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# Optimal stabilization using Lyapunov measure

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*Abstract*— **The focus of the paper is on the computation of optimal feedback stabilizing control for discrete time control system. We use Lyapunov measure, dual to the Lyapunov function, for the design of optimal stabilizing feedback controller. The linear Perron-Frobenius operator is used to pose the optimal stabilization problem as an infinite dimensional linear program. Finite dimensional approximation of the linear program is obtained using set oriented numerical methods. Simulation results for the optimal stabilization of periodic orbit in one dimensional logistic map are presented.**

## I. INTRODUCTION

Stability analysis and stabilization of nonlinear systems are two of the most important and extensively studied problems in control theory. Lyapunov function and Lyapuov function based methods have played an important role in providing solutions to these problems. In particular, the Lyapunov function is used for stability analysis and the control Lyapunov function (CLF) is used for the design of stabilizing feedback controllers. Another problem which is extensively studied in controls literature is the optimal control problem (OCP). Optimal control for the OCP can be obtained from the solution of the Hamilton Jacobi Bellman (HJB) equation. Under the additional assumption of detectability and stabilizability of nonlinear system, the optimal cost function if positive can also be used as control Lyapunov function. This establishes the connection between stability (Lyapunov function) and optimality (HJB equation). The HJB equation is a nonlinear partial different equation and hence, difficult to solve analytically and one has to resort to approximate numerical schemes for its solution. We review some of the literature particularly relevant to this paper on the approximation of HJB equation and OCP.

In [1], an adaptive space discretization scheme is used to obtain the solution of deterministic and stochastic discrete time HJB (dynamic programming) equation. Optimal cost function is obtained as a fixed point solution of a linear dynamic programming operator. In [2], [3], cell mapping approach is used to construct approximate numerical solutions for deterministic and stochastic optimal control problems. In [4], [5], set oriented numerical methods are used to underestimate the optimal one-step cost for transition between different state-space discretizations in the context of optimal control and optimal stabilization. This allows to represent the minimal cost control problem as one of finding the minimum cost path to reach the invariant set on a graph with edge costs derived from the under-estimation procedure. Djikstra's algorithm is used to construct an approximate solution to the HJB equation. In [6], [7], [8] solutions to deterministic optimal control problems are proposed by casting them as infinite dimensional linear programs. Approximate solution to the infinite dimensional linear program is then obtained using finite dimensional approximation of the linear programming problems or using sequence of LMI relaxation.

In this paper we propose the use of *Lyapunov measure* for the optimal stabilization of nonlinear systems. Lyapunov measure is introduced in [9], to study weaker set wise notion of almost everywhere stability and is shown to be dual to the Lyapunov function. Existence of Lyapunov measure guarantees stability from almost every with respect to Lebesgue measure initial conditions in the phase space. *Control Lyapunov measure* is introduced in [10] to provide Lyapunov measure based framework for the stabilization of nonlinear systems. In [10], problem of stabilization is posed as a co-design problem of jointly obtaining the control Lyapunov measure and the controller. The co-design problem is formulated as an infinite dimensional linear program using suitable change of coordinates. Computational method based on set oriented numerical approach and mixed integer linear program is proposed for the finite dimensional approximation and to obtain deterministic control respectively. Since the approach depends on discretizing the state and control spaces, solving an integer program of those dimensions is clearly computationally prohibitive. The goal of this paper is to extend the Lyapunov measure based framework for the optimal stabilization of an attractor set. One of the main differences between the results in [10] and this paper is that the finite deterministic optimal control is obtained as a solution of finite linear program as opposed to a mixed integer linear program in [10]. We must emphasize that our approach alo relies on discretization of the state and control spaces but hope that the linear program complexity allows better handling of the computational aspects.

This paper is organized as follows. In section II, we provide a brief overview of some of the key results from [9],[11],[10] for stability analysis and stabilization of nonlinear systems using Lyapunov measure. The framework for optimal stabilization using Lyapunov measure and transfer operators is posed as an infinite dimensional linear program in section III. A computational approach based on set oriented numerical methods in proposed for the finite dimensional approximation of the linear program in section IV. Simulation results for optimal stabilization of periodic orbit in one dimensional logistic map are presented in section V, followed by conclusion and discussion in section VI.

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## II. Lyapunov measure, stability and stabilization

In [9], [11], [10], *Lyapunov measure* is introduced for stability verification and for stabilizing controller design of an invariant set in nonlinear dynamical systems. Stability and stabilization problems of an attractor set $A$ for a nonlinear system $T : X \to X$, where $X \subset \mathbb{R}^{n_x}$ is compact, were studied using a weaker notion of almost everywhere stability.

*Definition 1 (Attractor set):* A closed set $A$ is said to be $T$ invariant if $T(A) = A$. A closed $T$ invariant set $A$ is called an attractor set if there exists a local neighborhood $V$ of $A$ such that $T(V) \subset V$.

We use following definition of almost everywhere (a.e.) stability with geometric decay of the attractor set in this paper.

*Definition 2:* An attractor set $A$ is said to be a.e. stable with geometric decay with respect to measure $m$ if given $\delta > 0$, there exists $K(\delta) < \infty$ and $\beta < 1$ such that

$$m\{x \in A^c : T^n(x) \in B\} < K(\delta)\beta^n$$

for all set $B \subset X \setminus U_\delta$, where $U_\delta$ is the $\delta$ neighborhood of an attractor set $A$ and $A^c := X \setminus A$ is the complement of the invariant set.

*Remark 3:* In the subsequent section we use the notation $m$ for the Lebesgue measure, $m_S$ for the Lebesgue measure supported on set $S$ and $U_\delta$ for the $\delta$ neighborhood of the attractor set $A$ for a given $\delta > 0$.

This weaker notion of a.e. stability is studied using a Linear transfer operator called as Perron-Frobenius (P-F) operator. P-F operator is used to study the propagation of sets or the measure supported on the sets. For any given continuous mapping $T : X \to X$, linear Perron-Frobenius (P-F) operator, denoted by $\mathbb{P}_T : \mathscr{M}(X) \to \mathscr{M}(X)$ is given by

$$\mathbb{P}_T[\mu](B) = \int_X \chi_B(T(x))d\mu(x) \tag{1}$$

where $\mathscr{M}(X)$ is the vector space of all measures supported on $X$, $\chi_B(x)$ is the indicator function supported on the set $B \subset \mathscr{B}(X)$, the Borel sigma-algebra of $X$. For more details on the P-F operator refer to [12]. Since the stability property of an attractor set in definition (2) is stated in terms of the transient behavior of the system on the complement of the attractor set $A^c$, we define sub-stochastic Markov operator as a restriction of the P-F operator on $A^c$ as follows:

$$\mathbb{P}^1_T[\mu](B) := \int_{A^c} \chi_B(T(x))d\mu(x) \tag{2}$$

for any set $B \in \mathscr{B}(A^c)$ and $\mu \in \mathscr{M}(A^c)$. Necessary and sufficient condition for almost everywhere uniform stability of an invariant set $A$ with respect to any finite measure $m$ were obtained in the form of existence of the positive solution, *Lyapunov measure*, to the following *Lyapunov measure equation*:

$$\mathbb{P}^1_T\bar{\mu}(B) - \bar{\mu}(B) = -m(B) \tag{3}$$

The precise theorem for stability as proved in [11] is:

*Theorem 4:* The attractor set $A$ for the dynamical system $T : X \to X$ is a.e. stable with geometric decay with respect to measure $m$ if and only if there exists a non-negative measure $\bar{\mu}$ which is finite on $\mathscr{B}(X \setminus U_\delta)$ and satisfies

$$\gamma\mathbb{P}^1_T\bar{\mu}(B) - \bar{\mu}(B) = -m(B)$$

for every set $B \subset X \setminus U_\delta$ and for some $\gamma > 1$. Measure $m$ is absolutely continuous with respect to Lyapunov measure $\bar{\mu}$.

Stability of the attractor set with respect to Lebesgue almost every initial condition starting from a given set $S$ can be studied by taking $m = m_S$ in the Lyapunov measure equation. In [13], set oriented numerical approach is used for the finite dimensional approximation of the Lyapunov measure $\bar{\mu}$. This finite dimensional approximation leads to further weaker notion of stability, which is referred to as *coarse stability*. Unlike almost everywhere stability, *coarse stability* of an invariant set allows for the existence of stable dynamics in the complement of an invariant set however the domain of attraction of the stable dynamics is strictly smaller than the size of the partition used in the finite dimensional approximation.

In [10], Lyapunov measure is used for the design of stabilizing feedback controller. For the stabilization problem we consider the control dynamical system of the form

$$x_{n+1} = T(x_n, u_n)$$

where $x_n \in X$ and $u_n \in U$ is the state space and control space respectively. The objective is to design feedback controller $u_n = K(x_n)$ to stabilize the invariant set $A$, which is assumed to be locally stable and hence forms an attractor set. The stabilization problem is solved using Lyapunov measure by extending the P-F operator formalism to the control dynamical system as follows. We define the feedback control mapping $C : X \to Y := X \times U$ as $C(x) = (x, K(x))$. Using the definition of feedback mapping $C$, we write the feedback control system as

$$x_{n+1} = T(x_n, K(x_n)) = T \circ C(x_n)$$

With the system mapping $T : Y \to X$ and the control mapping $C : X \to Y$, we can associate Perron-Frobenius operators $\mathbb{P}_T : \mathscr{M}(Y) \to \mathscr{M}(X)$ and $\mathbb{P}_C : \mathscr{M}(X) \to \mathscr{M}(Y)$ respectively, and are defined as follows:

$$\mathbb{P}_T[\theta](B) = \int_Y \chi_B(T(y))d\theta(y)$$

$$\mathbb{P}_C[\mu](D) = \int_D f(a|x)dm(a)d\mu(x)$$

where $\theta \in \mathscr{M}(Y), \mu \in \mathscr{M}(X)$ and $B \subset X, D \subset Y$. $f(a|x)$ is the conditional probability density function and is introduced to incorporate the particular form of feedback controller mapping $C(x) = (x, K(x))$. The advantage of writing the feedback control dynamical system as the composition of two maps $T : Y \to X$ and $C : X \to Y$ is that the P-F operator for the composition $T \circ C : X \to X$ can be written as a product of $\mathbb{P}_T$ and $\mathbb{P}_C$ as follows (refer to [10]):

$$\mathbb{P}_{T\circ C} = \mathbb{P}_T \cdot \mathbb{P}_C : \mathscr{M}(X) \to \mathscr{M}(X).$$

In [10], control Lyapunov measure is introduced for the stabilization of nonlinear systems. Control Lyapunov measure is defined as any non-negative measure $\bar{\mu} \in \mathscr{M}(A^c)$, which is finite on $\mathscr{B}(X \setminus U_\delta)$ and satisfies

$$\mathbb{P}_T^1 \cdot \mathbb{P}_C^1 \bar{\mu}(B) < \beta \bar{\mu}(B) \tag{4}$$

for every set $B \subset X \setminus U_\delta$ and $\beta < 1$. Operators $\mathbb{P}_T^1$ and $\mathbb{P}_C^1$ are the restrictions of the P-F operator $\mathbb{P}_T$ and $\mathbb{P}_C$ to the complement of the invariant set $A^c$ respectively and are defined similar to the restriction of the P-F operator in the autonomous case in equation (2).

## III. Optimal stabilization

The objective is to stabilize the invariant set $A$ using feedback control input $u = K(x)$, while minimizing a relevant cost function. We assume that the invariant set is locally stabilized and hence forms an attractor set. In the context of the controlled system, the invariant set $A$ satisfies $T(A, K(A)) = A$. Let, $G : Y \to \mathbb{R}^+$ be a continuous non-negative real valued function such that $G(A, 0) = 0$. The cost of stabilization of the invariant set $A$ with respect to initial conditions starting from the set $B \subset X_1 := X \setminus U_\delta$ is denoted by $\mathscr{C}(B)$ and is given by the following formula

$$\mathscr{C}(B) = \int_B \sum_{n=0}^{\infty} \gamma^n G(x_n, u_n) dm(x) \tag{5}$$

where $x_0 = x$, $x_{n+1} = T(x_n, u_n)$ for $n \geq 0$. For a given stabilizing feedback controller mapping $C(x) = (x, K(x))$, the cost of stabilization $\mathscr{C}_C(B)$ is given by

$$\mathscr{C}_C(B) = \int_B \sum_{n=0}^{\infty} \gamma^n G \circ C(x_n) dm(x) \tag{6}$$

Note that the cost of global stabilization of the invariant set $A$ is the special case of $B = X_1$. For (6) to be finite, when $\gamma > 1$, we require that the controller $C(x)$ is not just stabilizing but stabilizing the invariant set $A$ with geometric decay rate $\beta < \frac{1}{\gamma}$. In the following theorem we show that the cost of stabilization of the invariant set $A$ can be expressed using the control Lyapunov measure. We first introduce the Koopman operator which is to be used in the proof.

*Definition 5 (Koopman operator):* For a continuous mapping $F : X_1 \to X_2$, Koopman operator $\mathbb{U}_F : \mathscr{C}^0(X_2) \to \mathscr{C}^0(X_1)$ is given by

$$(\mathbb{U}_F f)(x) = f(F(x))$$

where $f \in \mathscr{C}^0(X_2)$ and $\mathscr{C}^0(X_i)$ is the space of all continuous functions on compact spaces $X_i$ for $i = 1, 2$.

The P-F and Koopman operator are dual to each other and the duality is expressed using the following inner product

$$\begin{aligned} \langle \mathbb{U}_F f, \mu_1 \rangle_{X_1} &= \int_{X_1} f(F(x_1)) d\mu_1(x_1) \\ &= \int_{X_2} f(x_2) d[\mathbb{P}_F \mu_1](x_2) = \langle f, \mathbb{P}_F \mu_1 \rangle_{X_2} \end{aligned} \tag{7}$$

The result on the cost of stabilization follows.

*Theorem 6:* Let $\gamma > 1$ in the cost function and the controller mapping $C(x)$ is designed such that the invariant set $A$ is a.e. stable with geometric decay rate $\beta < \frac{1}{\gamma}$. The cost of stabilization of an invariant set $A$ w.r.t. Lebesgue measure initial conditions starting from set $B$ is given by

$$\begin{aligned} \mathscr{C}_C(B) &= \int_B \sum_{n=0}^{\infty} \gamma^n G(x_n, u_n) dm(x) \\ &= \int_{A^c \times U} G(y) d[\mathbb{P}_C^1 \bar{\mu}_B](y) = \left\langle G, \mathbb{P}_C^1 \bar{\mu}_B \right\rangle_{A^c \times U} \end{aligned} \tag{8}$$

where $\bar{\mu}_B$ is the solution of the following *control Lyapunov measure equation*

$$\gamma \mathbb{P}_T^1 \cdot \mathbb{P}_C^1 \bar{\mu}_B(D) - \bar{\mu}_B(D) = -m_B(D) \tag{9}$$

for every set $D \subset X_1$.

*Proof:* From the assumption of a.e. geometric stability of the controller mapping $C : X \to Y$, we have by Theorem 4 that there exists non-negative measure $\bar{\mu}$ which is finite on $\mathscr{B}(X_1)$ and satisfies

$$\gamma[\mathbb{P}_T^1 \cdot \mathbb{P}_C^1 \bar{\mu}](D) - \bar{\mu}(D) = -m(D).$$

For the cost of stabilization of a set $B$, we have

$$\mathscr{C}_C(B) = \int_B \sum_{n=0}^{\infty} \gamma^n G \circ C(x_n) dm(x) = \int_B \lim_{N\to\infty} f_N(x) dm(x)$$

where $f_N(x) = \sum_{n=0}^{N} \gamma^n G \circ C(x_n)$ and $x_0 = x$. Using monotone convergence theorem and $G \geq 0$, $f_N(x) \leq f_{N+1}(x)$, we have

$$\int_B \lim_{N\to\infty} f_N(x) dm(x) = \lim_{N\to\infty} \sum_{n=0}^{N} \langle \gamma^n G \circ C(x_n), m_B \rangle_{A^c}$$

$$\lim_{N\to\infty} \sum_{n=0}^{N} \langle \gamma^n G \circ C(x_n), m_B \rangle_{A^c} = \lim_{N\to\infty} \left\langle \mathbb{U}_C^1 G, \sum_{n=0}^{N} \gamma^n [\mathbb{P}_{T\circ C}^1]^n m_B \right\rangle_{A^c}$$

where we have used the fact that $x_n = (T \circ C)^n(x)$ and the duality between the operators $\mathbb{U}_{T\circ C}^1$ and $\mathbb{P}_{T\circ C}^1$. Let $\bar{\mu}_B^N = \sum_{n=0}^{N} \gamma^n [\mathbb{P}_{T\circ C}^1]^n m_B$. The measure $\bar{\mu}_B^N$ is absolutely continuous with respect to Lebesgue measure $m$ for all $N$ since for any set $D \subset X_1$ if $m(D) = 0$ then $([\mathbb{P}_{T\circ C}^1]^n m_B)(D) = m((T \circ C)^{-n}(D) \cap B) = 0$ for all $n$ and every set $B \subset X_1$. The latter is true because of the non-singularity assumption of the closed loop map $T \circ C$. Moreover $\bar{\mu}_B^N(D) \leq \bar{\mu}_B^{N+1}(D)$ for every set $D, B \subset X_1$. Hence there exists an integrable function $g_N(x) \geq 0$ such that $g_N(x) \leq g_{N+1}(x)$ and $d\bar{\mu}_B^N(x) = g_N(x) dm(x)$. Hence we have

$$\lim_{N\to\infty} \left\langle \mathbb{U}_C^1 G, \sum_{n=0}^{N} \gamma^n [\mathbb{P}_{T\circ C}^1]^n m_B \right\rangle_{A^c} = \lim_{N\to\infty} \int_{A^c} (\mathbb{U}_C^1 G)(x) g_N(x) dm$$

$$= \int_{A^c} (\mathbb{U}_C^1 G)(x) \lim_{N\to\infty} g_N(x) dm(x) = \left\langle \mathbb{U}_C^1 G, \bar{\mu}_B \right\rangle_{A^c}$$

where $\bar{\mu}_B := \sum_{n=0}^{\infty} \gamma^n [\mathbb{P}_{T\circ C}^1]^n m_B = \sum_{n=0}^{\infty} \gamma^n [\mathbb{P}_T \cdot \mathbb{P}_C^1]^n m_B]$ and $\bar{\mu}_B$ is known to finite on any set $D \subset X_1$ because of a.e. stability property of the invariant set $A$ with geometric decay rate $\beta < \frac{1}{\gamma}$. Furthermore $\bar{\mu}_B$ satisfies following control Lyapunov measure equation (3). Finally using the duality between $\mathbb{U}_C^1$ and $\mathbb{P}_C^1$, we get

$$\left\langle \mathbb{U}_C^1 G, \bar{\mu}_B \right\rangle_{A^c} = \left\langle G, \bar{\mathbb{P}}_C^1 \bar{\mu}_B \right\rangle_{A_c^u}$$

which proves the claim. ■

The minimum cost of stabilization is defined as the minimum over all a.e. stabilizing controller mapping $C$ as follows:

$$\mathscr{C}^*(B) = \min_C \mathscr{C}_C(B) \tag{10}$$

Defining $\theta(O) := [\mathbb{P}^1_C \mu](O)$ for any set $O \subset X_1 \times U$, $\theta \in \mathscr{M}(A^c \times U)$, $\mu \in \mathscr{M}(A^c)$, the inner product $\langle f, \mu \rangle_X := \int_X f d\mu(x)$, and a projection map $P_1 : A^c \times U \to A^c$, $P_1(x,u) = x$, $P_1^{-1}(x) = (x, U)$ along with its corresponding P-F operator

$$[\mathbb{P}^1_{P_1}\theta](D) = \int_{A^c \times U} \chi_D(P_1(y))d\theta(y) = \int_{D\times U} d\theta(y) = \mu(D)$$

Using these definition, we can pose the infinite dimensional optimal linear program for stabilization as follows

$$\min_{\theta \geq 0} \langle G, \theta \rangle_{A^c \times U} \tag{11a}$$

$$\text{s.t. } \gamma[\mathbb{P}^1_T \theta](D) - [\mathbb{P}^1_{P_1}\theta](D) = -m_B(D) \quad \forall D \subset X_1 \tag{11b}$$

In the next section, we propose a computational framework based on the set oriented numerical methods for the finite dimensional approximation of the optimal stabilization problem. Optimal control for stabilization is obtained using the finite dimensional approximation of the linear program (11).

## IV. COMPUTATIONAL APPROACH

For the purposes of computations, the infinite-dimensional problem is replaced by its finite-dimensional approximations as in [10]. We assume that the controls belong to finite set $u \in \mathscr{U}_M$ where $\mathscr{U}_M = \{u^1, \ldots, u^a, \ldots, u^M\}$. This set may be taken after quantization of the control input space. We also assume a finite partition of $X$, and denote it by $\mathscr{X}_N := \{D_1, ..., D_i, ..., D_N\}$, together with the associated measure space $\mathbb{R}^N$. We assume without loss of generality that $D_N = A$. The partition for the joint space $Y$, denoted by $\mathscr{Y}_{N\times M} = \mathscr{X}_N \times \mathscr{U}_M$ has cardinality $M \cdot N$ and is identified with an associated vector space $\mathbb{R}^{NM}$. We use the notation $P_T : \mathbb{R}^{NM} \to \mathbb{R}^N$ to denote the discrete counterpart of $\mathbb{P}_T$. Since $T : Y \to X$, so $P_T$ is a Markov matrix. Further, $P_{T^a} : \mathbb{R}^N \to \mathbb{R}^N$ denotes the Markov matrix that is obtained by fixing the control input to $u^a$ for all $D \subset \mathscr{X}_N$. Additionally, $\overline{P}_{T^a} : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ will denote the sub-stochastic transition matrix where the control input is fixed to $u^a$. It is easily seen that $\overline{P}_{T^a}$ consists of the first $(N-1)$ rows and columuns of $P_{T^a}$.

With the above quantization of the control space and partition of the state space, the determination of the control $u(x) \in U$ (or equivalently $K(x)$) for all $x \in A^c$ has now been cast as a problem of choosing $u(D_i) \in \mathscr{U}_M$ for all sets $D_i \subset \mathscr{X}_N$. The finite dimensional approximation of the minimum cost of stabilization (11) is equivalent to solving the following finite-dimensional linear program:

$$\min_{\theta^a, \mu \geq 0} \sum_{a=1}^{M} (G^a)'\theta^a \tag{12a}$$

$$\text{s.t. } \gamma \sum_{a=1}^{M} (\overline{P}^a_T)'\theta^a - \sum_{a=1}^{M} \theta^a = -m \tag{12b}$$

where $m \in \mathbb{R}^{N-1}$, $m_j \geq 0$ denotes the discrete counterpart of the Lebesgue measure $m(B)$ and $(\cdot)'$ denotes the transpose operation; $G^a \in \mathbb{R}^{N-1}$ and $G^a_j$ is the cost associated with using control action $u^a$ on set $D_j$; $\theta^a \in \mathbb{R}^{N-1}$ is the discrete counter-part of infinte-dimensonal measure quantity in (11). The linear program (12) does not enforce the constraint

$$\theta^a_j > 0 \text{ for exactly one } a \in \{1, ..., M\} \tag{13}$$

for each $j = 1, ..., (N-1)$. The above constraint ensures that the control obtained is deterministic. We prove in the following that a deterministic controller can always be obtained provided the linear program (12) has a solution. To this end, we introduce the dual linear program [14] associated with the linear program in (12). The dual to (12) is,

$$\begin{aligned} &\max_V \ m'V \\ &\text{s.t. } V \leq \gamma \overline{P}_{T^a} V + G^a \ \forall a = 1, ..., M. \end{aligned} \tag{14}$$

In the above linear program (14), $V$ are the dual variables to the equality contraints (12a).

### A. Existence of stabilizing controls for a partition

We now make following assumption on the existence of stable fine- partition using finite set of controls ($\mathscr{U}_M$).

*Assumption 7 (Existence of a stable fine-partition):* There exists a partition of the state-space $\mathscr{X}_{N'} := \{E_1, \ldots, E_i, \ldots, E_{N'}\}$ with $N'$ sufficiently large and associated controls $u(E_i) \in \mathscr{U}_M$ such that the system is coarse stable.

*Remark 8:* Although, we do not have a proof, intuition suggests that assumption (7) is likely to hold true and is necessary for the existence of the finite dimensional controller. The assumption can be interpreted as a requirement of stabilizability using finite controls ($\mathscr{U}_M$) of the original infinite dimensional system.

We now introduce the concept of a sub-partition which will be used in the main result on existence of stabilizing control.

*Definition 9 (Sub-partition):* A partition $\mathscr{X}_{N'} := \{E_1, .., E_{N'}\}$ of the state-space is said to be a sub-partition of $\mathscr{X}_N$, if $N' > N$, $D_N = E_{N'}$, and for each $E_i$ there exists a unique $j \in \{1, ..., N\}$ such that $E_i \cap D_j \neq \emptyset$ and $E_i \cap D_k = \emptyset$ for all $k \neq j$.

We now state the main result on the existence of stabilizing controls for any partition.

*Theorem 10:* Suppose Assumption 7 holds. Then, there exists stabilizing controls for any partition $\mathscr{X}_N := \{D_1, \ldots, D_N\}$ of the state-space such that $\mathscr{X}_{N'}$ is a sub-partition of $\mathscr{X}_N$.

For the proof of this theorem refer to [15]. The critical piece in the proof is the requirement that $\mathscr{X}_{N'}$ be a sub-partition of $\mathscr{X}_N$. Using this we established [15] that existence of stabilizing controls for a sub-partition $\mathscr{X}_{N'}$ using finite set of controls ($\mathscr{U}_M$) always allows to choose controls on the coarse partition $\mathscr{X}_N$ so that stability continues to be preserved. Observe that the sub-partition assumption will be satisfied if the system is for example, everywhere stabilizable using controls in ($\mathscr{U}_M$).

### B. Deterministic control solution to the finite linear program

In this section, we show the existence of optimal deterministic control solutions to the finite linear program (12). The results presented here can easily be extended to the case where $m \geq 0, m \neq 0$. We will first derive conditions under which the linear program (12) is feasible.

*Lemma 11:* Suppose that Assumption (7) holds and $m > 0$. Then, for any partition $\mathscr{X}_N$ there exists $\overline{\gamma} > 1$ such that for all $\gamma \in [1, \overline{\gamma})$ there exists a feasible solution to (12).

Refer to [15] for the proof.

In the following we will derive conditions under which a solution to linear program (12) exists and then, show that condition for deterministic control (13) can be satisfied under the assumption of feasibility of linear program (12). The main result is stated in Theorem 14.

*Lemma 12:* Suppose the Assumption 7 holds and $m > 0, G(\cdot,\cdot) \geq 0$ on the complement of the invariant set. Then for all $\gamma \in [1, \overline{\gamma})$, there exists an optimal solution $\theta$ to linear program (12) and an optimal solution $V$ to the dual linear program (14) with equal objective values.

Refer to [15] for the proof.

The next result shows that linear program (12) always admits a determinstic control action as an optimal solution. In the following, we will assume that the cost is positive on the complement of the invariant set $G(\cdot,\cdot) > 0$. This assumption is crucial in order to obtain deterministic controls.

*Lemma 13:* Suppose Assumption 7 holds and $m > 0$, $G(\cdot,\cdot) > 0$ on the complement of the invariant set. Let $\theta$ solve (12) and $V$ solve (14) for some $\gamma \in [1, \overline{\gamma})$. Then the following hold at the solution:

1) For each $j = 1, ..., (N-1)$ there exists at least one $a_j \in 1, ..., M$ such that $V_j = \gamma(\overline{P}_{T^{a_j}} V)_j + G_j^{a_j}$ and $\theta_j^{a_j} > 0$ where $G_j^{a_j} := G(D_j, u^{a_j})$

2) There exists a $\tilde{\theta}$ that solves (12) and is such that for each $j = 1, ..., (N-1)$, there is exactly one $a_j \in 1, ..., M$ such that $\tilde{\theta}_j^{a_j} > 0$ and $\tilde{\theta}_j^{a'} = 0$ for $a' \neq a_j$.

*Proof:* From the assumptions, we have that Lemma 12 holds. Hence, there exists $(V, \theta)$ that satisfy the first-order optimality conditions [14],

$$\begin{gathered}\sum_{a=1}^{M} \theta^a - \gamma \sum_{a=1}^{M} (\overline{P}_{T^a})' \theta^a = m \\ V \leq \gamma \overline{P}_{T^a} V + G^a \perp \theta^a \geq 0 \ \forall a = 1, ..., M.\end{gathered} \tag{15}$$

We will prove each of the claims in order.

1) Suppose, there exists $j \in 1, ..., (N-1)$ such that $\theta_j^a = 0$ for all $a = 1, ..., M$. Substituting in the optimality conditions (15) one obtains $\gamma \sum_{a=1}^{M} ((\overline{P}_{T^a})' \theta^a)_j = -m_j$ which cannot hold since, $\overline{P}_{T^a}$ has non-negative entries, $\gamma > 0$ and $\theta^a \geq 0$. Hence, there exists at least one $a_j$ such that $\theta_j^{a_j} > 0$. The complementarity condition in (15) then requires that $V_j = \gamma(\overline{P}_{T^{a_j}} V)_j + G_j^{a_j}$. This proves the first claim.

2) Denote $a(j) = \min\{a | \theta_j^a > 0\}$ for each $j = 1, ..., (N-1)$. The existence of $a(j)$ for each $j$ follows from statement 1. Define $\overline{P}_{T^u} \in \mathbb{R}^{(N-1)\times(N-1)}$ and $G^u \in \mathbb{R}^{N-1}$ as follows:

$$(\overline{P}_{T^u})_{ji} := (\overline{P}_{T^{a(j)}})_{ji} \ ; \ G_j^u := G_j^{a(j)} \tag{16}$$

for all $j = 1, ..., (N-1)$. From the definition of $\overline{P}_{T^u}$ and $G^u$ and complementarity condition in (15) it is easily seen that $V$ satisfies,

$$V = \gamma \overline{P}_{T^u} V + G^u = \lim_{n\to\infty} ((\gamma \overline{P}_{T^u})^n V + \sum_{k=0}^{n} (\gamma \overline{P}_{T^u})^k G^u). \tag{17}$$

Since $V$ is bounded and $G^u > 0$ it follows that $\rho(\overline{P}_{T^u}) < 1/\gamma$. Define $\tilde{\theta}$ for $j = 1, ..., (N-1)$ as follows,

$$\tilde{\theta}_j^a = 0 \quad a \neq a(j) \ ; \ \tilde{\theta}_j^{a(j)} = \left( (I_{N-1} - \gamma (\overline{P}_{T^u})')^{-1} m \right)_j. \tag{18}$$

The above is well-defined since we have already shown that $\rho(\overline{P}_{T^u}) < 1/\gamma$. From the construction of $\tilde{\theta}$, we have that for each $j$ there exists only one $a_j$, namely $a(j)$, for which $\tilde{\theta}_j^{a(j)} > 0$. It remains to show that $\tilde{\theta}$ solves (12). For this observe that,

$$\begin{aligned}\sum_{a=1}^{M} (G^a)' \tilde{\theta}^a &= \sum_{j=1}^{N-1} G_j^{a(j)} \tilde{\theta}_j^{a(j)} = (G^u)' (I_{N-1} - \gamma (\overline{P}_{T^u})')^{-1} m \\ &= ((I_{N-1} - \gamma \overline{P}_{T^u})^{-1} G^u)' m = V' m\end{aligned}$$

The primal and dual objectives are equal with above definition of $\tilde{\theta}$ and hence, $\tilde{\theta}$ solves (12). The claim is proved. ■

Lemma 13 shows that if there exists a solution to linear program (12) then, the there exists a deterministic controller for the same. The following theorem states the main result.

*Theorem 14:* Given the system $T : \mathscr{X}_N \times \mathscr{U}_M \to \mathscr{X}_N$ and $m > 0$, $G(\cdot,\cdot) > 0$ on the complement of the invariant set. If Assumption 7 holds, then the following statements hold for all $\gamma \in [1, \overline{\gamma})$:

1) there exists a $\theta$ which is a solution to (12) and a $V$ which is a solution to (14)

2) the optimal control for each set $j = 1, ..., (N-1)$ is given by,

$$u(D_j) = u^{a(j)} \text{ where } a(j) := \min\{a | \theta_j^a > 0\}$$

3) $\mu$ is the closed-loop Lyapunov measure satisfying

$$\gamma (\overline{P}_{T^u})' \mu - \mu = -m \text{ where } (\overline{P}_{T^u})_{ji} = (\overline{P}_{T^{a(j)}})_{ji}. \tag{19}$$

For proof refer to [15].

## V. Example

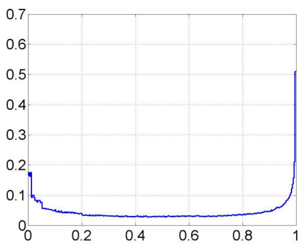
We present the simulation result for optimal stabilization of period two orbit in quadratic Logistic map. The controlled Logistic map is described by the following equation.

$$x_{n+1} = a x_n (1 - x_n) + u_n \tag{20}$$

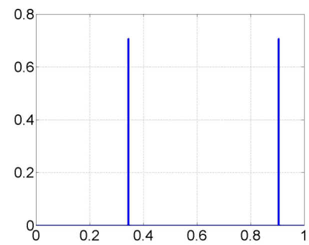
where $x_n \in [0, 1]$ is the state, $u_n$ is the control and the parameter $a = 4$. Figure (1a) shows the invariant measure for the uncontrolled Logistic map for the parameter value $a = 4$. Invariant measure gives us the steady state distribution of the points in the phase space. Invariant measure shows chaotic behavior with no stable periodic orbit of any period. Our goal is to stabilize a period two orbit. For Logistic map one can derive an analytical expression for the period-2 orbit in terms of the parameter $a$. The period-2 orbit points are given by the expression

$$x_{01} = \frac{a + 1 - \sqrt{a^2 - 2a - 3}}{2a}, \ x_{02} = \frac{a + 1 + \sqrt{a^2 - 2a - 3}}{2a}$$

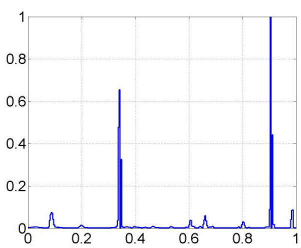
Hence for the parameter value $a = 4$ we get $x_{01} = 0.3455$ and $x_{02} = 0.9045$ as the unstable period-2 orbit. Figure (1) shows the simulation result for the stabilization of periodic two orbit for the Logistic map. The stabilization objective is achieved while minimizing the control input i.e., the cost function $G = x^2 + u^2$. For the finite dimension approximation we divide the interval $[0,1]$ into 300 equal length intervals. Similarly the control values ranges from $-0.05$ to 0.05 in the steps of 0.01. Figure (1b) shows the plot of closed loop invariant measure. Figure (1c) shows the plot of control Lyapunov measure. Figure (1d) shows the control values used for stabilization. From this plot we see control is used only at discrete set of points thus exploiting the natural dynamics of the system. The presence of eigenvalues at 1 and $-1$ for the closed loop system in figure (1e) implies the existence of stable period two orbit.



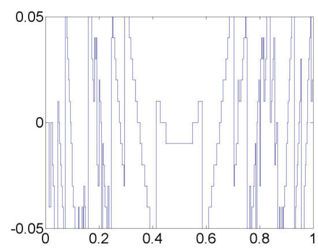

(a)



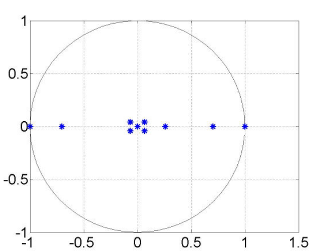

(b)




(c)




(d)




(e)

Fig. 1. (a) Open loop invariant measure (b) Closed- loop Invariant measure (c) log(Lyapunov measure) (d) Control values (e) Eigenvalues plot for closed loop system

## VI. Conclusions

The problem of optimal stabilization for discrete time nonlinear system is solved using linear transfer operator and Lyapunov measure based framework. Duality between Perron-Frobenius and Koopman operators is used to pose the primal and dual optimal stabilization problem as a infinite dimensional linear program. Computational framework based on set oriented numerical methods is used for the finite dimensional approximation of the optimal stabilization problem. This finite dimensional approximation of the optimal stabilization problem lead to solving finite number of linear inequalities. The highlight of the solution approach for the finite dimensional linear program is that the controller obtained is deterministic although the approximation of the linear transfer operators is stochastic. Simulation results for the optimal stabilization of period two orbit is presented. One of the main botlenecks in the approach is that the size of the linear program scales as a function of the state-space discretization. Clearly, this becomes a huge problem for higher dimensional systems. The solution of higher dimensional systems will require development of algorithms that exploit the structure of the problem. These and other ideas on computational complexity management will be addressed in a subsequent paper.

## References

[1] L. Grune, "Error estimation and adaptive discretization for the discrete stochastic Hamilton-Jacobi-Bellman equation," *Numerische Mathematik*, vol. 99, pp. 85–112, 2004.

[2] L. G. Crespo and J. Q. Sun, "Solution of fixed final state optimal control problem vis simple cell mapping," *Nonlinear dynamics*, vol. 23, pp. 391–403, 2000.

[3] ——, "Stochastic optimal control via bellman principle," *Automatica*, vol. 39, pp. 2109–2114, 2003.

[4] O. Junge and H. Osinga, "A set oriented approach to global optimal control," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 10, no. 2, pp. 259–270, 2004.

[5] L. Grune and O. Junge, "A set oriented approach to optimal feedback stabilization," *Systems Control Lett.*, vol. 54, no. 2, pp. 169–180, 2005.

[6] D. Hernandez-Hernandez, O. Hernandez-Lerma, and M. Taksar, "A linear programming approach to deterministic optimal control problems," *Applicationes Mathematicae*, vol. 24, no. 1, pp. 17–33, 1996.

[7] V. Gaitsgory and S. Rossomakhine, "Linear programming approach to deterministic long run average optimal control problems," *SIAM J. Control ad Optimization*, vol. 44, no. 6, pp. 2006–2037, 2006.

[8] J. Lasserre, C. Prieur, and D. Henrion, "Nonlinear optimal control: Numerical approximation via moment and lmi- relaxations," in *Proceeding of IEEE Conference on Decision and Control*, Seville, Spain, 2005.

[9] U. Vaidya and P. G. Mehta, "Lyapunov measure for almost everywhere stability," *IEEE Transactions on Automatic Control*, vol. 53, no. 1, pp. 307–323, 2008.

[10] U. Vaidya, P. Mehta, and U. Shanbhag, "Nonlinear stabilization using control lyapunov meausre," in *Proceedings of IEEE Conference on Decision and Control*, New Orleans, LA, 2007.

[11] U. Vaidya, "Converse theorem for almost everywhere stability using Lyapunov measure," in *Proceedings of American Control Conference*, New York, NY, 2007.

[12] A. Lasota and M. C. Mackey, *Chaos, Fractals, and Noise: Stochastic Aspects of Dynamics*. New York: Springer-Verlag, 1994.

[13] U. Vaidya and P. G. Mehta, "Computation of Lyapunov measure for almost everywhere stability," in *Proceedings of IEEE Conference on Decision and Control*, San Diego, CA, 2006, pp. 5228–5233.

[14] O. L. Mangasarian, *Nonlinear programming*, ser. Classics in Applied Mathematics. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM), 1994, vol. 10, corrected reprint of the 1969 original.

[15] A. Raghunathan and U. Vaidya, *Optimal stabilization using Lyapunov measure*. http://home.eng.iastate.edu/ ugvaidya/: Preprints, 2008.