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Integral Controllability and Integrity for Uncertain Systems

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***Abstract*—It is well known that the relative gain array (RGA) and the determinant of the gain matrix provide useful information about integral controllability and integrity (e.g., failure tolerance), which are important issues in decentralized control. The RGA also gives information about robustness with respect to modeling errors and input uncertainty. Almost exclusively only nominal models have been considered in previous studies and applications of these methods. Not until recently have there been attempts to consider model uncertainty more explicitly. However, the methods proposed in these studies tend to find uncertainty bounds that are too wide. In this paper a more accurate procedure based on sensitivity analysis is developed for studying the effect of model errors. Independent gain uncertainty as well as more structured uncertainty can be handled. The method is well suited for deriving tight bounds on the tolerable uncertainty.**

I. Introduction

The choice of control configuration for decentralized control has long been a challenging topic of great practical relevance. One of the earliest tools for this purpose was the relative gain array (RGA) [1], and it is still by far the most widely used tool. Originally, the RGA was introduced as a measure of interaction and the idea was to choose a control configuration where the interactions are minimized.

Further developments have shown that the RGA is not a very reliable interaction measure. However, it provides solid information about fundamental properties such as system integrity (failure tolerance) and robustness with respect to modeling errors and input uncertainty [2]–[4]. All these are important issues in decentralized control. Another crucial issue in decentralized control is the question of integral controllability. Useful necessary conditions (e.g., [5], [6]) as well as sufficient ones (e.g., [7]) have been derived.

Model uncertainty has not received much attention in the study of integral controllability and integrity. Only recently, the subject and in particular the usability of the RGA in the face of model uncertainty has been tackled [8], [9]. However, the proposed procedures are not accurate; the obtained bounds on the tolerable uncertainty tend to be too wide. This was also noted in [10].

In this paper, we deal with the problem of model uncertainty in a more direct way than done in [8] and [9]. Useful expressions for the sensitivity to model uncertainty are derived both for the RGA and the determinant of the gain matrix. These expressions are instrumental in the study of the effects of model uncertainty. In particular, an exact bound on the tolerable uncertainty is derived for an example studied in [8], [9]. The procedure is applicable to independent uncertainties in the process gains as well as to more structured uncertainty.

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II. Integral Control and Integrity

In this section we discuss some basic issues in decentralized control and summarize known conditions for integral controllability and integrity.

A. Desirable Features in Decentralized Control

In process control it is desirable to use controllers with integral action to enable good disturbance rejection and tight setpoint following. The possibility of retuning controllers and even taking controllers in a decentralized control system out of service (and back) without endangering the stability of the remaining control system are also important issues.

In a fully decentralized control system the plant is controlled by a set of single-input single-output (SISO) controllers. We consider a given control configuration, where the variable parings between the plant and the controllers are fixed (for the moment). It is assumed that all controllers contain integral action. We now define some relevant control system properties (adapted from [7]).

The system (i.e., the plant with the given control configuration) is *Integral Controllable* (IC) if it is stable with all controllers operating and suitably tuned and remains stable when the gains of all controllers are detuned simultaneously by the same factor ε, $0 < \varepsilon \le 1$. The system is *Integral Controllable with Integrity* (ICI), if it is IC, and remains IC, if any number and combination of controllers are taken out of service (i.e., put on manual). The system is *Decentralized Integral Controllable* (DIC) if it is ICI and remains stable when any number of controllers are detuned by individual factors ε_i, $0 \le \varepsilon_i \le 1$.

Usually DIC is the desired property, but besides being the most demanding property, it is also the one most difficult to ascertain because of the complexity of the problem (see, e.g.,[11]). It is easier to investigate ICI properties, and because ICI is a necessary condition for DIC, one often starts with an ICI analysis.

B. Conditions for Integral Controllability

The stability of the controlled system is obviously determined by the roots of the closed-loop characteristic equation. Let $G(s)$ denote the transfer function of the plant and $s^{-1}K(s)$ the transfer function of the controller. If $G(s)K(s)$ is stable (and contains no pole-zero cancellations) the characteristic equation may be written as

$$\det\left(sI + G(s)K(s)\right) = 0 \ . \tag{1}$$

Here, we also assume that $G(s)K(s)$ is proper. It is an interesting fact that it is then sufficient to consider the real-valued matrix $G(0)K(0)$ to determine the closed-loop stability and various controllability properties defined above [2], [5], [6], [11]. In the sequel, we only consider cases where it is sufficient to use steady-state gain matrices. Hence, the argument "0" will be omitted.

A system is integral controllable if all eigenvalues of GK are in the open right-half complex plane [2],[6]. According to [12], nonzero eigenvalues on the imaginary axis can also be allowed if the controllers contain a proportional part. Since the determinant of a matrix is equal to the product of the eigenvalues of the matrix, a necessary IC condition is

$$\det(GK) > 0 \ . \tag{2}$$

We are mainly interested in decentralized control, which means that the controller has a certain multiloop SISO structure. This implies that G and K are square. To simplify the notation and the formulation of various statements and interpretations, it is assumed that the inputs and the outputs are ordered so that K is diagonal for the control configuration under consideration. If needed, we also scale columns of G by –1 so that all gains along the diagonal of G are nonnegative. Apart from other requirements, we want every control loop to be stable by itself. With the above conditioning of G, this means that all controller gains have to be positive. Since $\det(GK) = \det(G)\det(K)$, (2) then simplifies to

$$\det(G) > 0 \, . \tag{3}$$

Equation (3) applies to the case when all loops are closed. If certain loops are open, a similar criterion applies for the subsystem of closed control loops. Let M denote the set of all possible subsystems along the diagonal of G and let G_m be the gain matrix of a given subsystem $m \in M$. Then

$$\det(G_m) > 0 \ , \quad \forall m \in M \ , \tag{4}$$

is a necessary condition for IC with full integrity.

III. Basic Controllability Analysis

In this section we summarize the basic methodology for analysis of integral controllability and integrity by means of the Relative Gain Array (RGA) and related methods.

A. *The Relative Gain Array*

When control loops are closed, the gains in other parts of the system are affected due to interaction. As a measure of this interaction, the relative gain was introduced [1]. The relative gain for the input-output pairing $u_j - y_i$ is defined

$$\lambda_{ij} = g_{ij} \big/ \tilde{g}_{ij} \ , \tag{5}$$

where g_{ij} is the open-loop gain between y_i and u_j, $\tilde{g}_{ij}$ is the gain between y_i and u_j when all other outputs are controlled by the remaining inputs. This control is perfect in the steady state due to integral action. If the number of inputs/outputs is n, the relative gains for all possible input-output pairings can be arranged into an $n \times n$ matrix Λ, called the relative gain array (RGA). Because $\tilde{g}_{ij}(G^{-1})_{ji} = 1$, it follows that the RGA can be calculated conveniently according to

$$\Lambda(G) = G \circ G^{-\mathrm{T}} \ , \tag{6}$$

where $G^{-\mathrm{T}}$ is the transpose of G^{-1} and $\circ$ denotes multiplication of corresponding elements in the two matrices.

The RGA can be used for pairing inputs and outputs in a decentralized control configuration. The basic rule is to choose pairings on positive relative gains as close to 1 as possible. It is well known that this rule is not very reliable for systems larger than 2×2, even when dynamics is not an issue, but it usually provides a good starting point. After a control configuration has been chosen for further study, inputs can be rearranged and G can be conditioned as described above to obtain a diagonal pairing and positive diagonal gains.

According to Cramer's Rule

$$(G^{-1})_{ji} = (-1)^{i+j} \det(G^{ij}) \big/ \det(G) \ , \tag{7}$$

where G^{ij} is the matrix obtained by deleting row i and column j from G. From (6) it then follows that the relative gain λ_{ij} can be expressed as

$$\lambda_{ij}(G) = (-1)^{i+j} g_{ij} \det(G^{ij}) \big/ \det(G) \ , \tag{8}$$

which gives a connection to the integral controllability and integrity conditions introduced above.

Assume that we choose pairings with $\lambda_{ii}(G) > 0$, $i = 1, \ldots, n$, and that $\det(G) > 0$ as required by (3). Since $g_{ii} > 0$ and $(-1)^{i+i} = 1$, it follows from (8) that all principal subsystems G^{ii} satisfy $\det(G^{ii}) > 0$, which is a necessary condition for integral controllability of every subsystem G^{ii} when loop $y_i - u_i$ is open.

We can calculate the RGA $\Lambda(G^{ii})$ for every subsystem of size $(n-1) \times (n-1)$. If all relative gains on the diagonal of these subsystems are positive, we know from the previous paragraph that every subsystem of size $(n-2) \times (n-2)$ has a positive determinant. We can continue this process down to subsystems of size 2×2 to check the necessary determinant criterion for integral controllability of all possible subsystems. If any determinant is nonpositive, the system does not have full integrity. In that case, we also find out in which subsystem the problem lies. Furthermore, it reveals whether a sign change of a relative gain occurs via 0 or $\pm\infty$. The problem of detecting sign changes was also considered in [8] and [9].

Of course, we could also check the sign of the determinant of all possible subsystems by direct calculation of the determinant. Computationally, this would probably be less demanding than RGA calculation. However, the RGA also gives useful information besides integral controllability and integrity. Large or very small RGA values in some subsystem, or values that imply another pairing than the chosen one, could indicate performance problems.

As mentioned, the determinant criterion is only a necessary condition. A necessary and sufficient condition is that a system is ICI if and only if there is a K such that all eigenvalues of GK are in the closed, nonzero, right half plane. For 3×3 systems it has been possible to derive

$$\sqrt{\lambda_{11}}+\sqrt{\lambda_{22}}+\sqrt{\lambda_{33}}>1 \tag{9}$$

as a necessary and sufficient DIC condition involving only relative gains [13]. Obviously, DIC also requires that (9) applies to all subsystems of size 3×3 in a larger system.

B. The Partial Relative Gain

The RGA can be used to obtain information about a subsystem G_m when the rest of the system is uncontrolled. One might also obtain useful information by considering G_m when the rest of the system is controlled. For this purpose, [14], [15] introduced the partial relative gain (PRG) array

$$\Lambda_m^{\mathrm{P}}(G)=\Lambda(\tilde{G}_m)=\tilde{G}_m\circ(\tilde{G}_m)^{-\mathrm{T}}, \tag{10}$$

where $\tilde{G}_m$ denotes the gain matrix of subsystem m when the rest of the loops are closed. Since $\tilde{G}_m$ is related to G by

$$(\tilde{G}_m)^{-\mathrm{T}}=(G^{-\mathrm{T}})_m, \tag{11}$$

and $\Lambda(G)=\Lambda(G^{-\mathrm{T}})$, the PRG can also be expressed as

$$\Lambda_m^{\mathrm{P}}(G)=\Lambda((G^{-\mathrm{T}})_m). \tag{12}$$

According to this relationship, large PRG values for subsystem m implies an ill-conditioned gain matrix inverse and thus an ill-conditioned subsystem which is difficult to control. This was demonstrated by an example in [14], [15]. An ill-conditioned subsystem m does not necessarily mean that the relative gains of $\Lambda(G_m)$ or $\Lambda_m(G)$ have to be large.

The PRG can also be related to relative gains of the open-loop system according to

$$\lambda_{ij}(\tilde{G}_m)=\lambda_{ij}(G)\big/\lambda_{ij}(G^{m^{ij}}), \tag{13}$$

where $G^{m^{ij}}$ is the gain matrix for the subsystem outside m except for the loop $i-j$, which is included (and which belongs to m). Here the index "ij" refers to the same loop in all (sub)systems, not necessarily to the position in a given matrix. If $\lambda_{ij}(G^{m^{ij}})$ is smallest of the three relative gains in (13), $\lambda_{ij}(\tilde{G}_m)$ will be the largest, and *vice versa*.

As can also be seen from (13), the PRG does not provide any information about ICI that cannot be obtained from the ordinary RGA for various subsystems. However, if $\lambda_{ij}(G)$ and $\lambda_{ij}(\tilde{G}_m)$ are calculated, $\lambda_{ij}(G^{m^{ij}})$ can be determined from (13) instead of from a separate RGA calculation. If the full system has size $n\times n$, there are n subsystems of size $(n-1)\times(n-1)$ and $n(n-1)/2$ subsystems of size 2×2. The relative gains in all these subsystems are given by n RGA and n PRG calculations.

C. The Block Relative Gain

Fully decentralized control is a special case of block decentralized control, where multiple-input, multiple-output (MIMO) control is used for part of the system. The block relative gain (BRG), defined as

$$\Lambda_m^{\mathrm{B}}(G)=G_m(\tilde{G}_m)^{-1}=G_m(G^{-\mathrm{T}})_m^{\mathrm{T}}, \tag{14}$$

has been introduced as a measure for studying the feasibility of such control structures [16], [17].

It is clear that determinant criteria such as (3) and (4) are necessary conditions for integral stability and integrity also in the case of block-decentralized control [17]. It can be shown that

$$\det\left(\Lambda_{ij}^{\mathrm{B}}(G)\right)=(-1)^{i+j}\det(G_{ij})\det(G^{ij})\big/\det(G), \tag{15}$$

where "ij" refers to the ij th block. This expression is analogous with that for the relative gain, (8). Results for the RGA concerning the determinant of gain matrices are thus directly applicable to the BRG.

IV. Independent Gain Uncertainty

In this section we study the sensitivity of the determinant conditions and the relative gain to independent variations in the open-loop gains. This provides useful information also for simultaneous gain variations.

A. Preliminary Analysis

The determinant of a matrix G can be expressed in terms of submatrix determinants (i.e., minors) by expansion along a row or a column of the matrix. If we choose to expand along a row, we have

$$\det(G)=\sum_{\forall\ell}(-1)^{i+\ell}g_{i\ell}\det(G^{i\ell}), \quad \text{any } i. \tag{16}$$

If g_{ij} changes by the amount Δg_{ij}, the new gain matrix $G+\Delta G$ satisfies

$$\det(G+\Delta G)=(-1)^{i+j}\Delta g_{ij}\det(G^{ij})+\det(G). \tag{17}$$

This means that $\det(G+\Delta G)=0$, i.e., the gain matrix becomes singular if $\det(G^{ij})\neq 0$ and

$$\Delta g_{ij}=-(-1)^{i+j}\det(G)\big/\det(G^{ij})=-g_{ij}\big/\lambda_{ij}. \tag{18}$$

This condition has been derived before, of course [3].

The determinant of G may become zero only if the gain g_{ij} changes in the direction indicated by (18). It is also clear that only a small change can be tolerated if $|\lambda_{ij}|$ is large, but if $|\lambda_{ij}|<1$, the change can be larger than 100 %.

Of course, if there are simultaneous changes in several gains, G may become singular for smaller changes. However, even in the case of simultaneous changes, (18) gives the worst direction.

B. Effect of Gains on Determinants

In this section we show how bounds on the allowable gain changes can be determined in the case of independent simultaneous changes. Proofs of the theorems are omitted due to space limitations.

Theorem 1. Sensitivity of $\det(G)$ *to gain variations.*

$$\partial \det(G) \big/ \partial g_{ij} = (-1)^{i+j} \det(G^{ij}) \tag{19a}$$

If $\det(G) \neq 0$, we can equivalently write[1]

$$\partial \det(G) \big/ \partial G^{\mathrm{T}} = \det(G) G^{-\mathrm{T}} \tag{19b}$$

$$\partial \det(G) \big/ \partial g_{ij} = \det(G) \lambda_{ij} \big/ g_{ij} \,,\ g_{ij} \neq 0 \tag{19c}$$

$$G \circ (\partial \det(G) \big/ \partial G^{\mathrm{T}}) = \det(G) \Lambda(G) \,,\ \forall g_{ij} \neq 0 \,. \tag{19d}$$

We note that (19a) and (19b) directly show in which direction a gain change will move $\det(G)$. Furthermore, (19c) and (19d) show that the relative sensitivity is given by the RGA.

Let us now assume that g_{ij} is bounded by

$$\underline{g}_{ij} \leq g_{ij} \leq \overline{g}_{ij} \ ,\ \ \forall g_{ij} \,. \tag{20}$$

Let G_0 denote the nominal gain matrix and consider

$$G = G_0 + \alpha W \tag{21}$$

$$w_{ij} = \begin{cases} \overline{g}_{ij} - (G_0)_{ij} & \text{if } (G^{-\mathrm{T}})_{ij} < 0 \\ 0 & \text{if } (G^{-\mathrm{T}})_{ij} = 0 \\ \underline{g}_{ij} - (G_0)_{ij} & \text{if } (G^{-\mathrm{T}})_{ij} > 0. \end{cases} \tag{22}$$

When α is increased from 0, the value of $\det(G)$ will start to decrease until $\det(G) = 0$ is reached for some $\alpha = \overline{\alpha}$. From (19b) it follows that $\alpha = \overline{\alpha}$ denotes the largest uncertainty of the form (21) that can be tolerated if there are no sign changes in the elements of $G^{-\mathrm{T}}$ for $\alpha \in (0, \overline{\alpha})$. If $\overline{\alpha} > 1$, $\det(G) > 0$ for all uncertainties restricted by (20).

If there are sign changes in $G^{-\mathrm{T}}$ for some $\alpha \in (0, \overline{\alpha})$, there is another set of uncertainty directions which will yield $\det(G) = 0$ for some $\alpha < \overline{\alpha}$. In this case, $\overline{\alpha}$ is only an upper bound of the tolerable uncertainty.

A sign change of $(G^{-\mathrm{T}})_{ij}$ indicates that the opposite bound of g_{ij} should be considered in the definition of w_{ij}. As shown in the application, this is an effective way of reducing the number of "matrix vertices" that need to be checked. For an $n \times n$ matrix there is originally 2^n possibilities, but if there are two sign changes, for example, the number is reduced to 2.

Obviously, $\alpha = \overline{\alpha}$ satisfying $\det(G_0 + \alpha W) = 0$ can be found numerically or graphically by plotting $\det(G_0 + \alpha W)$ against α. Furthermore, $G^{-\mathrm{T}}$ could be plotted against α to check for sign changes of the sensitivities.

[1] Here we use the convention that the derivative of a scalar with respect to a column vector results in a row vector.

However, $\overline{\alpha}$ can also be determined by an eigenvalue calculation. Since

$$\det(G) = \det(G_0 + \alpha W) = \det(G_0) \det(I + \alpha W G_0^{-1}) \,, \tag{23}$$

the smallest $\alpha = \overline{\alpha} > 0$ that satisfies $\det(I + \alpha W G_0^{-1}) = 0$ is

$$\overline{\alpha} = -1 / \underline{\sigma}(W G_0^{-1}) \,, \tag{24}$$

where $\underline{\sigma}(\cdot)$ is the smallest eigenvalue of $(\cdot)$, which in this case will be negative.

C. Effect of Gains on Relative Gains

As shown by (8), relative gain calculation can also be used to determine sign changes of subsystem determinants. However, apart from that, it may also be of interest to determine the range of relative gain variations.

Theorem 2. Sensitivity of RGA to gain variations. Here we restrict the theorem to variations of nonzero gains. Then,

$$\partial \lambda_{ij} \big/ \partial G^{\mathrm{T}} = \lambda_{ij} \left((G^{\tilde{ij}})^{-\mathrm{T}} - G^{-\mathrm{T}} \right) \tag{25a}$$

$$G \circ (\partial \lambda_{ij} \big/ \partial G^{\mathrm{T}}) = \lambda_{ij} \left(\Lambda(G^{\tilde{ij}}) - \Lambda(G) \right) \tag{25b}$$

where $G^{\tilde{ij}}$ is the matrix obtained from G by replacing the elements of row i and column j, except element ij, by zeros.

The sensitivities in Theorem 2 can be used to determine the maximum relative gain variations as illustrated in the application.

V. Structured Uncertainty

A large class of structured uncertainty descriptions can be formulated as

$$G = G_0 + \sum_{\forall i} \delta_i G_i \ ,\quad -1 \leq \delta_i \leq 1 \,. \tag{26}$$

We can study the relevant sensitivities with respect to every δ_i. If they are treated independently, it suffices to consider one uncertainty, and for notational convenience we use (21) as an uncertainty model. Thus, $\alpha = \delta_i$ and $W = G_i$ in the equations below.

A. Effect of Scalar Uncertainty on Determinants

Theorem 3. Sensitivity of $\det(G)$ *to scalar uncertainty.*

$$\partial \det(G) \big/ \partial \alpha = \sum_{\forall k,\ell} (-1)^{k+\ell} \det(G^{k\ell}) w_{k\ell} \tag{27a}$$

If $\det(G) \neq 0$, we can write

$$\partial \det(G) \big/ \partial \alpha = \det(G) \cdot \mathrm{trace}(W G^{-1}) \tag{27b}$$

$$\partial \det(G) \big/ \partial \alpha = \det(G) \sum_{\forall k,\ell} \lambda_{k\ell}\, w_{k\ell} \big/ g_{k\ell} \ , \forall g_{k\ell} \neq 0 \,. \tag{27c}$$

Here, (27) directly give the worst direction since there is only one free variable (α). We can choose the sign of W so that $\partial \det(G) / \partial \alpha < 0$, which means that the worst direction corresponds to a positive α. According to (27b), this sign is given by the requirement $\mathrm{trace}(W G^{-1}) < 0$. The smallest $\alpha = \bar{\alpha}$ such that $\det(G) = 0$ can now be determined by the procedure of Section IV-B.

If we are interested in finding the smallest $\alpha = \bar{\alpha}$ such that $\forall \delta_i < \bar{\alpha}$, we can apply the same procedure to the uncertainty description

$$G = G_0 + \alpha \sum_{\forall i} G_i \ , \tag{28}$$

where the signs of every G_i are (initially) chosen as above. However, now the sign of every $\mathrm{trace}(G_i G^{-1})$ needs to be checked as α is increased. A positive $\mathrm{trace}(G_i G^{-1})$ indicates that a sign change of G_i should be considered.

B. Effect of Scalar Uncertainty on Relative Gains

Theorem 4. Sensitivity of RGA to scalar uncertainty. We restrict the theorem to nonzero gains. Then,

$$\partial\lambda_{ij} / \partial\alpha = \lambda_{ij} \cdot \mathrm{trace}\Big(W((G^{\tilde{ij}})^{-1} - G^{-1}) \Big) \tag{29a}$$

$$\partial\lambda_{ij} / \partial\alpha = \lambda_{ij} \sum_{\forall k,\ell} \Big(\lambda_{k\ell}(G^{\tilde{ij}}) - \lambda_{k\ell}(G) \Big) w_{k\ell} / g_{k\ell} \ . \tag{29b}$$

VI. Application to Robustness Analysis

As an application we shall investigate the robustness properties for decentralized control of a system that has been introduced in [18]. The system has also been used in previous studies of the effect of uncertainty on the RGA [8]–[10].

A. Problem Formulation and Previous Studies

When conditioned as described in Section II-B for the proposed control configuration, the system has the nominal gain matrix

$$G_0 = \begin{bmatrix} 0.66 & 0.61 & -0.0049 \\ 1.11 & 2.36 & -0.012 \\ -33.68 & -46.2 & 0.87 \end{bmatrix} \tag{30}$$

and the RGA is

$$\Lambda(G_0) = \begin{bmatrix} 1.94 & -0.67 & -0.27 \\ -0.66 & 1.90 & -0.23 \\ -0.28 & -0.23 & 1.51 \end{bmatrix}. \tag{31}$$

According to the RGA, the diagonal control configuration is the only one that can possess integrity. In addition to $\det(G_0) = 0.51$, we have $\det(G_0^{11}) = 1.50$, $\det(G_0^{22}) = 0.41$, and $\det(G_0^{33}) = 0.88$, which means that the nominal system satisfies necessary integrity and integral controllability conditions. Moreover, the diagonal relative gains satisfy (9), so the nominal system is decentralized integral controllable.

In [8] it was assumed that every gain is uncertain according to

$$g_{ij} = g_{0,ij} + \Delta g_{ij} \ , \ \ |\Delta g_{ij}| \le \alpha \, |g_{0,ij}| \tag{32}$$

and the main issue was to determine a bound on α such that the recommended nominal variable pairings for decentralized control would not be affected and the control system would remain robust. Expressions for $\partial\lambda_{ij} / \partial g_{kl}$, $\forall i,j,k,l$, similar to (25), were derived, and the total effect on λ_{ij} was estimated for various values of α by applying the formula for a total differential. The estimates indicated that the system could tolerate $\alpha = 0.5$.

In [9] a technique based on structured singular values for evaluation of relative gains of norm-bounded uncertain systems was developed. It was stated that the method can be used for calculation of a tight bound on the worst-case relative gain and derivation of necessary and sufficient conditions for a sign change of a relative gain. For this example, the calculations gave $\alpha = 0.302$ as the largest value with no sign changes of the diagonal relative gains.

However, in [10] a minimization of $\det(G)$ subject to the constraints in (32), combined with a numerical search over α , showed that $\det(G) = 0$ was reached for $\alpha = 0.178$. According to (8), the worst-case relative gains are undefined at this point. This also means that the system does not have the property of robust integral controllability and integrity if $\alpha \ge 0.178$.

As such, this does not prove that the system is ICI for all $\alpha < 0.178$. It is still possible that some principal subsystem G_m might violate (4) for some $\alpha < 0.178$. However, for a 3×3 system, this would be revealed by a sign change of a diagonal relative-gain value as shown by (8).

B. Integral Controllability Bounds

The worst-case uncertainty directions for $\det(G)$ can be determined according to (19b) in Theorem 1. We have

$$G_0^{-\mathrm{T}} = \begin{bmatrix} 2.948 & -1.104 & 55.47 \\ -0.598 & 0.805 & 19.56 \\ 0.008 & 0.005 & 1.732 \end{bmatrix}, \tag{33}$$

which means that an increase of the gains corresponding to negative elements will decrease $\det(G)$ while the opposite applies for the other gains. In accordance with (22) we then choose

$$W_{\mathrm{a}} = \begin{bmatrix} -0.66 & 0.61 & -0.0049 \\ 1.11 & -2.36 & -0.012 \\ -33.68 & -46.2 & -0.87 \end{bmatrix} \tag{34}$$

after which (24) gives

$$\bar{\alpha}_{\mathrm{a}} = 0.205 \ . \tag{35}$$

A check with $\alpha = 0.2$ in (21) shows that elements (2,3) and (3,2) in the inverse of the resulting gain matrix have other signs than in G_0^{-1}. The opposite directions should therefore be checked by changing the signs of these elements in W_{a} .

Of the three possibilities, the weight

$$W_{\mathrm{b}} = \begin{bmatrix} -0.66 & 0.61 & -0.0049 \\ 1.11 & -2.36 & 0.012 \\ -33.68 & 46.2 & -0.87 \end{bmatrix} \tag{36}$$

gives the lowest bound

$$\bar{\alpha}_{\mathrm{b}} = 0.178 \ . \tag{37}$$

In principle, there could be an even number of sign changes in some element of $G^{-\mathrm{T}}$, which means that they might not be detected by this procedure. The correct bound could thus be lower than that in (37). There are various ways of tackling this problem. According to (7), the sign of $(G^{-1})_{ji}$ is changed if an only if the sign of $\det(G^{ij})$ changes for some $\alpha < \bar{\alpha}_{\mathrm{b}}$. Such sign changes could be checked for every G^{ij} by a similar eigenvalue calculation as above.

A more practical approach is first to consider only those gains which have the strongest effect on the determinant. Equation (19d) and the RGA in (31) suggests

$$W_{\mathrm{c}} = \begin{bmatrix} -0.66 & 0.61 & 0 \\ 1.11 & -2.36 & 0 \\ 0 & 0 & -0.87 \end{bmatrix}, \tag{38}$$

which gives the same bound as in (35). A Calculation of $G^{-\mathrm{T}}$ with W_{c} and $\alpha = 0.2$ now shows the directions in the remaining unchanged elements that decrease $\det(G)$. This gives W_{b} and the solution in (37).

C. Bounds on Relative Gains

We can calculate bounds on the relative gains for various α values. Let us choose $\alpha = 0.1$, which is one of the values considered in [8] and [10]. Direct calculation from the gain matrices obtained with the weight W_{b} and the values $\alpha = -0.1$ and $\alpha = 0.1$ gives

$$\begin{aligned} 1.48 \le \lambda_{11} \le 3.65 \\ 1.64 \le \lambda_{22} \le 2.85 \\ 1.44 \le \lambda_{33} \le 1.75. \end{aligned} \tag{39}$$

However, these are not necessarily the correct bounds for $|\alpha| \le 0.1$ because the worst-case directions for $\det(G)$ might not be the worst-case directions for the relative gains.

The sensitivities in (25) in Theorem 2 show that the worst-case directions for λ_{11} coincide with W_{b} in (36), but for λ_{22} and λ_{33} they give, respectively,

$$W_{\mathrm{d}} = \begin{bmatrix} -0.66 & 0.61 & 0.0049 \\ 1.11 & -2.36 & -0.012 \\ 33.68 & -46.2 & -0.87 \end{bmatrix} \tag{40}$$

$$W_{\mathrm{e}} = \begin{bmatrix} -0.66 & -0.61 & -0.0049 \\ -1.11 & -2.36 & -0.012 \\ -33.68 & -46.2 & -0.87 \end{bmatrix}. \tag{41}$$

For $\alpha = -0.1$ and $\alpha = 0.1$ these weights give

$$\begin{aligned} 1.46 \le \lambda_{22} \le 3.42 \\ 1.29 \le \lambda_{33} \le 2.01 \end{aligned} \tag{42}$$

which are the same as those obtained in [10] by much more demanding nonconvex optimization methods.

VII. Conclusion

We have developed a method for analyzing the effect of model uncertainty on integral controllability and integrity in decentralized control. Useful uncertainty sensitivity expressions for the determinant of the gain matrix as well as for the RGA have been derived. By means of these, the number of possible worst-case combinations of the model uncertainties can be drastically reduced to a manageable number. In this way, integral controllability and integrity issues can be effectively studied and tight bounds on the tolerable uncertainties can be obtained. The method allows independent uncertainties in all process gains as well as more structured uncertainties.

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