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Approximate Low Rank Solutions of Lyapunov Equations via Proper Orthogonal Decomposition

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***Abstract*— We present an algorithm to approximate the solution Z of a stable Lyapunov equation $AZ+ZA^*+BB^*=0$ using proper orthogonal decomposition (POD). This algorithm is applicable to large-scale problems and certain infinite dimensional problems as long as the rank of B is relatively small. In the infinite dimensional case, the algorithm does not require matrix approximations of the operators A and B. POD is used in a systematic way to provide convergence theory and simple a priori error bounds.**

I. Introduction

Lyapunov equations are one of the fundamental equations in systems and control theory, see e.g., [1]. For example, Lyapunov equations arise in Newton iterations for Riccati equations, which are used to compute optimal feedback control laws for linear systems [2]. We propose an algorithm based on proper orthogonal decomposition (POD) to compute approximate low rank solutions of stable Lyapunov equations of the form

$$AZ + ZA^* + BB^* = 0. \tag{1}$$

Recent research has focused on approximating the solution of large-scale Lyapunov equations, such as those arising from the discretization of an infinite dimensional system (e.g., see the recent paper [3] and the references therein). The solution of a matrix Lyapunov equation is often a full (dense) matrix, thus many recent large-scale algorithms compute factored low rank approximations to the Lyapunov solution.

The POD-based algorithm presented here also computes an approximate low rank solution to the Lyapunov equation. Unlike many other large-scale algorithms, the POD-based approach is not iterative; instead, the solution is constructed by simulating m linear differential equations, where m is the rank of B, and then computing POD eigenvalues and modes. The main computational cost of the algorithm is approximating the solutions of the linear differential equations. Thus, the proposed algorithm is applicable to large-scale systems when the rank of B is relatively small.

Some attractive features of the POD-based Lyapunov algorithm are:

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1) The algorithm is directly applicable to certain infinite dimensional problems even if matrix approximations of the A and B operators are not available.
2) The algorithm produces an approximation to a best low rank approximate solution, even in the infinite dimensional case.
3) Simple, computable a priori error bounds indicate the quality of the approximation and can guide the order of the approximation. The error bounds are again valid even in the infinite dimensional case.

This algorithm has great potential for infinite dimensional problems. In this case, one must solve infinite dimensional linear differential equations. These computations can be performed using existing simulation code; furthermore, tools such as adaptive solvers, parallel algorithms, multigrid methods, etc. can be used to increase computational efficiency and accuracy. Again, matrix approximations of A and B are not required. For some problems, such as linearized fluid flow, it may not be a simple task to obtain approximating matrices.

The convergence theory for the infinite dimensional case considered here simply requires convergence of the solutions of the infinite dimensional linear differential equations. In contrast, if one solves the Lyapunov equation using matrix approximations of A and B, then the convergence theory for this procedure is more complex (see, e.g., [4, Corollary 4.11]). It is possible that a "natural" discretization scheme may fail to satisfy the requirements of the theory and produce an incorrect approximation. For an example with a delay equation that is not dual convergent (a standard theoretical requirement), see [5].

II. The Algorithm

We now present the algorithm which is applicable to the matrix case and a certain infinite dimensional case. Throughout this work, we let X be a Hilbert space with inner product $(\cdot,\cdot)$ and corresponding norm $\|\cdot\|_X = (\cdot,\cdot)^{1/2}$. For the matrix Lyapunov equation, X is taken to be $\mathbb{R}^n$ and the inner product can be taken as the standard dot product, $(a,b) = a^T b$, or a weighted dot product, $(a,b) = a^T M b$, where $M \in \mathbb{R}^{n\times n}$ is symmetric positive definite.

We suppose A and B have the following properties. In the matrix case, $A \in \mathbb{R}^{n\times n}$ is exponentially stable and $B \in \mathbb{R}^{n\times m}$. In the infinite dimensional case, $A : D(A) \subset$

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$X \to X$ generates an exponentially stable C_0-semigroup e^{At} over X and $B : \mathbb{R}^m \to X$ is finite rank and bounded. This assumption implies that B must take the form

$$Bu = \sum_{j=1}^{m} b_j u_j,$$

where each $b_j \in X$ and $u = [\, u_1, \ldots, u_m \,]^T \in \mathbb{R}^m$ (see [6, Theorem 6.1]). Note that this representation for B also holds for the matrix problem; in this case, b_j is the jth column of B.

The algorithm to approximate the solution $Z : X \to X$ of the Lyapunov equation (1) can be briefly summarized as follows.

Main Algorithm:

1) Let w_j^N be an approximation to the solution w_j of the linear differential equation

$$\dot{w}_j(t) = Aw_j(t), \quad w_j(0) = b_j, \tag{2}$$

for $j = 1, \ldots, m$, where $Bu = \sum_{j=1}^m b_j u_j$.

2) Compute $\{\lambda_k^N\}$ and $\{\varphi_k^N\}$, the POD eigenvalues and modes of the dataset $\{w_j^N\}_{j=1}^m$, e.g., by method of snapshots (see Section III-B).
3) Choose r and form the rth order approximate Lyapunov solution $Z_r^N : X \to X$ given by

$$Z_r^N x = \sum_{k=1}^{r} \lambda_k^N (x, \varphi_k^N) \varphi_k^N, \tag{3}$$

where $(\cdot, \cdot)$ is the inner product over the Hilbert space.

In Section V below we discuss the choice of the order r and the approximation level N.

If desired, the approximate solution can be factored as $Z_r^N = R^* R$, where $R : X \to \mathbb{R}^r$ and its adjoint $R^* : \mathbb{R}^r \to X$ are defined by

$$\begin{aligned} Rx &= [\, (\lambda_1^N)^{1/2} (x, \varphi_1^N), \ldots, (\lambda_r^N)^{1/2} (x, \varphi_r^N) \,]^T, \\ R^* a &= \sum_{k=1}^{r} a_k (\lambda_k^N)^{1/2} \varphi_k^N, \quad a = [\, a_1, \ldots, a_r \,]^T. \end{aligned}$$

We note that for the dual Lyapunov equation

$$A^* Z + ZA + C^* C = 0, \tag{4}$$

one must instead approximate the solutions of the dual linear evolution equations

$$\dot{z}_j(t) = A^* z_j(t), \quad z_j(0) = c_j. \tag{5}$$

In the matrix case, c_j is the jth row of the matrix C. In the infinite dimensional case, we assume $C : X \to \mathbb{R}^p$ is bounded and finite rank so that C must have the form $Cx = [(x, c_1), \ldots, (x, c_p)]$, where each $c_j \in X$ (again, see [6, Theorem 6.1]). The remainder of the algorithm remains unchanged.

We now review proper orthogonal decomposition and the method of snapshots. We discuss the approximation error and the choice of r and N in Section V below.

III. The Continuous Proper Orthogonal Decomposition

The key to the proposed algorithm is that the Lyapunov operator is exactly the *continuous POD operator* of the set of functions $\{w_j\}_{j=1}^m$. This is shown in Proposition 1 below. This property is used to construct the approximate low rank Lyapunov solution.

We summarize the continuous proper orthogonal decomposition from the recent works of Kunisch and Volkwein [7], [8] and Henri and Yvon [9], [10], [11]. These works focus on the continuous POD for a finite time interval, however the theory extends naturally to the case of an infinite time interval.

Section III-A reviews properties of the continuous proper orthogonal decomposition and Section III-B focuses on approximating the POD eigenvalues and modes using the method of snapshots [12].

A. Properties of the Continuous POD

Let $L^2(0, \infty; X)$ be the set of all functions w such that $w(t) \in X$ for all $t \geq 0$ and whose X norm is square integrable, i.e.,

$$\|w\|_{L^2(0,\infty;X)} = \left(\int_0^\infty \|w(t)\|_X^2 \, dt \right)^{1/2} < \infty.$$

A sequence of functions $\{w_k\} \subset L^2(0, \infty; X)$ converges to $w \in L^2(0, \infty; X)$ if $\|w_k - w\|_{L^2(0,\infty;X)} \to 0$ as $k \to \infty$.

We now define the continuous proper orthogonal decomposition and discuss its properties.

Definition 1: The continuous POD operator $Z : X \to X$ for a dataset $\{w_j\}_{j=1}^m \subset L^2(0, \infty; X)$ is defined by

$$Zx = \int_0^\infty \sum_{j=1}^{m} (x, w_j(t)) w_j(t) \, dt. \tag{6}$$

The continuous POD operator is self adjoint, compact, and nonnegative; thus, the eigenvalues of Z may be ordered $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and the corresponding orthonormal eigenvectors $\{\varphi_k\} \subset X$ form a complete set.

Definition 2: The eigenvalues $\{\lambda_k\}$ of the continuous POD operator Z are called the POD eigenvalues of $\{w_j\}$ and the orthonormal eigenvectors $\{\varphi_k\} \subset X$ of Z are called the POD modes of $\{w_j\}$.

The POD eigenvalues are an indication of "energy content" and the POD modes are optimal for data reconstruction. First, the "total energy" in the dataset $\{w_j\}$ is contained in the POD eigenvalues:

$$\sum_{j=1}^{m} \int_0^\infty \|w_j(t)\|_X^2 \, dt = \sum_{k \geq 1} \lambda_k < \infty.$$

The POD modes $\{\varphi_k\}$ can be used to give an optimal reconstruction of the set $\{w_j\}$ in the following manner. Let w_j^r be the rth order projection of w_j onto the POD basis, i.e.,

$$w_j^r(t) = \sum_{k=1}^{r} (w_j(t), \varphi_k) \varphi_k. \tag{7}$$

Then the reconstruction error is given in terms of the sum of the neglected POD eigenvalues

$$\sum_{j=1}^{m} \int_0^\infty \left\| w_j(t) - w_j^r(t) \right\|_X^2 \, dt = \sum_{k>r} \lambda_k.$$

In the case of a finite time interval, no other orthonormal basis yields a smaller reconstruction error. This optimal reconstruction property likely extends to the case of an infinite time interval, however the author has not examined this case as it is not required for the current work.

B. Computing the Continuous POD via the Method of Snapshots

An important feature of proper orthogonal decomposition is that the POD eigenvalues and modes of a time varying dataset $\{w_j\}_{j=1}^m \subset L^2(0,\infty;X)$ can be computed using the method of snapshots. The main idea is to approximate each w_j with functions whose POD eigenvalues and modes are easily computable. The following result guarantees that these approximate POD eigenvalues and modes converge to the POD eigenvalues and modes of $\{w_j\}$.

Theorem 1: Let $w_j^N \in L^2(0,\infty;X)$ be a sequence of functions converging to $w_j \in L^2(0,\infty;X)$ for each $j = 1,\dots,m$. Let $\{\lambda_k^N, \varphi_k^N\}$ and $\{\lambda_k, \varphi_k\}$ denote the POD eigenvalues and modes of $\{w_j^N\}_{j=1}^m$ and $\{w_j\}_{j=1}^m$, respectively. Then for each k,

$$\lim_{N\to\infty} |\lambda_k^N - \lambda_k| = 0, \quad \lim_{N\to\infty} \|\varphi_k^N - \varphi_k\|_X = 0.$$

Furthermore, as $N \to \infty$,

$$\sum_{k\geq 1} \lambda_k^N \to \sum_{k\geq 1} \lambda_k. \tag{8}$$

A popular approach to the method of snapshots is to use piecewise constant functions (in time) to approximate the functions w_j. For simplicity, we focus on the case $m = 1$, i.e., there is only one function in the dataset. The algorithm is similar for $m > 1$.

Method of Snapshots (for $m = 1$):

1) Let $a_j \approx w(t_j)$ be approximate snapshots of $w(t)$ at times $0 = t_0 < t_1 < \cdots < t_N = T$ for $j = 0,\dots,N$.
2) Let $v_j = (a_j + a_{j-1})/2$ be the approximate average value of $w(t)$ over the jth time interval for $j = 1,\dots,N$.
3) Let $\delta_j = t_j - t_{j-1}$ be the jth time step for $j = 1,\dots,N$.
4) Let Γ be the symmetric $N \times N$ matrix whose entries are the inner products $\Gamma_{ij} = (\delta_j^{1/2} v_j, \delta_i^{1/2} v_i)$.
5) Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$ be the ordered eigenvalues of Γ with corresponding orthonormal eigenvectors $\{\gamma_k\}_{k=1}^N$.
6) The (approximate) POD eigenvalues are given by $\{\lambda_k\}$ and, if $\lambda_k \neq 0$, the (approximate) kth POD mode is

$$\varphi_k = \lambda_k^{-1/2} \sum_{j=1}^{N} \delta_j^{1/2} [\gamma_k]_j v_j,$$

where $[\gamma_k]_j$ is the jth component of γ_k.

We note that this algorithm is often implemented using an equally spaced time grid.

Remark: Another method of computing POD eigenvalues and modes is to approximate the time integral in the continuous POD operator (6) by quadrature. This approach leads to a similar algorithm.

IV. Derivation of the Algorithm

We now give a derivation of the algorithm.

Given the assumptions of Section II, the exact solution $Z : X \to X$ of the Lyapunov equation is given by [13, Theorem 4.1.23]

$$Zx = \int_0^\infty e^{At} B B^* e^{A^* t} x \, dt.$$

We now show that the Lyapunov solution equals the continuous POD operator for the dataset $\{w_j\}$ given in the main algorithm.

Proposition 1: The unique solution $Z : X \to X$ of the Lyapunov equation (1) takes the form

$$Zx = \int_0^\infty \sum_{j=1}^{m} (x, w_j(t)) w_j(t) \, dt, \tag{9}$$

where each w_j is the exact solution of the linear evolution equation (2).

Proof: The solution may be factored as $Z = \mathcal{B}\mathcal{B}^*$, where $\mathcal{B} : L^2(0,\infty;\mathbb{R}^m) \to X$ is defined by

$$\mathcal{B}u = \int_0^\infty e^{At} B u(t) \, dt$$

and $\mathcal{B}^* : X \to L^2(0,\infty;\mathbb{R}^m)$, the adjoint of $\mathcal{B}$, is given by $\mathcal{B}^* x = B^* e^{A^* t} x$. Again, given the assumptions above on B, the operator must have the form $Bu = \sum_{j=1}^m b_j u_j$, where $u = [\, u_1, \dots, u_m \,]^T \in \mathbb{R}^m$, and each b_j is in X. Then we have

$$\mathcal{B}u = \int_0^\infty e^{At} B u(t) \, dt = \int_0^\infty \sum_{j=1}^{m} u_j(t) w_j(t) \, dt,$$

where $w_j(t) = e^{At} b_j$ for $j = 1,\dots,m$. This implies that each $w_j \in L^2(0,\infty;X)$ is the solution of the linear evolution equation (2). The adjoint operator $\mathcal{B}^* : X \to L^2(0,\infty;\mathbb{R}^m)$ is easily computed to be

$$[\mathcal{B}^* x](t) = [\, (x, w_1(t)), \dots, (x, w_m(t)) \,]^T.$$

Again using $Z = \mathcal{B}\mathcal{B}^*$ gives the expression (9). ∎

Remark: The representation (9) could be very useful if one only needed to compute the product of the Lyapunov operator with a few vectors in X. This could be accomplished by computing the solutions of the linear differential equations (2) and approximating the time integral in (9) by quadrature or some other method. However, if one obtained an approximate Lyapunov solution in this fashion, the result would likely not have low rank.

Since the Lyapunov solution Z equals the continuous POD operator for $\{w_j\}$, the POD eigenvalues and modes equal, by definition, the eigenvalues and orthonormal eigenvectors of Z.

Corollary 1: Let $w_j \in L^2(0,\infty;X)$ be the exact solution of the linear evolution equations (2), for $j = 1,\dots,m$. The POD eigenvalues $\{\lambda_k\}$ and modes $\{\varphi_k\} \subset X$ of the dataset $\{w_j\}$ are the eigenvalues and orthonormal eigenvectors of the unique solution $Z : X \to X$ of the Lyapunov equation (1).

The truncated eigenvalue expansion of Z is given by

$$Z_r x = \sum_{k=1}^{r} \lambda_k (x, \varphi_k)\varphi_k. \tag{10}$$

To complete the algorithm, we simply approximate the POD eigenvalues and modes to use in the truncated eigenvalue expansion of the Lyapunov solution Z.

V. Approximation Theory and Error Bounds

A. Notation and Background

In order to discuss the properties of the approximate Lyapunov solution, we first introduce some notation and background material.

Let K be a compact linear operator from a Hilbert space X_1 to a Hilbert space X_2. The operator norm of K is defined by

$$\|K\| = \sup_{x\in X_1, x\neq 0} \frac{\|Kx\|}{\|x\|} = \sigma_1,$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$ denote the singular values of K in decreasing order. The much stronger trace (or nuclear) norm of K equals the sum of all of the singular values of K, i.e.,

$$\|K\|_{\mathrm{tr}} = \sum_{k\geq 1} \sigma_k.$$

A best rank r approximation, K_r, to K is given by a solution of the following problem: find the minimizer over all rank r operators F_r of the operator norm error $\|K - F_r\|$. A solution of this problem (which may not be unique) is given by the rth order truncated singular value decomposition of K. The best value of the operator norm error is σ_{r+1}, the first neglected singular value. The truncated singular value decomposition also gives a best rank r approximation of K if the norm is taken to be the trace norm. In this case, the best trace norm error is given by $\sum_{k>r} \sigma_k$, the sum of the neglected singular values.

In this work, many of the operators we consider map a Hilbert space into itself and are compact, self adjoint, and nonnegative. The eigenvalues of such an operator can be ordered $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. Furthermore, the eigenvalues are equal to the singular values and the truncated eigenvalue expansion is equal to the truncated singular value decomposition. Thus, the truncated eigenvalue expansion provides the best low rank approximation in this case.

B. Main Results

We now state the main theoretical results. The proofs will be given in a later work.

As in the algorithm in Section II, we let $\{w_j^N\}_{j=1}^m$ be approximations of the solutions $\{w_j\}_{j=1}^m$ of the solutions of the differential equations (2). We let $\{\lambda_k^N, \varphi_k^N\}$ and $\{\lambda_k, \varphi_k\}$ denote the POD eigenvalues and modes of $\{w_j^N\}$ and $\{w_j\}$, respectively. Recall from Corollary 1 above that $\{\lambda_k, \varphi_k\}$ are also the eigenvalues and orthonormal eigenvectors of the Lyapunov operator.

Let $Z_r^N : X \to X$ as defined in (3) denote the approximate Lyapunov solution. Also let $Z_r : X \to X$ as defined in (10) denote the rth order truncated eigenvalue expansion of the Lyapunov solution.

Our first result is that the approximate POD eigenvalues and modes converge to the eigenvalues and orthonormal eigenvectors of the Lyapunov solution Z; thus, Z_r^N converges to Z_r, a best rank r approximation of Z.

Theorem 2: Let r be given. Suppose, for $j = 1,\dots,m$, $w_j^N \to w_j$ in $L^2(0,\infty;X)$ as $N \to \infty$. Then $\lambda_k^N \to \lambda_k$ and $\varphi_k^N \to \varphi_k$ for $1 \leq k \leq r$. Also, as the POD eigenvalues and modes converge, $Z_r^N \to Z_r$ in the operator norm.

Furthermore, the approximation error between Z and Z_r^N in the operator norm depends on the speed of the convergence of the POD eigenvalues and modes.

Theorem 3: The operator norm error between Z_r^N and Z, the exact solution to the Lyapunov equation (1), is bounded as follows:

$$\|Z - Z_r^N\| \leq \lambda_{r+1} + \sum_{k=1}^{r} \left(|\lambda_k - \lambda_k^N| + 2\lambda_k^N \|\varphi_k - \varphi_k^N\|_X \right).$$

By Theorem 2, the second term in the error bound converges to zero as each $w_j^N \to w_j$ in $L^2(0,\infty;X)$. Also, by Theorem 1, the first term in the error bound, λ_{r+1}, can be approximated by λ_{r+1}^N. Thus, if the first $r+1$ POD eigenvalues and the first r POD modes are converged, then λ_{r+1}^N is a good approximation of the operator norm error bound between Z and Z_r^N.

The following result gives a bound on the approximation error in the stronger trace norm.

Theorem 4: The trace norm error between Z_r^N and Z, the exact solution to the Lyapunov equation (1), is bounded as follows:

$$\|Z - Z_r^N\|_{\mathrm{tr}} \leq \sum_{k>r} \lambda_k^N + C^N \left(\sum_{j=1}^{m} \|w_j - w_j^N\|_{L^2(0,\infty;X)}^2 \right)^{1/2},$$

where the constant C^N is given by

$$C^N = \left(\sum_{k\geq 1} \lambda_k^N \right)^{1/2} + \left(\sum_{k\geq 1} \lambda_k \right)^{1/2}.$$

As each $w_j^N \to w_j$ in $L^2(0,\infty;X)$, the last term in the error bound tends to zero, and also

$$\sum_{k>r} \lambda_k^N \to \sum_{k>r} \lambda_k, \quad C^N \to 2\left(\sum_{k\geq 1} \lambda_k \right)^{1/2}.$$

We note that both terms in the error bound can be approximately computed or estimated. First, the sum of the neglected POD eigenvalues, $\sum_{k>r} \lambda_k^N$, is computable. For the second term in the error bound, the constant C^N cannot be computed exactly; however it can be approximated by

$$C^N \approx 2\left(\sum_{k\geq 1} \lambda_k^N \right)^{1/2}.$$

This is due to equation (8) in Theorem 1. The $L^2(0,\infty;X)$ errors between w_j and w_j^N are not computable. However, error bounds or estimators can often be used to approximate this error term.

This error bound also points to one potential strength of this algorithm. Since the $L^2(0,\infty;X)$ errors between w_j and w_j^N appear in the error bound, error estimators or adaptive solvers can be used to guide refined computations to reduce the approximation error. Of course, increasing the order, r, will decrease the first term in the error bound.

VI. Numerical Results for a Model Problem

In this section, we present numerical results for an infinite dimensional model problem. The results are compared with matrix Lyapunov computations using matrix approximations of the infinite dimensional operators.

A. The Model Problem

We take the A and B operators from the one dimensional convection diffusion equation

$$\begin{aligned} w_t(t,x) &= \mu w_{xx}(t,x) - \kappa w_x(t,x) + b(x)u(t), \\ w(t,0) &= 0, \quad w(t,1) = 0, \quad w(0,x) = w_0(x), \end{aligned}$$

where subscript denote partial derivatives, μ is a positive constant, and κ is a real constant. The function $b(x)$ is in $L^2(0,1)$.

Let the Hilbert space X equal $L^2(0,1)$, the space of square integrable functions, with the standard inner product $(f,g) = \int_0^1 f(x)g(x)\,dx$. The A operator is defined by

$$Aw = \mu w_{xx} - \kappa w_x, \quad D(A) = H^2 \cap H_0^1$$

and B is given by $[Bu](x) = b(x)u$. Here, H^m is the standard Sobolev space of functions with m derivatives all of which are square integrable; also, any function $w \in H_0^1$ must satisfy the Dirichlet boundary conditions $w(0) = 0$ and $w(1) = 0$.

The eigenvalues of the convection diffusion operator A are given by $\lambda_n = -\mu n^2\pi^2 - \kappa^2/4\mu$. Since the eigenvalues are all negative and bounded away from the imaginary axis, the results in [14] and [13, Section 2.3] can be used to show that A generates an exponentially stable C_0-semigroup.

B. Numerical Results

We now compare the numerical results of the POD-based algorithm with matrix Lyapunov computations using matrix approximations of the A and B operators.

For the computations, we chose $\mu = 0.1$, $\kappa = 1$, and $b(x) = 5(1-x)^2\sin(\pi x)$. Standard piecewise linear finite elements were used for the spatial discretization of the partial differential equation (2). The discretized equations were integrated over $0 \le t \le 2$ using Matlab's `ode15s` solver with default error tolerances; at $t = 2$, the numerical solution is nearly zero. The time points returned from `ode15s` were used in the method of snapshots to approximate the POD eigenvalues and modes.

Standard piecewise linear finite elements were also used to provide the matrix approximations of the A and B operators for the matrix Lyapunov computations. Matlab's `lyap` function was used to solve the resulting matrix Lyapunov equations.

Figure 1 shows the POD eigenvalues computed by the method of snapshots for $N = 64$, 128, and 256 equally spaced finite element nodes. Eigenvalue computations for the matrix Lyapunov solution using the standard matrix approximations produced similar results. The larger POD

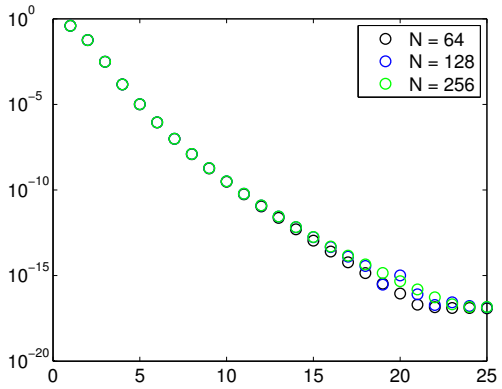

Fig. 1. POD eigenvalues computed using $N = 64$, 128, and 256 equally spaced finite element nodes.

eigenvalues are converged at this level of refinement; the POD eigenvalues nearer to machine precision (10^{-16}) have not yet converged. Further refinement is unnecessary since only the larger POD eigenvalues are used to construct the approximate Lyapunov solution.

Figure 2 shows the first POD mode computed by the method of snapshots for $N = 32$ equally spaced finite element nodes. The mode is converged at this level of refinement. The other POD modes converged in a similar

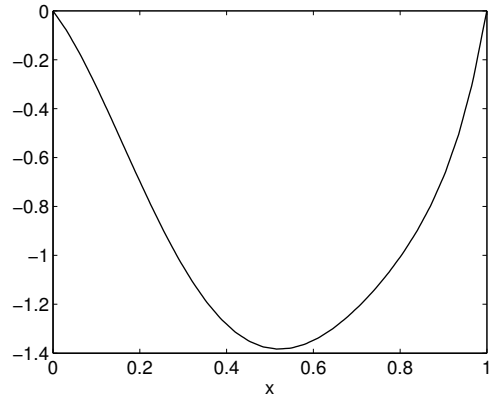

Fig. 2. The first POD mode computed using $N = 32$ equally spaced finite element nodes.

fashion, however the higher numbered modes were slower to converge under refinement. This behavior is likely due to the fact that the higher numbered modes tend to oscillate more

than the lower numbered modes. Eigenvector computations for the matrix Lyapunov solution using the standard matrix approximations produced similar results.

Figure 3 shows approximate Lyapunov solutions acting on $w = \exp(x)$. POD-based approximations are shown with $N = 32$ equally spaced finite element nodes with orders $r = 1$ and $r = 2$. The matrix Lyapunov computations using the standard matrix approximations is shown with $N = 128$ equally spaced finite element nodes for comparison. The low

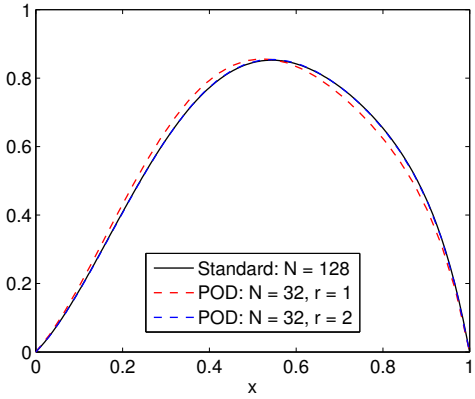

Fig. 3. Approximate Lyapunov solutions acting on $w = \exp(x)$.

order POD-based approximations give excellent agreement with the refined standard matrix approximation computations. In particular, for $r = 2$ the POD approximation is indistinguishable from the result of the standard computation.

The operator norm error bound in Theorem 3 gives a good indication of the accuracy of the POD-based approximation without comparison to other computations. Recall $\|(Z - Z_r^N)w\| \leq \|Z - Z_r^N\|\|w\|_X$. As discussed after Theorem 3, we approximate $\|Z - Z_r^N\|$ by λ_{r+1}^N. For $w(x) = \exp(x)$, $\|w\|_X \approx 1.7873$. For $r = 1$, $\|Z - Z_r^N\| \approx 0.0569$; for $r = 2$, $\|Z - Z_r^N\| \approx 0.0031$. These values give approximate error bounds for $\|(Z - Z_r^N)w\|$ of 0.1016 for $r = 1$ and 0.0055 and $r = 2$. The above computations agree with these approximate error bounds.

We also look at the the trace norm error bound in Theorem 4. For $r = 1$, the sum of the neglected eigenvalues is approximately 0.0601; for $r = 2$, this sum is approximately 0.0032. These values also give a good estimate of the approximation error. Of course, the full error bound involves the $L^2(0, \infty; X)$ error between the exact and approximate solution to the partial differential equation (2); we do not attempt to estimate this here.

VII. Conclusion

We presented a POD-based algorithm to compute approximate low rank solutions of Lyapunov equations. The algorithm is applicable to large-scale matrix problems as well as a class of infinite dimensional problems. Since the algorithm is based on approximating the solutions of linear evolution equations, the computations can use existing simulation code as well as tools such as adaptive solvers and parallel algorithms. The quality of the approximate solution can be ascertained by simple, computable a priori error bounds. Numerical results confirmed the convergence theory.

In future work, we will compare this approach with other large-scale matrix Lyapunov solvers. We also will consider other classes of infinite dimensional systems, such as those with an unbounded B operator.

We also note that the solution of Lyapunov equations plays an important role in standard methods to compute truncated balanced reduced order models of linear systems (see, e.g., [1], [15]). Although the POD-based algorithm presented here could be used for these Lyapunov computations, we propose that it is more natural to use Rowley's POD-based algorithm for approximate balanced truncation [16]. (In fact, Rowley's algorithm inspired the present work and also [17], which extends the algorithm in [16] to an infinite dimensional case.) This method requires the solution of the linear differential equations (2) and (5) and bypasses the solution of Lyapunov equations (1) and (4).

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