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Lyapunov Redesign Approach to Output Regulation of Nonlinear Systems Using Conditional Servocompensators*

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Abstract— **This paper studies the output regulation of nonlinear systems using conditional servocompensators. Previous work introduced the conditional servocompensator that acts as a traditional servocompensator in a neighborhood of the zero-error manifold, while acting as a stable system outside a boundary layer, leading to improvement in the transient response while achieving zero steady-state tracking error in the presence of time-varying exogenous signals. The conditional servocompensator tool was introduced for sliding-mode feedback controllers. This paper extends the technique to more general feedback controllers by using Lyapunov redesign and saturated high-gain feedback.**

I. INTRODUCTION

The nonlinear servomechanism problem deals with the design of a controller to make the output of an uncertain plant asymptotically track reference signals and reject disturbance signals, both produced by an autonomous external system called the *exosystem*. Over the past two decades, several researchers have contributed toward the investigation of this important, yet challenging, problem. A good account of the available results for nonlinear systems can be found in [2], [5], [6], [7]. In this paper, we focus our attention on the earlier work [10], [11], [12] of Khalil and co-workers, where the idea of conditional servocompensators is introduced. The key feature of this idea is that the conditional servocompensator acts as a traditional servocompensator only in a neighborhood of the zero-error manifold, while it is a bounded-input-bounded-state system whose state is guaranteed to be of the order of a small design parameter. The use of conditional servocompensators enables us to achieve zero steady-state tracking error without degrading the transient response of the system. The idea was introduced in [10] and [11] in a sliding mode control framework, where [10] dealt with the special case of conditional integrator for constant exogenous signals, while the more general case of time-varying signals was treated in [11]. To extend the design beyond the sliding mode control, [12] developed the conditional integrator using Lyapunov redesign and saturated high-gain feedback. Starting with any stabilizing state feedback controller, [12] shows how to include a conditional integrator by modifying the original controller. The objective of this paper is to complete the development of [12] by addressing the servomechanism problem for time-varying exogenous signals.

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II. SYSTEM DESCRIPTION AND ASSUMPTIONS

Consider the nonlinear system

$$\begin{aligned} \dot{\zeta} &= \tilde{f}(\zeta, w) + \tilde{G}(\zeta, w)u \\ e &= \tilde{h}(\zeta, w) \end{aligned} \tag{1}$$

where $\zeta \in R^n$ is the state and $u \in R^m$ is the control input. The plant is subjected to a set of *exogenous* input variables w that belong to a compact set $\mathcal{W} \in R^w$, which include unknown disturbances to be rejected and references to be tracked. The variable $e \in R^p$ denotes the error, which is a function of the state ζ and the exogenous input w. The functions $\tilde{f}$, $\tilde{G}$ and $\tilde{h}$ are sufficiently smooth in ζ on a domain $\Xi \subset R^n$ and are continuous in w for $w \in \mathcal{W}$. Our goal is to design a controller to asymptotically regulate e to zero.

Assumption 1: $w(t)$ is generated by the known exosystem

$$\dot{w} = S_0 w \tag{2}$$

where S_0 has distinct eigenvalues on the imaginary axis and $w(t)$ belongs to a compact set $\mathcal{W}$.

Assumption 2: There exist a continuously differentiable mapping $\zeta = \pi(w)$, with $\pi(0) = 0$, and a continuous mapping $\chi(w)$ that solve the equations

$$\begin{aligned} \frac{\partial \pi(w)}{\partial w} S_0 w &= \tilde{f}(\pi, w) + \tilde{G}(\pi, w)\chi(w) \\ 0 &= h(\pi, w) \end{aligned} \tag{3}$$

for all $w \in \mathcal{W}$.

Assumption 3: There exists a set of real numbers $c_0, ..., c_{q-1}$ such that $\chi(w)$ satisfies the identity

$$L_s^q \chi = c_0 \chi + c_1 L_s \chi + \cdots + c_{q-1} L_s^{q-1} \chi \tag{4}$$

for all $w \in \mathcal{W}$, where $L_s \chi = (\partial \chi / \partial w) S_0 w$ and the characteristic polynomial

$$p^q - c_{q-1} p^{q-1} - \cdots - c_0$$

has distinct roots on the imaginary axis.

Defining

$$S = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ c_0 & \cdots & \cdots & \cdots & c_{q-1} \end{bmatrix}, \quad \tau = \begin{bmatrix} \chi \\ L_s \chi \\ \vdots \\ L_s^{q-2} \chi \\ L_s^{q-1} \chi \end{bmatrix}$$

and $\Gamma = [1 \ 0 \ \cdots \ 0]_{1 \times q}$, it can be shown [6] that $\chi(w)$ is generated by the internal model

$$\frac{\partial \tau(w)}{\partial w} S_0 w = S\tau(w), \quad \chi(w) = \Gamma \tau(w) \tag{5}$$

With the change of variables $x = \zeta - \pi$, the system (1) can be represented by

$$\dot{x} = f(x, w) + G(x, w)[u - \chi(w)] \tag{6}$$

where $f(x, w) = \tilde{f}(x + \pi, w) - \tilde{f}(\pi, w) + [\tilde{G}(x + \pi, w) - \tilde{G}(\pi, w)]\chi(w)$ and $G(x, w) = \tilde{G}(x + \pi, w)$.

The system (6) is in the form where the state feedback regulation problem can be formulated as a state feedback stabilization problem by treating $\chi(w)$ as a matched uncertainty. We start by assuming a stabilizing state feedback control for

$$\dot{x} = f(x, w) + G(x, w)u$$

and a corresponding Lyapunov function for the closed loop system.

Assumption 4: There exists a locally Lipschitz function $\psi(x, w)$, with $\psi(0, w) = 0$, and a continuously differentiable Lyapunov function $V(x, w)$, possibly unknown, such that

$$\alpha_1(\|x\|) \le V(x, w) \le \alpha_2(\|x\|) \tag{7}$$

$$\frac{\partial V}{\partial w} S_0 w + \frac{\partial V}{\partial x}[f(x, w) + G(x, w)\psi(x, w)] \le -W(x) \tag{8}$$

$\forall x \in \mathcal{X} \subset R^n$ and $\forall w \in \mathcal{W}$, where α_1 and α_2 are class $\mathcal{K}$ functions and $W(x)$ is a continuous positive definite function.

The system (6) can also be written as

$$\begin{aligned} \dot{x} \;\; = \;\; & f(x, w) + G(x, w)\psi(x, w) \\ & + G(x, w)u - G(x, w)[\chi(w) + \psi(x, w)] \end{aligned} \tag{9}$$

We wish to design a saturated high-gain feedback controller for this system to deal with the uncertain term $\chi(w)$. Towards that end, let $\Omega = \{V(w, x) \le c_1\} \subset \mathcal{X}$ be a compact set for some $c_1 > 0$ and $\delta(x)$ be a function such that

$$\|\chi(w) + \psi(x, w)\| \le \delta(x) \qquad \forall x \in \Omega, \quad \forall w \in \mathcal{W} \tag{10}$$

Assumption 5: $(\partial V/\partial x)G(x, w)$ can be expressed as

$$(\partial V/\partial x)G(x, w) = \upsilon^T(x)H(x, w) \tag{11}$$

where $\upsilon(x)$ is a known, locally Lipschitz function, with $\upsilon(0) = 0$, and $H(x, w)$ is a, possibly unknown, function that satisfies

$$H^T(x, w) + H(x, w) \ge 2\lambda I_m, \;\; \|H(x, w)\| \le k; \; k \ge \lambda > 0$$

$\forall x \in \Omega$ and $\forall w \in \mathcal{W}$, where I_m is $m \times m$ identity matrix.

III. Control Design

We introduce the *conditional servocompensator* [10], [11] via the saturated high gain feedback controller

$$u = -\alpha(x)\phi\left(\frac{s}{\mu}\right) \tag{12}$$

where $s = \upsilon(x) + K_1\sigma$,

$$\phi\left(\frac{s}{\mu}\right) = \begin{cases} \frac{s}{\|s\|} & \text{if} \quad \|s\| \ge \mu \\ \frac{s}{\mu} & \text{if} \quad \|s\| \le \mu \end{cases} \tag{13}$$

and σ is output of the conditional servocompensator

$$\dot{\sigma} = (S - JK_1)\sigma + \mu J\phi\left(\frac{s}{\mu}\right) \tag{14}$$

with $\mu > 0$ being the width of the boundary layer, $J = [0, \cdots, 0, 1]^T$ and K_1 chosen such that $S - JK_1$ is Hurwitz, which is always possible since the pair (S, J) is controllable. The function $\alpha(x)$ satisfies

$$\alpha(x) \ge \frac{k}{\lambda}\delta(x) + \alpha_0, \quad \alpha_0 > 0 \tag{15}$$

Equation (14) is a perturbation of the exponentially stable system $\dot{\sigma} = (S - JK_1)\sigma$, with the norm of the perturbation bounded by μ. Our next step is to show that the saturated high-gain controller (12) achieves practical stabilization of the system (6). We define the Lyapunov function

$$V_0(\sigma) = \sigma^T P_0 \sigma \tag{16}$$

where the symmetric positive definite matrix P_0 is the solution of $P_0 A_\sigma + A_\sigma^T P_0 = -I$ and $A_\sigma \triangleq S - JK_1$. Consider the compact set

$$\left\{\sigma : V_0(\sigma) \le \mu^2 c_2\right\}$$

where c_2 is a positive constant. Using the inequality

$$\dot{V}_0 \le -\|\sigma\|^2 + 2\mu\|\sigma\|\|P_0 J\|$$

it is easy to show that $\dot{V}_0 \le 0$ on the boundary $V_0(\sigma) = \mu^2 c_2$ for the choice $c_2 = 4\|P_0 J\|^2 \lambda_{max}(P_0)$. Hence, $\left\{\sigma : V_0(\sigma) \le \mu^2 c_2\right\}$ is positively invariant. We require $\sigma(0)$ to belong to this set.

IV. Analysis

In this section we will show that, for sufficiently small μ, every trajectory of the closed-loop system (2), (6), (12) and (14) asymptotically approaches an invariant manifold on which the error is zero. The forthcoming analysis shares many points in common with the ones in [12], apart from various technical differences due to nature of the problem.

The closed-loop system is given by

$$\begin{aligned} \dot{w} \;\; &= \;\; S_0 w \\ \dot{x} \;\; &= \;\; f(x, w) + G(x, w)\psi(x, w) - \alpha(x)G(x, w)\phi\left(\frac{s}{\mu}\right) \\ & \quad\;\; - G(x, w)[\chi(w) + \psi(x, w)] \qquad (17) \\ \dot{\sigma} \;\; &= \;\; A_\sigma \sigma + \mu J\phi\left(\frac{s}{\mu}\right) \end{aligned}$$

We start by showing that the set $\Psi = \Omega \times \left\{V_0(\sigma) \le \mu^2 c_2\right\}$ is positively invariant and every trajectory in Ψ reaches the positively invariant set $\Psi_\mu = \{V(x) \le \rho(\mu)\} \times \left\{V_0(\sigma) \le \mu^2 c_2\right\}$

in finite time, where ρ is a class $\mathcal{K}$ function.

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial w}S_0 w + \frac{\partial V}{\partial x}[f(x,w)+G(x,w)\psi(x,w)] \\ &\quad - \frac{\partial V}{\partial x}G(x,w)\alpha(x)\phi\left(\frac{s}{\mu}\right) \\ &\quad - \frac{\partial V}{\partial x}G(x,w)[\chi(w)+\psi(x,w)] \\ &\leq -W(x) - \alpha(x)(s-K_1\sigma)^T H(x,w)\phi\left(\frac{s}{\mu}\right) \\ &\quad - (s-K_1\sigma)^T H(x,w)[\chi(w)+\psi(x,w)]\end{aligned}$$

Inside Ψ, $\|\sigma\| \leq \mu\sqrt{c_2/\lambda_{min}(P_0)}$. Using this along with (13) and (15), it can be shown that when $\|s\| \geq \mu$, we have

$$\begin{aligned}\dot{V} &\leq -W(x) - \lambda\alpha(x)\|s\| + k\delta(x)\|s\| \\ &\quad + \|K_1\|\,\|\sigma\|\,k[\alpha(x)+\delta(x)] \\ &\leq -W(x) + \mu\gamma_1 \qquad (18)\end{aligned}$$

where $\gamma_1 = \max_{x\in\Omega} kk_0[\alpha(x)+\delta(x)]$ and $k_0 = \|K_1\|\sqrt{c_2/\lambda_{min}(P_0)}$. Similarly, when $\|s\| \leq \mu$, we have

$$\begin{aligned}\dot{V} &\leq -W(x) - \lambda\alpha(x)\frac{\|s\|^2}{\mu} + k\delta(x)\|s\| \\ &\quad + \alpha(x)\|K_1\|\,\|\sigma\|\,k\frac{\|s\|}{\mu} + \delta(x)\|K_1\|\,\|\sigma\|\,k \\ &\leq -W(x) + \mu\gamma_2 \qquad (19)\end{aligned}$$

where $\gamma_2 = \max_{x\in\Omega}\ kk_0[\alpha(x)+\delta(x)(1+1/k_0)] \geq \gamma_1$. From (18) and (19), $\dot{V} \leq -W(x) + \mu\gamma_2$, $\forall (x,\sigma) \in \Psi$. Hence, from [8, Theorem 4.18], for sufficiently small μ, Ψ is positively invariant and all trajectories starting in Ψ enter a positively invariant set $\Psi_\mu = \{V(x) \leq \rho(\mu)\} \times \{V_0(\sigma) \leq \mu^2 c_2\}$ in finite time.

Next, we use $V_1 = \frac{1}{2}s^T s$ and Assumption 6, below, to show that the trajectories reach the boundary layer $\{\|s\| \leq \mu\}$ in finite time.

Assumption 6: $N(x,w) \triangleq (\partial\upsilon/\partial x)G(x,w)$ satisfies

$$N(x,w) + N^T(x,w) \geq 2\lambda_p I_m,\ \ \|N(x,w)\| \leq k_p$$

where $k_p \geq \lambda_p > 0$, for all $x \in \{V(x) \leq \rho(\mu)\}$ and $w \in \mathcal{W}$. Moreover, $\alpha(0) \geq \frac{k_p}{\lambda_p}\delta(0) + \alpha_0,\ \alpha_0 > 0$.

For $(x,\sigma) \in \Psi_\mu$ and $\|s\| \geq \mu$, we have

$$\begin{aligned}s^T\dot{s} &\leq -\alpha(x)\lambda_p\|s\| + \|N(x,w)\|\,\|\chi(w)+\psi(x,w)\|\,\|s\| \\ &\quad + \left\|\frac{\partial\upsilon}{\partial x}[f(x,w)+G(x,w)\psi(x,w)]\right\|\|s\| \\ &\quad + (\|\sigma\|\,\|K_1\|\,\|A_\sigma\| + \mu\|K_1\|\,\|J\|)\|s\| \qquad (20)\end{aligned}$$

Inside Ψ_μ, $\|\sigma\| \leq \mu\sqrt{c_2/\lambda_{min}(P_0)}$. Also, the function $\frac{\partial\upsilon}{\partial x}[f(x,w)+G(x,w)\psi(x,w)]$ is continuous such that $\frac{\partial\upsilon}{\partial x}[f(0,w)+G(0,w)\psi(0,w)] = 0$. Therefore, the norm $\left\|\frac{\partial\upsilon}{\partial x}[f(x,w)+G(x,w)\psi(x,w)]\right\|$ together with the norms $\|\sigma\|\,\|K_1\|\,\|A_\sigma\|$ and $\mu\|K_1\|\,\|J\|$ can be bounded by a class $\mathcal{K}$ function $\rho_1(\mu)$. Hence,

$$\begin{aligned}s^T\dot{s} &\leq -\alpha(x)\lambda_p\|s\| + k_p\delta(x)\|s\| + \rho_1(\mu)\|s\| \\ \implies\quad \dot{V}_1 &\leq -\lambda_p\left[\alpha(x) - \frac{k_p}{\lambda_p}\delta(x) - \frac{\rho_1(\mu)}{\lambda_p}\right]\|s\| \\ &\leq -\lambda_p\left[\alpha_0 - \frac{\rho_1(\mu)}{\lambda_p}\right]\|s\| \qquad (21)\end{aligned}$$

Thus, for sufficiently small μ, all trajectories inside Ψ_μ reach the boundary layer $\{\|s\| \leq \mu\}$ in finite time.

Finally, under Assumption 7, below, we show that inside the boundary layer the trajectories of the closed-loop system asymptotically approach an invariant manifold on which the error is zero.

Assumption 7: There exist non-negative constants k_1 to k_6 such that

$$\|\psi(x,w)\| \leq k_1\|\upsilon(x)\| + k_2\sqrt{W(x)}$$

$$\left\|\frac{\partial\upsilon}{\partial x}[f(x,w)+G(x,w)\psi(x,w)]\right\| \leq k_3\|\upsilon(x)\| + k_4\sqrt{W(x)}$$

$$\left|\frac{\alpha(x)-\alpha(0)}{\alpha(0)}\right| \leq k_5\|\upsilon(x)\| + k_6\sqrt{W(x)}$$

$\forall w \in \mathcal{W}$, in some neighborhood of $x = 0$.

Inside the boundary layer, the closed-loop system (17) is given by

$$\begin{aligned}\dot{w} &= S_0 w \\ \dot{x} &= f(x,w) + G(x,w)\psi(x,w) - \alpha(x)G(x,w)(s/\mu) \\ &\quad - G(x,w)[\chi(w)+\psi(x,w)] \qquad (22) \\ \dot{\sigma} &= S\sigma + J\upsilon\end{aligned}$$

From [10], there exists a unique matrix Λ such that

$$S\Lambda = \Lambda S \qquad \text{and} \qquad -K_1\Lambda = \Gamma$$

We define

$$\mathcal{M}_\mu = \{x = 0, \sigma = \bar{\sigma}\}$$

where $\bar{\sigma} = (\mu/\alpha(0))\Lambda\,\tau(w)$. It is easy to verify that $\mathcal{M}_\mu$ is an invariant manifold of (23) for all $w \in \mathcal{W}$.

Defining $\tilde{\sigma} = \sigma - \bar{\sigma}$ and $\tilde{s} = \upsilon + K_1\tilde{\sigma}$, the closed-loop system inside the boundary layer can be written as

$$\begin{aligned}\dot{w} &= S_0 w \\ \dot{x} &= f(x,w) + G(x,w)\psi(x,w) - \alpha(x)G(x,w)\frac{\tilde{s}}{\mu} \\ &\quad + G(x,w)\left(\frac{\alpha(x)-\alpha(0)}{\alpha(0)}\right)\chi(w) \\ &\quad - G(x,w)\psi(x,w) \qquad (23) \\ \dot{\tilde{\sigma}} &= A_\sigma\tilde{\sigma} + J\tilde{s} \;=\; S\tilde{\sigma} + J\upsilon\end{aligned}$$

Consider the Lyapunov function candidate

$$V_2 = V(x) + \frac{b}{\mu}\tilde{\sigma}^T P_0\tilde{\sigma} + \frac{c}{2}\tilde{s}^T\tilde{s} \qquad (24)$$

where b and c are positive constants to be chosen. Calculating $\dot{V}_2$ along the trajectories of the system (25), we obtain

$$\dot{V}_2 = \dot{V} + \frac{b}{\mu}\left[\tilde{\sigma}^T P_0\dot{\tilde{\sigma}} + \dot{\tilde{\sigma}}^T P_0\tilde{\sigma}\right] + c\tilde{s}^T\dot{\tilde{s}}$$

Using Assumptions 4 - 7 yields

$$\begin{aligned}\dot{V} &\leq -W(x) - [\alpha_0(\lambda/\mu) - k_7]\|\upsilon\|^2 + k_8\|\upsilon\|\sqrt{W(x)} \\ &\quad + (k_9/\mu)\|\upsilon\|\,\|\tilde{\sigma}\| \qquad (25)\end{aligned}$$

where k_7 to k_9 are some positive constants. Similarly, the second term of $\dot{V}_2$ can be written as

$$\frac{b}{\mu}\left[\tilde{\sigma}^T P_0 \dot{\tilde{\sigma}} + \dot{\tilde{\sigma}}^T P_0 \tilde{\sigma}\right] \;\leq\; -\frac{b}{\mu}\|\tilde{\sigma}\|^2 + \frac{2bk_{10}}{\mu}\|\tilde{\sigma}\|\,\|\tilde{s}\| \quad (26)$$

where $k_{10} = \lambda_{max}(P_0)$. Next, we have

$$c\tilde{s}^T\dot{\tilde{s}} \;\leq\; -c\alpha_0(\lambda_p/\mu)\|\tilde{s}\|^2 + ck_{11}\|\tilde{s}\|\,\|\tilde{\sigma}\| + ck_{12}\|\tilde{s}\|\,\|\upsilon\| + ck_{13}\|\tilde{s}\|\sqrt{W(x)} \quad (27)$$

where k_{11} to k_{13} are some positive constants. From (25), (26) and (27), we have

$$\begin{aligned}\dot{V}_2 \;\leq\; & -W(x) - [\alpha_0(\lambda/\mu) - k_7]\|\upsilon\|^2 - \frac{b}{\mu}\|\tilde{\sigma}\|^2 \\ & - c\alpha_0(\lambda_p/\mu)\|\tilde{s}\|^2 + k_8\|\upsilon\|\sqrt{W(x)} \\ & + ck_{13}\|\tilde{s}\|\sqrt{W(x)} + (k_9/\mu)\|\upsilon\|\,\|\tilde{\sigma}\| \\ & + (2bk_{10}/\mu + ck_{11})\|\tilde{\sigma}\|\,\|\tilde{s}\| \\ & + ck_{12}\|\tilde{s}\|\,\|\upsilon\| \end{aligned} \quad (28)$$

It can be seen that the right-hand side of (28) can be arranged in the following quadratic form of $\Pi = [\sqrt{W}\;\; \|\upsilon\|\;\; \|\tilde{\sigma}\|\;\; \|\tilde{s}\|]^T$:

$$\dot{V}_2 \;\leq\; -\Pi^T \Delta \Pi \quad (29)$$

where the symmetric matrix Δ has the form

$$\Delta = \begin{bmatrix} 1 & \frac{-k_8}{2} & 0 & \frac{-ck_{13}}{2} \\ \frac{-k_8}{2} & \frac{\alpha_0\lambda}{\mu} - k_7 & -\frac{k_9}{2\mu} & \frac{-ck_{12}}{2} \\ 0 & -\frac{k_9}{2\mu} & \frac{b}{\mu} & -\frac{bk_{10}}{\mu} - \frac{ck_{11}}{2} \\ \frac{-ck_{13}}{2} & \frac{-ck_{12}}{2} & -\frac{bk_{10}}{\mu} - \frac{ck_{11}}{2} & \frac{c\alpha_0\lambda_p}{\mu} \end{bmatrix}$$

If the principal leading minors of Δ can be made positive by choosing the constants b and c appropriately, and by choosing μ sufficiently small, then $\dot{V}_2$ will be negative definite. This would imply that, inside the boundary layer, the trajectories of the closed-loop system will asymptotically approach $\mathcal{M}_\mu$ as $t \to \infty$. Towards that end, we partition the matrix Δ as

$$\Delta = \begin{bmatrix} 1 & -q_{12}^T \\ -q_{12} & \frac{1}{\mu}Q_{22} + \Delta_{22} \end{bmatrix} \quad (30)$$

where

$$q_{12} = \begin{bmatrix} \frac{k_8}{2} & 0 & \frac{c\;k_{13}}{2} \end{bmatrix}^T \quad (31)$$

$$Q_{22} = \begin{bmatrix} \alpha_0\lambda & -\frac{k_9}{2} & 0 \\ -\frac{k_9}{2} & b & -bk_{10} \\ 0 & -bk_{10} & c\;\alpha_0\lambda_p \end{bmatrix} \quad (32)$$

and

$$\Delta_{22} = \begin{bmatrix} -k_7 & 0 & \frac{-c\;k_{12}}{2} \\ 0 & 0 & -\frac{ck_{11}}{2} \\ \frac{-c\;k_{12}}{2} & -\frac{ck_{11}}{2} & 0 \end{bmatrix} \quad (33)$$

From (32), it is easy to see that by choosing b and c, we can successively make the principal leading minors of Q_{22} positive. First, b is chosen large enough to make the 2×2 minor positive, and then, c is chosen large enough to make the 3×3 minor positive. Finally, choosing μ small enough will render

$$det \begin{bmatrix} 1 & -q_{12}^T \\ -q_{12} & \frac{1}{\mu}Q_{22} + \Delta_{22} \end{bmatrix} > 0 \quad (34)$$

Consequently, $\dot{V}_2$ will be negative definite. Therefore, the trajectories of the closed-loop system will asymptotically approach $\mathcal{M}_\mu$ as $t \to \infty$. Our conclusions can be summarized in the following theorem.

Theorem 1: Suppose Assumptions 1 - 7 are satisfied and consider the closed-loop system formed of the system (6), the servocompensator (14) and the state feedback control (12). Then, there exists $\mu^* > 0$ such that $\forall \mu \in (0, \mu^*]$, the state variables of the closed-loop system are bounded and $\lim_{t\to\infty}\; e(t) = 0$.

V. Output Feedback

In this section, we consider the design of an output feedback controller for a class of minimum-phase, input-output linearizable systems, which can recover the asymptotic properties of the state feedback controller of the previous section using a high-gain observer. Consider the system (6)

$$\begin{aligned} \dot{x} &= f(x,w) + G(x,w)[u - \chi(w)] \\ e &= h(x,w) \end{aligned} \quad (35)$$

where $h(x,w) = \tilde{h}(\zeta,w)$ is the measured output that we want to regulate to zero. It is well known [6] that if the above system has a well-defined vector relative degree and the distribution $\mathcal{G} = span\{g_1,\ldots,g_m\}$ is involutive, where $g_1,\ldots,g_m$ are columns of the matrix G, then it can be transformed into the *normal form*

$$\begin{aligned} \dot{\xi} &= A\xi + B\{f_1(\xi,z,w) + G_1(\xi,z,w)[u - \chi(w)]\} \\ \dot{z} &= f_2(\xi,z,w) \\ e &= C\xi \end{aligned} \quad (36)$$

where ξ and z belong to the sets $\mathcal{X}_\xi \subset R^{n-r}$ and $\mathcal{X}_z \subset R^r$, respectively and the $r \times r$ matrix A, the $r \times m$ matrix B and the $m \times r$ matrix C, given by

$$A = blkdiag[\mathrm{A}_1,\ldots,A_m], \;\; B = blkdiag[\mathrm{B}_1,\ldots,B_m]$$

$$A_i = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{r_i \times r_i}, \; B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{r_i \times 1}$$

$C = \; blockdiag[\mathrm{C}_1, \ldots, C_m]$, $C_i = \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \end{array}\right]_{1\times r_i}$ where $1 \le i \le m$ and $r = r_1 + \ldots + r_m$, represent m chains of integrators. In the new coordinates, the zero-error manifold is given as

$$\mathcal{M}_\mu = \{\xi = 0, z = 0, \sigma = \bar{\sigma}\}$$

Assumption 8: The function $f_2(\xi, z, w)$ is continuously differentiable for all $(\xi, z, w) \in \mathcal{X}_\xi \times \mathcal{X}_z \times \mathcal{W}$. Moreover, the following inequalities hold over the domain of interest:

$$G_1(\xi, z, w) + G_1^T(\xi, z, w) > 2\lambda I_m$$
$$B^T P_1 B G_1(0, 0, w) + G_1^T(0, 0, w) B^T P_1 B > 0$$

where $P_1 = P_1^T$ is the solution of the Lyapunov equation $P_1(A - BK) + (A - BK)^T P_1 = -I_n$ and K is chosen such that $A - BK$ is Hurwitz.

Assumption 9: There exists a Lyapunov function $V_z(z, w)$ such that for all $(\xi, z, w) \in \mathcal{X}_\xi \times \mathcal{X}_z \times \mathcal{W}$

$$k_{14} \|z\|^2 \le V_z(z, w) \le k_{15} \|z\|^2 \tag{37}$$

$$\frac{\partial V_z}{\partial w} S_0 w + \frac{\partial V_z}{\partial \tilde{z}} f_2(\xi, z, w) \le -k_{16} \|z\|^2 + k_{17} \|z\| \|\xi\| \tag{38}$$

for some positive constants k_{14}, k_{15}, k_{16} and k_{17}.

With $f_2(\xi, z, w)$ being continuously differentiable, if $z = 0$ is an exponentially stable equilibrium point of $\dot{z} = f_2(0, z, w)$, then the existence of such Lyapunov function is ensured by the converse Lyapunov theorem [8, Theorem 4.14]. Let $V(\xi, z, w) = V_z(z, w) + k_{18}[\xi^T P_1 \xi]$ for some positive constant k_{18}. It can be seen that Assumption 4 is satisfied with $\psi(\xi, z, w) = G_1^{-1}(\xi, z, w)[-f_1(\xi, z, w) - K\xi]$ and inequalities (7) and (8) are satisfied with α_1, α_2 and w which are quadratic in $\left\|\left[\begin{array}{cc} \xi & z \end{array}\right]^T\right\|$. Assumption 5 is satisfied with $\upsilon^T(\xi) = 2k_{18}\xi^T P_1 B$ and $H(\xi, z, w) = G_1(\xi, z, w)$.

Since $[\chi(w) + \psi(\xi, 0, w)]$ is independent of z, (10) is satisfied with a function $\delta(\xi)$. Thus, (15) is modified as

$$\alpha(\xi) \ge \frac{k}{\lambda}\delta(\xi) + \alpha_0, \quad \alpha_0 > 0 \tag{39}$$

and from (12), a partial state feedback control is taken as

$$u = -\alpha(\xi)\phi\left(\frac{s_1}{\mu}\right) \tag{40}$$

where $s_1 = \upsilon(\xi) + K_1\sigma$. It can be verified that the closed-loop system under the state feedback control (40) is uniformly exponentially stable with respect to the manifold $\mathcal{M}_\mu$. The control (40) can be implemented as an output feedback controller that uses the high-gain observer

$$\dot{\hat{\xi}} = A\hat{\xi} + H(e - C\hat{\xi}) \tag{41}$$

to estimate ξ, where the observer gain H is chosen as

$$H = \; blkdiag[\mathrm{H}_1, \ldots, H_m], \;\; H_i^T = \left[\begin{array}{ccc} \frac{\alpha_1^i}{\epsilon} & \frac{\alpha_2^i}{\epsilon^2} & \ldots \frac{\alpha_{r_i}^i}{\epsilon^{r_i}} \end{array}\right]$$

in which ϵ is a positive constant and the positive constants α_j^i are chosen such that the roots of $s^{r_i} + \alpha_1^i s^{r_i - 1} + \cdots + \alpha_{r_i}^i = 0$ are in open left-half plane, for all $i = 1, \ldots, m$. The output feedback control is given by

$$u = -\alpha(\hat{\xi})\phi\left(\frac{\hat{s}_1}{\mu}\right) \tag{42}$$

where $\hat{s}_1 = \upsilon(\hat{\xi}) + K_1\sigma$, in which the estimate $\hat{\xi}$ is provided by the high-gain observer (41) and σ is the output of the conditional servocompensator

$$\dot{\sigma} = (S - JK_1)\sigma + \mu J\phi\left(\frac{\hat{s}_1}{\mu}\right) \tag{43}$$

The proofs of Theorems 2 and 5 of [1] show that there is a neighborhood $\mathcal{N}$ of the origin, independent of ϵ, and $\epsilon_1 > 0$ such that for every $0 < \epsilon \le \epsilon_1$, the origin is exponentially stable and every trajectory in $\mathcal{N}$ converges to the origin as $t \to \infty$. From [1, Theorems 1,2,5], for any compact set $\mathcal{B}$ which contains $\mathcal{M}_\mu$, and any compact set $\mathcal{Q} \subseteq R^r$, there is $\epsilon_2 > 0$ such that for every $0 < \epsilon \le \epsilon_2$, the solutions starting in $\mathcal{B} \times \mathcal{Q}$ enter $\mathcal{N}$ in finite time. Hence, for every $0 < \epsilon \le \epsilon_3 = min\{\epsilon_1, \epsilon_2\}$, the origin is exponentially stable and $\mathcal{B} \times \mathcal{Q}$ is a subset of the region of attraction. Therefore, from [1, Theorems 2,5], we conclude that, for sufficiently small ϵ, the closed-loop system under the output feedback (42) is uniformly exponentially stable with respect to the set $\mathcal{M}_\mu \times \{\xi - \hat{\xi} = 0\}$. Hence, $\lim_{t\to\infty} e(t) = 0$.

VI. Simulation Example

Consider a second-order system modeled by the equations

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= -\theta_1(\tilde{x}_1 - \tilde{x}_1^3/6) + \theta_2 u \\ y &= \tilde{x}_1 \end{aligned} \tag{44}$$

with the reference signal $r(t) = r_0 sin(\omega t)$, which is generated by the exosystem

$$\dot{w} = \left[\begin{array}{cc} 0 & \omega \\ -\omega & 0 \end{array}\right] w, \;\; w^T(0) = \left[\begin{array}{c} 0 \\ r_0 \end{array}\right], \;\; r(t) = w_1$$

With change of variables $x_1 = \tilde{x}_1 - r$, $x_2 = \tilde{x}_2 - \dot{r}$, we have

$$\begin{aligned} \dot{x} &= Ax + B[f(x, w) + G(x, w)(u - \chi(w))] \\ e &= Cx \end{aligned} \tag{45}$$

where $A = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]$, $B = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$, $C = [1 \quad 0]$, $f(x, w) = -\theta_1 x_1 + \frac{\theta_1}{6}\left[(x_1 + w_1)^3 - w_1^3\right]$, $G(x, w) = \theta_2$, and $\chi(w) = \frac{1}{\theta_2}\left[\theta_1 w_1 - \frac{1}{6}\theta_1 w_1^3 - \omega^2 w_1\right]$. It can be verified that $\chi(w)$ satisfies the identity $L_s^4\chi = -9\omega^4\chi - 10\omega^2 L_s^2\chi$.

We compare the performance of saturated high-gain feedback stabilizing controller without a servocompensator (Design 1), with two control designs that use servocompensators (Design 2 and Design 3). Design 2 uses the conditional servocompensator (43). In Design 3, we augment a fourth-order conventional servo-compensator $\dot{\sigma} = S\sigma + Je$, with the system (45) to obtain an *augmented system* of the form

$$\begin{aligned} \dot{\xi} &= A_1\xi + B_1[f(x, w) + G(x, w)\psi(x, w)] \\ e &= C_1\xi \end{aligned} \tag{46}$$

where $A_1 = \left[\begin{array}{cc} S & JC \\ 0 & A \end{array}\right]$, $B_1 = [0 \quad B]^T$, $C_1 = [0 \quad C]$ and $\xi = \left[\begin{array}{cc} \sigma & x \end{array}\right]^T$ is the state. A stabilizing feedback controller is designed via Lyapunov redesign [9], which yields

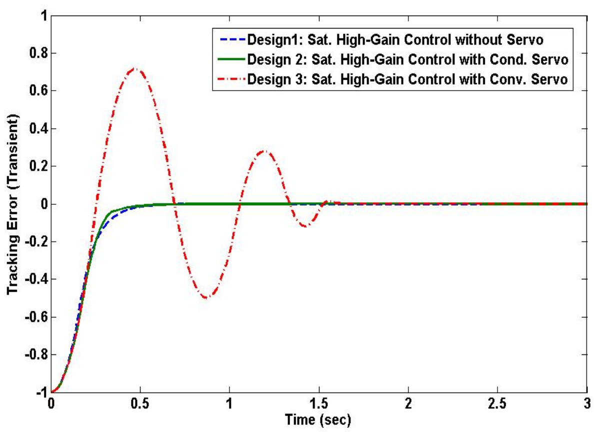

Fig. 1. Tracking error during the transient period

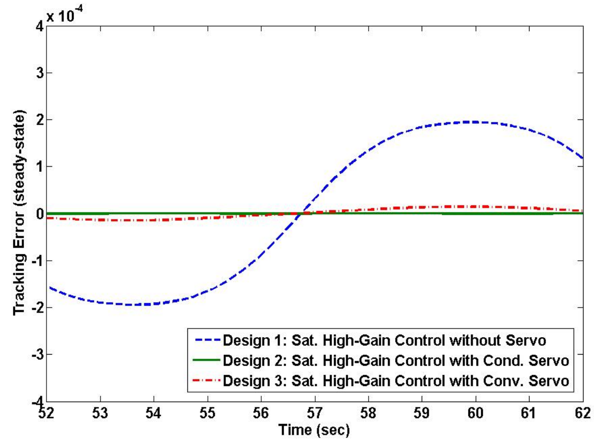

Fig. 2. Steady-state tracking error

$\psi(x,w) = \frac{1}{\theta_2}\left[\theta_1 x_1 - \frac{\theta_1}{6}\left((x_1+w_1)^3 - w_1^3\right) - \frac{1}{k}\phi(\varpi/\mu)\right]$, where $\varpi = 2B^T P\xi$, and $P = P^T > 0$ is the solution of

$$P(A_1 + B_1 K_2) + (A_1 + B_1 K_2)^T P = -I$$

in which K_2 is chosen such that $A_1 + B_1 K_2$ is Hurwitz. For the conditional servocompensator design (Design 2), Assumption 4 is satisfied with $V(x) = \frac{1}{2}(3x_1^2 + 2x_1x_2 + 2x_2^2)$ and $\psi(x,w) = \frac{1}{\theta_2}\left[\theta_1 x_1 - \frac{\theta_1}{6}[(x_1+w_1)^3 - w_1^3] - k_1x_1 - k_2x_2\right]$ where $k_1 = k_2 = 1$. Assumption 5 is satisfied with $\upsilon(x) = x_1 + 2x_2$ and $H(x,w) = 1$. The state estimate $\hat{x}_2$ is provided by the high-gain observer

$$\dot{\hat{x}}_1 = \hat{x}_2 + g_1(x_1 - \hat{x}_1)/\epsilon, \quad \dot{\hat{x}}_2 = g_2(x_1 - \hat{x}_1)/\epsilon^2$$

where g_1 and g_2 are chosen such that the polynomial $\lambda^2 + g_1\lambda + g_2$ is Hurwitz. We use the following numerical values in the simulation: $\theta_1 = 1$, $\theta_2 = 3$, $\omega = 0.5$ rad/s, $r_0 = 1$, $k = 1/5$, $\mu = 0.1$, $g_1 = 2$, $g_2 = 1$ and $\epsilon = 0.01$. For the conventional servocompensator design (Design 3), K_2 is chosen so as to assign the eigenvalues of $A_1 + B_1 K_2$ at $-0.5, -1, -1.5$, -2, -2.5 and -3. For the conditional servocompensator (Design 2), K_1 is chosen so as to assign the eigenvalues of $S - JK_1$ at $-0.5, -1, -1.5$ and -2. The control u is given by

$$u = -10\ sat\left(\frac{x_1 + 2x_2 + K_1\sigma}{\mu}\right)$$

Figure 1 shows the tracking error during the transient period and Figure 2 shows the steady state tracking error for the three designs. The transient response of the controller design without a servocompensator (Design 1) is close to the one with conditional servocompesator (Design 2), but it does not result in asymptotic error convergence. Asymptotic error convergence to zero is achieved with the conventional servocompensator design (Design 3), however, at the expense of a degraded transient performance.

VII. Conclusions

This paper extends the state feedback regulation of nonlinear systems using conditional integrators, Lyapunov redesign, and saturated high-gain feedback, to a more general case of time-varying signals by using conditional servocompensators. We showed that the use of conditional servocompensators enables us to achieve zero steady-state tracking error, in the presence of time-varying exogenous signals that are generated by a known exosystem. We also considered output feedback regulation of minimum-phase, input-output linearizable, nonlinear systems where the states of the system are regulated to a disturbance-dependent invariant manifold on which the tracking error is zero. The output feedback control is implemented using a high-gain observer. Analytical results are provided for a compact set of initial conditions, which can be chosen arbitrarily large if all the conditions hold globally. The performance improvement of the control design with a conditional servocompensator is demonstrated by a simulation example.

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