k,k-\tau_2) \le \beta_4 I_{3\times 3}. \tag{22}$$

In the following discussions, we derive constraints on the formations so that the uniformly complete controllability and observability conditions are satisfied. The general procedure is to show that there exist positive real numbers β_1, β_2, ..., β_{28} that serve as time-independent bounds for various quantities. The actual value for these bounds do not affect the correctness of our arguments.

For uniformly complete controllability the following lemma holds.

Lemma 3.5: The state dynamics (9) are uniformly completely controllable if the following conditions are satisfied:

(Cd1) The symmetric matrix M_{j-1} is uniformly bounded i.e. $\beta_5 I \le M_{j-1} \le \beta_6 I$ for all j and for some constants $\beta_5, \beta_6 > 0$.

(Cd2) The speed of each platform is uniformly bounded i.e. $\left\| \mathbf{r}_{i,j} - \mathbf{r}_{i,j-1} \right\| \le \beta_7$ for all time j, for $i = 1, ..., N$, and for some constant $\beta_7 > 0$.

Proof: Due to condition (Cd1), the controllability Grammian satisfies $\beta_5 \sum_{j=k-\tau_1}^{k} \Phi(k,j)\Phi^T(k,j) \le \mathscr{C}(k,k-\tau_1)$ and $\mathscr{C}(k,k-\tau_1) \le \beta_6 \sum_{j=k-\tau_1}^{k} \Phi(k,j)\Phi^T(k,j)$ for any k and τ_1 such that $k > \tau_1$. We first observe that $\Phi(k,j)\Phi^T(k,j)$ is a positive semi-definite symmetric matrix for each j such that $k - \tau_1 \le j \le k$. If we can find uniform bounds for each of these matrices i.e. $\Phi(k,j)\Phi^T(k,j)$, we obtain an overall bound for the controllability Grammian.

We apply Lemma 3.1 to compute $\Phi(k,j)\Phi^T(k,j)$ i.e.

$$\Phi(k,j)\Phi^T(k,j) = \begin{bmatrix} 1 + \| \delta\mathbf{r}(k,j) \|^2 & (\delta\mathbf{r}(k,j))^T \\ \delta\mathbf{r}(k,j) & I_{2\times 2} \end{bmatrix} \tag{23}$$

where we define $\delta\mathbf{r}(k,j) = \mathbf{r}_{c,k} - \mathbf{r}_{c,j}$. The minimum eigenvalue of matrix (23) is

$$\lambda_{\min} = \frac{1}{2}\left(\| \delta\mathbf{r}(k,j) \|^2 + 2 - \sqrt{(\| \delta\mathbf{r}(k,j) \|^2 + 2)^2 - 4} \right)$$

and the maximum eigenvalue is

$$\lambda_{\max} = \frac{1}{2}\left(\| \delta\mathbf{r}(k,j) \|^2 + 2 + \sqrt{(\| \delta\mathbf{r}(k,j) \|^2 + 2)^2 - 4} \right).$$

Since (Cd2) is satisfied and $\delta\mathbf{r}(k,j)$ is the averaged movement over all platforms between time j and k, we must have $\| \delta\mathbf{r}(k,j) \| \le (k-j)\beta_7 \le \tau_1\beta_7$ for all $j \in [k-\tau_1, k]$. It is straightforward to show that $\lambda_{\min}$ assumes its minimum value when $\| \delta\mathbf{r}(k,j) \| = \tau_1\beta_7$. This minimum value is $\beta_8 = \frac{1}{2}\left((\tau_1\beta_7)^2 + 2 - \sqrt{(\tau_1\beta_7)^2 + 2)^2 - 4} \right)$. We can see that $\beta_8 > 0$. On the other hand, $\lambda_{\max}$ assumes its maximum value also when $\| \delta\mathbf{r}(k,j) \| = \tau_1\beta_7$. This maximum value is $\beta_9 = \frac{1}{2}\left((\tau_1\beta_7)^2 + 2 + \sqrt{(\tau_1\beta_7)^2 + 2)^2 - 4} \right)$, and $\beta_9 > 0$. Therefore, we conclude that $\beta_8 I_{3\times 3} \le \Phi(k,j)\Phi^T(k,j) \le \beta_9 I_{3\times 3}$ for all $j \in [k-\tau_1, k]$. Thus $\beta_5\tau_1\beta_8 I_{3\times 3} \le \mathscr{C}(k,k-\tau_1) \le \beta_6\tau_1\beta_9 I_{3\times 3}$. Let $\beta_1 = \beta_5\tau_1\beta_8$ and $\beta_2 = \beta_6\tau_1\beta_9$. Since β_1 and β_2 do not depend on k, we have proved the uniformly complete controllability claim using Definition 3.3. ■

By the arguments for proving Lemma 3.5, we have also proved the following lemma.

Lemma 3.6: Suppose condition (Cd2) is satisfied. Then there exist constants $\tau_1 > 0$, $\beta_8 > 0$, and $\beta_9 > 0$ such that the state transition matrices satisfy

$$\beta_8 I_{3\times 3} \le \Phi(i,j)\Phi^T(i,j) \le \beta_9 I_{3\times 3} \tag{24}$$

for all $i, j \in [k-\tau_1, k]$ and for all $k > \tau_1$.

To prove uniformly complete observability, we also need an elementary lemma that we do not show the proof.

Lemma 3.7: Suppose two 2×1 vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ form an angle γ such that $0 < \gamma < \pi$. Then the minimum eigenvalue $\lambda_{\min}$ of the 2×2 matrix $A = \mathbf{a}_1\mathbf{a}_1^T + \mathbf{a}_2\mathbf{a}_2^T$ is strictly positive i.e. $\lambda_{\min} > 0$.

We have the following lemma regarding uniformly complete observability of a moving formation.

Lemma 3.8: Suppose $\mathbf{w}_k = 0$ for all k. The state dynamics (9) with the measurement equation (7) are uniformly completely observable if (Cd2) and the following conditions are satisfied:

(Cd3) The symmetric matrices R_j and U_j are uniformly bounded, i.e., $\beta_{10} I_{N\times N} \le R_j \le \beta_{11} I_{N\times N}$ and $0 \le U_j \le \beta_{12} I_{N\times N}$ for all j and for some constants $\beta_{10}, \beta_{11}, \beta_{12} > 0$.

(Cd4) The distance between each platform and the formation center is uniformly bounded from both above and below, i.e., $\beta_{13} \le \left\| \mathbf{r}_{i,j} - \mathbf{r}_{c,j} \right\| \le \beta_{14}$ for all j, for $i = 1, 2, ..., N$, and for some constants $\beta_{13}, \beta_{14} > 0$.

(Cd5) There exists a constant time difference τ_2 and for all $k > \tau_2$, there exist time instances $j_1, j_2 \in [k-\tau_2, k]$ where $j_1 < j_2$, as well as two platforms indexed by i_1 and i_2, such that *one* of the following two conditions is satisfied:

(Cd5.1) The two vectors, $\mathbf{r}_{i_1,j_1}-\mathbf{r}_{c,j_1}$ and $\mathbf{r}_{c,j_1}-\mathbf{r}_{c,j_2}$ form an angle γ_1 that is uniformly bounded away from 0 or π. In other words, there exists a positive constant $\beta_{15}<1$ such that $\sin(\gamma_1/2)\geq\beta_{15}$.

(Cd5.2) The two vectors, $\mathbf{r}_{i_1,j_1}-\mathbf{r}_{c,j_1}$ and $\mathbf{r}_{i_2,j_2}-\mathbf{r}_{c,j_2}$ form an angle γ_2 that is uniformly bounded away from 0 or π. In other words, there exists a positive constant $\beta_{15}<1$ such that $\sin(\gamma_2/2)\geq\beta_{15}$.

Proof: Condition (Cd3) implies that U_j is positive semi-definite, and condition (Cd4) implies that every component of D_j is bounded above. Hence the matrix $D_jU_jD_j^T$ is a positive semi-definite matrix with its maximum eigenvalue bounded above. Also from (Cd3), R_j is a positive definite symmetric matrix. Therefore, Weyl's theorem (c.f. [20], Theorem 4.3.1) that states the eigenvalues of the sum of two Hermitian matrices are bounded above by the sum of the two maximum eigenvalues and bounded below by the sum of the two minimum eigenvalues can be applied to $R_j+D_jU_jD_j^T$. This implies that there exist positive constants $\beta_{16},\beta_{17}>0$ such that $\beta_{16}I_{N\times N}\leq(R_j+D_jU_jD_j^T)\leq\beta_{17}I_{N\times N}$ where $\beta_{16}\geq\beta_{10}$ and $\beta_{17}\geq\beta_{11}$. Thus, one must have $\beta_{17}^{-1}\sum_{j=k-\tau_2}^{k}\Phi^T(j,k)C_j^TC_j\Phi(j,k)\leq\mathscr{J}(k,k-\tau_2)\leq\beta_{16}^{-1}\sum_{j=k-\tau_2}^{k}\Phi^T(j,k)C_j^TC_j\Phi(j,k)$ for all $k>\tau_2$. Next, we prove the existence of positive uniform upper and lower bounds for $\sum_{j=k-\tau_2}^{k}\Phi^T(j,k)C_j^TC_j\Phi(j,k)$ for all $k>\tau_2$.

First for the upper bound, according to Lemma 3.1, we can compute

$$\Phi^T(j,k)C_j^TC_j\Phi(j,k)=\begin{bmatrix} N & (\mathbf{r}_{c,j}-\mathbf{r}_{c,k})^T \\ (\mathbf{r}_{c,j}-\mathbf{r}_{c,k}) & \sum_{i=1}^N(\mathbf{r}_{i,j}-\mathbf{r}_{c,k})(\mathbf{r}_{i,j}-\mathbf{r}_{c,k})^T \end{bmatrix}.$$

The condition (Cd2) and (Cd4) imply that each component of the above matrix is bounded above. Hence there exists $\beta_{18}>0$ such that $\Phi^T(j,k)C_j^TC_j\Phi(j,k)\leq\beta_{18}I_{3\times3}$.

We now use condition (Cd5) to argue that there exists the lower bound $\beta_{19}>0$ such that $\beta_{19}I_{3\times3}\leq\sum_{j=k-\tau_2}^{k}\Phi^T(j,k)C_j^TC_j\Phi(j,k)$. Consider the two time instances indexed by j_1 and j_2 as given by condition (Cd5). It is sufficient to show that the matrix $\mathscr{I}$ defined by

$$\mathscr{I}=\Phi^T(j_1,k)C_{j_1}^TC_{j_1}\Phi(j_1,k)+\Phi^T(j_2,k)C_{j_2}^TC_{j_2}\Phi(j_2,k) \tag{25}$$

satisfies $\mathscr{I}\geq\beta_{19}I_{3\times3}$.

Because $\Phi(j_1,k)=\Phi(j_1,j_2)\Phi(j_2,k)$, we have $\mathscr{I}=\Phi^T(j_1,k)\mathscr{I}_1\Phi^T(j_1,k)$ where

$$\mathscr{I}_1=\Phi^T(j_1,j_2)C_{j_1}^TC_{j_1}\Phi(j_1,j_2)+C_{j_2}^TC_{j_2}. \tag{26}$$

By direct calculation one can verify that

$$C_{j_2}^TC_{j_2}=\begin{bmatrix} N & 0 \\ 0 & \sum_{i=1}^N(\mathbf{r}_{i,j_2}-\mathbf{r}_{c,j_2})(\mathbf{r}_{i,j_2}-\mathbf{r}_{c,j_2})^T \end{bmatrix}. \tag{27}$$

Using Lemma 3.1 and the fact that $\sum_{i=1}^N(\mathbf{r}_{i,j_1}-\mathbf{r}_{c,j_1})=0$, we have

$$\begin{aligned}&\Phi^T(j_1,j_2)C_{j_1}^TC_{j_1}\Phi(j_1,j_2)\\ &=\begin{bmatrix} 1 & (\mathbf{r}_{c,j_1}-\mathbf{r}_{c,j_2})^T \\ \mathbf{r}_{c,j_1}-\mathbf{r}_{c,j_2} & (\mathbf{r}_{c,j_1}-\mathbf{r}_{c,j_2})(\mathbf{r}_{c,j_1}-\mathbf{r}_{c,j_2})^T \end{bmatrix}\\ &+\begin{bmatrix} N-1 & 0 \\ 0 & \sum_{i=1}^N(\mathbf{r}_{i,j_1}-\mathbf{r}_{c,j_1})(\mathbf{r}_{i,j_1}-\mathbf{r}_{c,j_1})^T \end{bmatrix}\\ &+\begin{bmatrix} 0 & 0 \\ 0 & (N-1)(\mathbf{r}_{c,j_1}-\mathbf{r}_{c,j_2})(\mathbf{r}_{c,j_1}-\mathbf{r}_{c,j_2})^T \end{bmatrix}.\end{aligned} \tag{28}$$

Then the matrix $\mathscr{I}_1$ can be obtained by adding (27) and (28) together. Considering the platforms i_1 and i_2 in (Cd5.1) and (Cd5.2), we can further decompose $\mathscr{I}_1$ as the sum of two matrices: $\mathscr{I}_1=\mathscr{I}_2+\mathscr{I}_3$ where $\mathscr{I}_2=\begin{bmatrix} 1 & 0 \\ 0 & \mathscr{I}_4 \end{bmatrix}$ with

$$\begin{aligned}\mathscr{I}_4=&(\mathbf{r}_{i_1,j_1}-\mathbf{r}_{c,j_1})(\mathbf{r}_{i_1,j_1}-\mathbf{r}_{c,j_1})^T\\ &+(\mathbf{r}_{i_1,j_2}-\mathbf{r}_{c,j_2})(\mathbf{r}_{i_1,j_2}-\mathbf{r}_{c,j_2})^T\\ &+(\mathbf{r}_{c,j_1}-\mathbf{r}_{c,j_2})(\mathbf{r}_{c,j_1}-\mathbf{r}_{c,j_2})^T,\end{aligned} \tag{29}$$

and $\mathscr{I}_3$ is a positive semi-definite matrix.

Because either condition (Cd5.1) or condition (Cd5.2) is satisfied, according to Lemma 3.7, there exists $\beta_{21}>0$ such that the matrix $\mathscr{I}_4\geq\beta_{21}I_{2\times2}$. Therefore, using the Weyl's theorem (c.f. [20], Theorem 4.3.1) we conclude that there exists $\beta_{20}>0$ such that $\mathscr{I}_1\geq\beta_{20}I_{3\times3}$. Then Lemma 3.6 guarantees the existence of $\beta_{19}>0$ such that $\mathscr{I}\geq\beta_{19}I_{3\times3}$, which further implies that $\mathscr{J}(k,k-\tau_2)\geq\beta_{19}I_{3\times3}$.

Because both the uniform upper and lower bounds for the observability Grammian $\mathscr{J}(k,k-\tau_2)$ exist for all $k>\tau_2$, we have proved the uniformly complete observability claim. ■

We now consider the case when the colored noise $\mathbf{w}_k\neq0$. The following lemma establishes the sufficient conditions for uniformly complete observability.

Lemma 3.9: The state dynamics (9) and (10) with the measurement equation (13) are uniformly completely observable if (Cd2), (Cd4), and the following conditions are satisfied:

(Cd6) The symmetric matrix $\widetilde{R}_j$ is uniformly bounded i.e. $\beta_{22}I_{N\times N}\leq\widetilde{R}_j\leq\beta_{23}I_{N\times N}$ for all j and some positive constants β_{22} and β_{23}.

(Cd7) The matrix A_{j-1}^{w} and the matrix C_{j-1} satisfy $\beta_{24}I_{N\times N}\leq(I_{N\times N}-A_{j-1}^{\mathrm{w}})^T(I_{N\times N}-A_{j-1}^{\mathrm{w}})^T\leq\beta_{25}I_{N\times N}$ and $\beta_{26}I_{N\times N}\leq C_{j-1}^TC_{j-1}\leq\beta_{27}I_{N\times N}$ for some positive constants $\beta_{24},\beta_{25},\beta_{26}$ and β_{27}.

(Cd8) The constants β_7 in (Cd2) and the constants β_{24},β_{26} in (Cd7) satisfy $\beta_7\sqrt{N}+\beta_{28}<\sqrt{\beta_{24}\beta_{26}}$ for some positive constant β_{28}.

Proof: Condition (Cd6) implies that $\beta_{23}^{-1}\sum_{j=k-\tau_2}^{k}\Phi^T(j,k)\widetilde{C}_j^T\widetilde{C}_j\Phi^T(j,k)\leq\mathscr{J}^{\mathrm{w}}(k,k-\tau_2)$ and $\mathscr{J}^{\mathrm{w}}(k,k-\tau_2)\leq\beta_{22}^{-1}\sum_{j=k-\tau_2}^{k}\Phi^T(j,k)\widetilde{C}_j^T\widetilde{C}_j\Phi^T(j,k)$.

Consider $\widetilde{C}_j=C_jA_{j-1}^{\mathrm{s}}-A_{j-1}^{\mathrm{w}}C_{j-1}$. Using Lemma 3.1 we have $C_j A^{\mathrm{s}}_{j-1} = C_{j-1} + \delta C_j$ where

$$\delta C_j = \begin{bmatrix} 0 & (\mathbf{r}_{1,j} - \mathbf{r}_{\mathrm{c},j-1})^T \\ \vdots & \vdots \\ 0 & (\mathbf{r}_{N,j} - \mathbf{r}_{\mathrm{c},j-1})^T \end{bmatrix}. \quad (30)$$

Therefore, $\widetilde{C}_j = (I_{N\times N} - A^{\mathrm{w}}_{j-1})C_{j-1} + \delta C_j$. Applying the Hoffman-Wielandt theorem ([20], Theorem 7.3.8), we have

$$\Big| \sqrt{\lambda_{\min}(\widetilde{C}_j^T \widetilde{C}_j)} - \sqrt{\lambda_{\min}(C_{j-1}^T (I_{N\times N} - A^{\mathrm{w}}_{j-1})^T (I_{N\times N} - A^{\mathrm{w}}_{j-1}) C_{j-1})} \Big| \le \sqrt{\mathrm{trace}(\delta C_j \delta C_j^T)}. \quad (31)$$

Thus using condition (Cd8), we have

$$\sqrt{\lambda_{\min}(\widetilde{C}_j^T \widetilde{C}_j)} \ge \sqrt{\lambda_{\min}(C_{j-1}^T (I_{N\times N} - A^{\mathrm{w}}_{j-1})^T (I_{N\times N} - A^{\mathrm{w}}_{j-1}) C_{j-1})} - \mathrm{trace}(\delta C_j \delta C_j^T) \ge \sqrt{\beta_{24}\beta_{26}} - \beta_7 \sqrt{N} > \beta_{28}. \quad (32)$$

Therefore $\sum_{j=k-\tau_2}^{k} \Phi^T(j,k) \widetilde{C}_j^T \widetilde{C}_j \Phi^T(j,k)$ is uniformly bounded below, away from singular matrices. It is also uniformly bounded above by conditions (Cd2), (Cd4) and (Cd7). Hence $\mathscr{I}^{\mathrm{w}}(k, k-\tau_2)$ is uniformly bounded below, away from singular matrices, and above. ■

With Lemmas 3.5, 3.8, and 3.9 justified, the following theorems can be viewed as corollaries of Theorem 7.4 in [7].

Theorem 3.10: Suppose $\mathbf{w}_k = 0$ for all k. Consider the state dynamics (9) with the measurement equation (7). If the conditions (Cd1)-(Cd5) are satisfied, then the cooperative Kalman filter converges and the error covariance matrix P_k is bounded as $k \to \infty$.

Theorem 3.11: Consider the state dynamics (9) and (10) with the measurement equation (13). If the conditions (Cd1)-(Cd2), (Cd4) and (Cd6)-(Cd8) are satisfied, then the cooperative Kalman filter for this case converges and the error covariance matrix is bounded as $k \to \infty$.

C. Formation design principles

The conditions (Cd1)-(Cd8) have provided us the following intuitive guidelines for formation design to yield successful cooperative Kalman filters.

1) If $N \ge 3$, there is no penalty in fixing the orientation of the formation, as long as the shape is nonsingular. A singular formation occurs when all platforms are on a straight line or collapse to a point. In fact, if the formation is singular only occasionally, the Kalman filter will still converge.
2) If $N = 2$ or a line formation is desired, then one should make the orientation of the line change over time, such as in a rocking or rolling motion.
3) The speed of the platforms needs to be bounded from both above and below to guarantee the controllability and observability conditions at the same time. Such bounds depend on the strength of the error covariance matrices.

IV. Summary

We formulate the cooperative exploration problem and introduce a cooperative Kalman filter to estimate local structures of an unknown scalar field in the plane. We provide sufficient conditions for the cooperative Kalman filter to converge. To satisfy these conditions, we may use at least two sensor platforms. There is an advantage of using three or more platforms since the orientation of the formation can be kept fixed, reducing the speed variations for each platform.

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