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Optimal Quadratic Regulation for Discrete-time Switched Linear Systems: A Numerical Approach

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Abstract— **This paper studies the discrete-time linear quadratic regulation problem for switched linear systems (DLQRS) based on dynamic programming approach. The unique contribution of this paper is the analytical characterizations of both the value function and the optimal control strategies for the DLQRS problem. Based on the particular structures of these analytical expressions, an efficient algorithm suitable for solving an arbitrary DLQRS problem is proposed. The algorithm is also tested through simulations on a number of second-order DLQRS problems. Simulation results indicate that the proposed algorithm can solve the second-order DLQRS problems with very low computational complexity. The theoretical analysis in this paper dramatically simplifies the computation, making an NP hard problem numerically tractable.**

I. Introduction

In this paper, we study the optimal discrete-time quadratic regulation problem for switched linear systems (hereby referred to as the DLQRS problem). The goal is to develop a computationally appealing algorithm to construct an optimal control law that minimizes the given quadratic cost function. The problem is of fundamental importance in both theory and practice and has challenged researchers for many years. To the authors' knowledge, few constructive method for finding both the optimal switching strategy and the optimal control input has appeared in the literature. The bottleneck is mostly on the determination of the optimal switching strategy. Many methods have been proposed to tackle this problem, most of which are in a divide-and-conquer manner. Algorithms for optimizing the switching instants with fixed mode sequence have been derived for general switched systems in [1] and for autonomous switched systems in [2]. Although an algorithm for updating the switching sequence is discussed in [2], finding the best switching sequence is still an NP-hard problem, even for switched linear systems.

This paper studies the DLQRS problem from the dynamic programming (DP) perspective. The last few years have seen increasing interest in using DP to solve various optimal control problems of switched systems. In [3], DP is used to derive a search algorithm to find the optimal switching instants for fixed switching sequences. In [4], iterative algorithms are proposed to approximate the true value functions with guaranteed accuracy. These general algorithms are also used to study switched systems in [5], [6]. Compared with previous studies, the contributions of this paper are the following. First, we characterize analytically the value function and the optimal control strategy for general DLQRS problems. More specifically, we show that the value function at each time step of any DLQRS problem is a pointwise minimum of a finite number of quadratic functions, and that the optimal state-feedback gain is of a Kalman-type form with a state-dependent positive semi-definite matrix. Secondly, we prove that under certain conditions of subsystems, the value function converges exponentially fast as the control horizon increases. Finally, based on the particular structure of the value function and its convergence property, an efficient algorithm is proposed to solve general DLQRS problems. The algorithm has also been implemented for second-order DLQRS problem and tested through simulations. Simulation results indicate that the proposed algorithm can compute the optimal switching strategy and the optimal control input simultaneously with very low computational complexity for second-order DLQRS problems. It is worth mentioning that in [4], Lincoln *et al.* proposed a similar structure of the value function when they apply their general theory of relaxed dynamic programming to switched linear systems. However, the approach adopted in this paper is different and was developed independently of the one used in [4]. Moreover, in this paper, the value functions and the optimal control strategies are derived more explicitly, which provide more insights about the underlying problems.

This paper is organized as follows. In Section II, the DLQRS problem is formulated. The value function of the DLQRS problem is derived in a simple analytical form in Section III. An algorithm is developed in Section (IV) to compute the value function in an efficient way. Numerical simulations are performed in Section V to verify the algorithm. Finally, some concluding remarks are given in Section VI.

II. Problem Formulation

Consider the discrete-time switched linear system defined as:

$$x(t+1) = A_{v(t)}x(t) + B_{v(t)}u(t),\ t = 0, \dots, N-1, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the continuous state, $v(t) \in \mathbb{M} \triangleq \{1, \dots, M\}$ is the discrete control or switching strategy, and $u(t) \in \mathbb{R}^p$ is the continuous control. For each $i \in \mathbb{M}$, A_i and B_i are constant matrices of appropriate dimension, and

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the pair (A_i, B_i) is called a subsystem of (1). This switched linear system is time invariant in the sense that the set of available subsystems $\{(A_i, B_i)\}_{i=1}^{M}$ is independent of time t. We assume that there is no internal forced switchings, i.e., the system can stay at or switch to any mode at any time instant. In this paper, the terminal cost function $\psi(x)$ and the running cost function $L(x, u, v)$ are assumed to be in the following quadratic forms:

$$\psi(x) = x^T Q_f x, \quad L(x, u, v) = x^T Q_v x + u^T R_v u,$$

where $Q_f = Q_f^T \succeq 0$ is the terminal state weight, and $Q_v = Q_v^T \succeq 0$ and $R_v = R_v^T \succ 0$ are the running weights for the state and the control for subsystem $v \in \mathbb{M}$, respectively. The overall objective function to be minimized over the time horizon $[0, N]$ can thus be defined as

$$J(u, v) = \psi(x(N)) + \sum_{j=0}^{N-1} L(x(j), u(j), v(j))). \quad (2)$$

The goal of this paper is to solve the following discrete-time LQR problem for the switched linear system (1) (referred to as DLQRS problem hereby).

Problem 1 (DLQRS problem): Find the u and v that minimize $J(u, v)$ subject to the dynamic equation (1).

III. THE VALUE FUNCTION OF THE DLQRS PROBLEM

Following the idea of dynamic programming, for each time $t \in \{0, 1, \ldots, N\}$, we define the value function $V_{t,N} : \mathbb{R}^n \to \mathbb{R}$ as:

$$V_{t,N}(z) = \min_{\substack{v(j)\in\mathbb{M}, u(j),\\ t\leq j\leq N-1}} \Big\{ \psi(x(N) + \sum_{j=t}^{N-1} L(x(j), u(j), v(j)) \Big| \text{ subject to eq. (1)} \quad \text{with } x(t) = z \Big\}. \quad (3)$$

The $V_{t,N}(z)$ so defined is the minimum cost-to-go starting from state z at time t. The minimum cost for the DLQRS problem with a given initial condition $x(0) = x_0$ is simply $V_{0,N}(x_0)$. Due to the time-invariant nature of the switched system (1), its value function depends only on the number of remaining time steps, i.e.,

$$V_{t,N}(z) = V_{t+m,N+m}(z),$$

for all $z \in \mathbb{R}^n$ and all integers $m \geq -N$. In the rest of this paper, when no ambiguity arises, we will denote by $V_k(z)$ the value function at the time $t = N - k$ when there are k time steps left.

In the special case when $M = 1$, the switched system consists of only one subsystem, say, (A, B). Thus, the DLQRS problem degenerates into the classical LQR problem. Denote by Q and R the state and control weighting matrices in this degenerate case. Then, according to the LQR theory, the value function defined in (3) is of the following quadratic form:

$$V_k(z) = z^T P_k z, \quad k = 0, \ldots, N, \quad (4)$$

where $\{P_k\}_{k=0}^{N}$ is a sequence of positive semi-definite matrices satisfying the Difference Riccati Equation (DRE)

$$P_{k+1} = Q + A^T P_k A - A^T P_k B(R + B^T P_k B)^{-1} B^T P_k A, \quad (5)$$

with initial condition $P_0 = Q_f$. Some important facts about the matrices P_k's are summarized in the following lemma.

Lemma 1 ([7], [8]): Let $\mathcal{A}$ be the set of all positive semi-definite (p.s.d.) matrices, then

1) If $P_k \in \mathcal{A}$, then $P_{k+1} \in \mathcal{A}$.
2) If (A, B) is stabilizable, then the sequence $\{||P_k||_2\}_{k=0}^{\infty}$ is uniformly bounded.
3) Let C be the matrix such that $Q = C^T C$. If (A, B) is stabilizable and (A, C) is detectable, then $\lim_{k\to\infty} P_k = P^*$, where P^* is the unique stabilizing solution to the Algebraic Riccati Equation (ARE)

$$P = Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA.$$

In general, when $M \geq 2$, the value function $V_k(z)$ is no longer of a simple quadratic form as in (4). To derive the value function for the general switched linear system (1), define the Riccati mapping $\rho_i : \mathcal{A} \to \mathcal{A}$ for each subsystem $i \in \mathbb{M}$:

$$\rho_i(P) = Q_i + A_i^T PA_i - A_i^T PB_i(R_i + B_i^T PB_i)^{-1} B_i^T PA_i. \quad (6)$$

Let $\mathcal{H}_0 = \{Q_f\}$ be a set consisting of only one matrix Q_f. Define the set $\mathcal{H}_k$ for $k \geq 0$ iteratively as

$$\mathcal{H}_{k+1} = \rho_{\mathbb{M}}(\mathcal{H}_k) \triangleq \{P \in \mathcal{A} : P = \rho_i(P_k), \text{ for some } i \in \mathbb{M} \text{ and } P_k \in \mathcal{H}_k\}. \quad (7)$$

In other words, each matrix in $\rho_{\mathbb{M}}(\mathcal{H}_k)$ is obtained by taking the Riccati mapping for some matrix in $\mathcal{H}_k$ through some subsystem i. Denote by N_k the number of distinct matrices in $\mathcal{H}_k$ and let $\mathbb{N}_k = \{1, \ldots, N_k\}$. Then it can be easily seen that $N_0 = 1$ and $N_k \leq M^k$ for any $k \geq 0$.

Theorem 1: The value function for the LQRS problem at time $N - k$, i.e., with k time steps left, is

$$V_k(z) = \min_{j\in\mathbb{N}_k} z^T P_k^{(j)} z, \quad (8)$$

where $P_k^{(j)}$ is the j^{th} matrix in $\mathcal{H}_k$. Furthermore, for $k \geq 0$, the optimal mode (discrete control) and continuous control at time $N-(k+1)$ and state z are $v^*(N-(k+1)) = i^*(z)$ and $u^*(N - (k + 1)) = -K_{k+1}^{j^*(z),i^*(z)} z$, respectively, where $i^*(z)$ and $j^*(z)$ are defined as

$$(j^*(z), i^*(z)) = \arg\min_{(j\in\mathbb{N}_k, i\in\mathbb{M})} z^T \rho_i(P_k^{(j)}) z, \quad (9)$$

and $K_{k+1}^{j,i}$ is the Kalman gain for subsystem i with matrix $P_k^{(j)}$, i.e.,

$$K_{k+1}^{j,i} \triangleq (R_i + B_i^T P_k^{(j)} B_i)^{-1} B_i^T P_k^{(j)} A_i. \quad (10)$$

.

Proof: The theorem can be easily proved through induction. It is obvious that for $k = 0$ the value function is

$V_k(z) = z^T Q_f z$, satisfying (8). Now suppose equation (8) holds for a general integer k, i.e., $V_k(z) = \min_{j\in\mathbb{N}_k} z^T P_k^{(j)} z$, we shall show that it is also true for $k+1$. By the principle of dynamic programming and noting that $V_k(\cdot)$ represents the value function at time $N-k$, the value function at time $N-(k+1)$ can be recursively computed as

$$\begin{aligned}
&V_{k+1}(z) = \min_{i\in\mathbb{M},u} \left[z^T Q_i z + u^T R_i u + V_k(A_i z + B_i u)\right] \\
&= \min_{i\in\mathbb{M},u} \Big[z^T Q_i z + u^T R_i u \\
&\qquad + \min_{j\in\mathbb{N}_k} \Big((A_i z + B_i u)^T P_k^{(j)} (A_i z + B_i u)\Big)\Big] \\
&= \min_{i\in\mathbb{M},j\in\mathbb{N}_k,u} \Big[z^T Q_i z + u^T R_i u \\
&\qquad + (A_i z + B_i u)^T P_k^{(j)} (A_i z + B_i u)\Big] \\
&= \min_{i\in\mathbb{M},j\in\mathbb{N}_k,u} \Big[z^T (Q_i + A_i^T P_k^{(j)} A_i) z \\
&\qquad + u^T (R_i + B_i^T P_k^{(j)} B_i) u + 2 z^T A_i^T P_k^{(j)} B_i u\Big] \\
&\triangleq \min_{i\in\mathbb{M},j\in\mathbb{N}_k,u} f(i,j,u). \qquad (11)
\end{aligned}$$

With symmetric matrix $P_k^{(j)}$, it can be easily computed that

$$\frac{\partial f(i,j,u)}{\partial u} = 2(R_i + B_i^T P_k^{(j)} B_i) u + 2 B_i^T P_k^{(j)} A_i z.$$

Since u is unconstrained, its optimal value u^* must satisfy $\frac{\partial f(i,j,u^*)}{\partial u} = 0$, i.e.,

$$u^* = -(R_i + B_i^T P_k^{(j)} B_i)^{-1} B_i^T P_k^{(j)} A_i z = -K_{k+1}^{j,i} z,$$

where $K_{k+1}^{j,i}$ is the matrix defined in (10). Substitute u^* into (11), we obtain

$$\begin{aligned}
&V_{k+1}(z) = \min_{i\in\mathbb{M},j\in\mathbb{N}_k} f(i,j,u^*) \\
&= \min_{i\in\mathbb{M},j\in\mathbb{N}_k} \left[z^T \Big(Q_i + A_i^T P_k^{(j)} A_i - A_i^T P_k^{(j)} B_i K_{k+1}^{j,i}\Big) z\right] \\
&= \min_{i\in\mathbb{M},j\in\mathbb{N}_k} z^T \rho_i(P_k^{(j)}) z.
\end{aligned}$$

Let $i^*(z)$ and $j^*(z)$ be the indices that minimize $z^T \rho_i(P_k^{(j)}) z$, i.e., they are defined according to (9). Then the optimal continuous control and discrete control at time $N-(k+1)$ and state z are $u^*(N-(k+1)) = -K_{k+1}^{j^*(z),i^*(z)} z$ and $v^*(N-(k+1)) = i^*(z)$, respectively. Furthermore, observing that $\{\rho_i(P_k^{(j)}) : i \in \mathbb{M}, j \in \mathbb{N}_k\} = \rho_{\mathbb{M}}(\mathcal{H}_k) = \mathcal{H}_{k+1}$. Thus

$$V_{k+1}(z) = \min_{j\in\mathbb{N}_{k+1}} z^T P_{k+1}^{(j)} z,$$

where $P_{k+1}^{(j)}$ is the j^{th} element in $\mathcal{H}_{k+1}$. ∎

According to Theorem 1, comparing to the classical LQR problem, the value function of the DLQRS problem is no longer a single quadratic function; it actually becomes a pointwise minimum of a finite number of quadratic functions. In addition, at each time step, instead of having a single Kalman gain for the entire space, the optimal state feedback gain becomes state dependent. Furthermore, the minimizer $(j^*(z), i^*(z))$ of equation (9) is homogeneous, indicating that at each time step all the points along the same radial direction have the same optimal mode and optimal feedback gain.

IV. Computation of the Value Function

According to Theorem 1, the value function $V_k(\cdot)$ is completely characterized by the set $\mathcal{H}_k$, which can be obtained iteratively by (7). Since the size of the set $\mathcal{H}_k$ grows exponentially fast in general, it becomes unfeasible to compute $\mathcal{H}_k$ when k gets large. However, in terms of computing the value function, we only need to keep the matrices in $\mathcal{H}_k$ that give rise to the minimum of (8) for at least one $z \in \mathbb{R}^n$. Such matrices are called *effective* with respect to $\mathcal{H}_k$. More precisely, a matrix $P \in \mathcal{H}_k$ is called *effective* if there exists a $z \in \mathbb{R}^n$ such that $V_k(z) = z^T P z$. According to our simulation results in Section V, the number of effective matrices in $\mathcal{H}_k$ grows at a much slower rate than N_k as k increases. Therefore, the computation of the value function and in turn the optimal control strategy may become feasible if the set of effective matrices can be obtained efficiently. To find the set of effective matrices, the following definitions are introduced.

Definition 1 (Equivalent Sets of p.s.d. Matrices): Let $\mathcal{H}$ and $\hat{\mathcal{H}}$ be two sets of p.s.d. matrices. The set $\mathcal{H}$ is called *equivalent* to $\hat{\mathcal{H}}$, denoted by $\mathcal{H} \sim \hat{\mathcal{H}}$, if $\forall z \in \mathbb{R}^n$, $\min_{P\in\mathcal{H}} z^T P z = \min_{\hat{P}\in\hat{\mathcal{H}}} z^T \hat{P} z$.

Therefore, any equivalent sets of p.s.d. matrices will define the same value function of the DLQRS problem. To ease the computation, we are more interested in finding the smallest equivalent set of $\mathcal{H}_k$.

Definition 2 ((Minimum) Equivalent Subset (MES)): Let $\mathcal{H}$ and $\hat{\mathcal{H}}$ be two sets of symmetric p.s.d. matrices. $\hat{\mathcal{H}}$ is called an *equivalent subset* of $\mathcal{H}$, if $\hat{\mathcal{H}} \subseteq \mathcal{H}$ and $\hat{\mathcal{H}} \sim \mathcal{H}$. Furthermore, $\hat{\mathcal{H}}$ is called a *minimum equivalent subset* (MES) of $\mathcal{H}$ if it is the equivalent subset of $\mathcal{H}$ with the fewest elements.

Note that the MES of $\mathcal{H}$ many not be unique. Denote by $\Gamma[\mathcal{H}]$ one of the MESs of $\mathcal{H}$. It is also worth mentioning that due to its special structure, the value function is homogeneous of degree 2, namely,

$$V_k(\lambda z) = \lambda^2 V_k(z), \forall z \in \mathbb{R}^n, \text{ and } \forall \lambda \in \mathbb{R}^1. \qquad (12)$$

Therefore, it suffices to consider only the points on the unit sphere in checking the conditions of the above two definitions.

Lemma 2: $\hat{\mathcal{H}}$ is an equivalent subset of $\mathcal{H}$ if and only if

1) $\hat{\mathcal{H}} \subseteq \mathcal{H}$
2) $\forall P \in \mathcal{H}$ and $\forall z \in \mathbb{R}^n$, there exists a $\hat{P} \in \hat{\mathcal{H}}$ and such that $z^T \hat{P} z \le z^T P z$.

Proof: (a) (sufficiency): We only need to prove $\min_{P\in\mathcal{H}} z^T P z = \min_{\hat{P}\in\hat{\mathcal{H}}} z^T \hat{P} z$, $\forall z \in \mathbb{R}^n$. Obviously $\min_{P\in\mathcal{H}} z^T P z \le \min_{\hat{P}\in\hat{\mathcal{H}}} z^T \hat{P} z$, $\forall z \in \mathbb{R}^n$ because $\hat{\mathcal{H}} \subseteq \mathcal{H}$. On the other hand, by condition 2), for each $z \in \mathbb{R}^n$ and $P \in \mathcal{H}$, there exist a $\hat{P}$ such that $z^T \hat{P} z \le z^T P z$. Thus, $\min_{\hat{P}\in\hat{\mathcal{H}}} z^T \hat{P} z \le \min_{P\in\mathcal{H}} z^T P z$. (b) (necessity): straightforward by a standard contradiction argument. ∎

Remark 1: Lemma 2 can be used as an alternative definition of the equivalent subset. Although the original definition is conceptually simpler, the conditions given in this lemma provide a more explicit characterization of the equivalent subset, which finds more beneficial in our subsequent discussions.

For each $k \leq N$, let $\hat{\mathcal{H}}_k$ be an equivalent subset of $\mathcal{H}_k$ and $\hat{N}_k$ be the number of distinct elements in $\hat{\mathcal{H}}_k$. Define $\hat{\mathbb{N}}_k \triangleq \{1, \ldots, \hat{N}_k\}$. The following corollary follows immediately from Definition 2.

Corollary 1: The value function of the DLQRS problem at time $N-k$ is

$$V_k(z) = \min_{j \in \hat{\mathbb{N}}_k} z^T \hat{P}_k^{(j)} z,$$

where $\hat{P}_k^{(j)}$ is the j^{th} element in $\hat{\mathcal{H}}_k$. Furthermore, the optimal mode at time $N-(k+1)$ and state z is $v(N-(k+1))^* = i^*(z)$ and the optimal control at time $N-(k+1)$ and state z is $u(N-(k+1))^* = -\hat{K}_{k+1}^{j^*(z),i^*(z)} z$, where $i^*(z)$ and $j^*(z)$ are defined as

$$(j^*(z), i^*(z)) = \underset{(j \in \hat{\mathbb{N}}_k, i \in \mathbb{M})}{\arg\min} \; z^T \rho_i(\hat{P}_k^{(j)}) z,$$

and $\hat{K}_{k+1}^{j,i}$ is the Kalman gain defined in (10) with $P_k^{(j)}$ replaced by $\hat{P}_k^{(j)}$.

Corollary 1 says that to compute the value function $V_k(z)$ and the corresponding optimal control strategies, we only need to compute a version of the equivalent subsets of $\mathcal{H}_k$. In particular, $\Gamma(\mathcal{H}_k)$, namely, an MES of $\mathcal{H}_k$, will be sufficient. As mentioned early in this section, $\Gamma(\mathcal{H}_k)$ is usually much smaller than $\mathcal{H}_k$. If for a particular application, the size of $\Gamma(\mathcal{H}_k)$ grows reasonably slowly, the DLQRS problem can be solved numerically using Corollary 1. However, a direct computation of $\Gamma(\mathcal{H}_k)$ based on $\mathcal{H}_k$ could be very difficult because its complexity usually depends on N_k (the number of matrices in $\mathcal{H}_k$), which grows exponentially fast in general. Fortunately, this difficulty can be overcome by the following lemma.

Lemma 3 (Computation of MES): Let the sequece of sets $\{\mathcal{H}_k^*\}_{k=0}^N$ be generated by

$$\mathcal{H}_0^* = \mathcal{H}_0, \text{ and } \mathcal{H}_{k+1}^* = \Gamma(\rho_{\mathbb{M}}(\mathcal{H}_k^*)) \text{ for } k \leq N-1. \quad (13)$$

Then every $\mathcal{H}_k^*$ is an MES of $\mathcal{H}_k$, i.e., $\mathcal{H}_k^* = \Gamma(\mathcal{H}_k)$.

Proof: Obviously $\mathcal{H}_0^* = \Gamma(\mathcal{H}_0)$ as $\mathcal{H}_0$ contains only one element Q_f. Now assume $\mathcal{H}_k^* = \Gamma(\mathcal{H}_k)$ for some $k < N$. We need to prove that $\mathcal{H}_{k+1}^* = \Gamma(\mathcal{H}_{k+1}) \triangleq \Gamma(\rho_{\mathbb{M}}(\mathcal{H}_k))$. Clearly, $\mathcal{H}_{k+1}^* \subseteq \Gamma(\mathcal{H}_{k+1})$ as $\mathcal{H}_{k+1}^* \subseteq \mathcal{H}_{k+1}$. To prove the other direction, it suffices to show that $\mathcal{H}_{k+1}^*$ is an equivalent subset of $\mathcal{H}_{k+1}$. First, $\mathcal{H}_{k+1}^* \subseteq \Gamma(\mathcal{H}_{k+1}) \subseteq \mathcal{H}_{k+1}$ guarantees condition (i) of Lemma 2. To prove the other condition, take an arbitrary $P \in \mathcal{H}_{k+1}$ and an arbitrary $z \in \mathbb{R}^n$. Then $P = \rho_i(P_k^{(j)})$ for some $i \in \mathbb{M}$ and $P_k^{(j)} \in \mathcal{H}_k$. Hence,

$$\begin{aligned}
& z^T P z = z^T \rho_i(P_k^{(j)}) z \\
&= \min_u \left[z^T Q_i z + u^T R_i u + (A_i z + B_i u)^T P_k^{(j)} (A_i z + B_i u) \right] \\
&\geq \min_u \left[z^T Q_i z + u^T R_i u + (A_i z + B_i u)^T \hat{P}_k^{(j)} (A_i z + B_i u) \right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{for some } \hat{P}_k^{(j)} \in \mathcal{H}_k^*) \\
&= x^T \rho_i(\hat{P}_k^{(j)}) x \geq z^T P^* z. \qquad (\text{for some } P^* \in \mathcal{H}_{k+1}^*)
\end{aligned}$$

Thus it follows that $\mathcal{H}_k^*$ is an MES of $\mathcal{H}_k$. ∎

According to Lemma 2, $\Gamma(\mathcal{H}_k)$ can be obtained only based on $\rho_{\mathbb{M}}(\mathcal{H}_{k-1}^*)$ without referring to the original set $\mathcal{H}_k$. Denoted by N_k^* the size of $\mathcal{H}_k^*$. The set $\rho_{\mathbb{M}}(\mathcal{H}_{k-1}^*)$ contains only $M \cdot N_{k-1}^*$ matrices which is usually much smaller than $N_k = M^k$. Therefore, Lemma 2 has dramatically simplified the computation of $\Gamma(\mathcal{H}_k)$. However, it is still possible for N_k^* to grow out of hand when the time horizon N is large. The following theorem allows us to terminate the computation with guaranteed accuracy at some early stage for large time horizon N.

Theorem 2: If $Q_f \succ 0$, and for all $i \in \mathbb{M}$, (A_i, B_i) is stabilizable and $Q_i \succ 0$, then $V_k(z)$ converges exponentially fast for each $z \in \mathbb{R}^n$ as $k \to \infty$. Furthermore, the convergence is uniform on the unit sphere in $\mathbb{R}^n$.

Remark 2: Note that the conditions in this theorem are not so stringent since a randomly selected p.s.d. matrix is almost surely nonsingular. The proof of this theorem is quite involved and is out of the scope of this conference paper. Interested readers are referred to [9] for a complete proof.

The exponential convergence result is crucial for the efficient computation of the value function. For a reasonable tolerance on the accuracy, say 10^{-3}, the value function usually converges in only a few steps (usually less than 10) as observed in our simulations in Section V. This dramatically simplifies the computations of the value functions, especially for the case with large time horizon N. In practice the convergence is usually tested only on a finite set of sampling points on the unit sphere. These sampling points should be chosen dense enough to capture the behaviors of all the value functions on the entire unit sphere. The existence of such sampling points is guaranteed by the following corollary of Theorem 2.

Corollary 2: Under the same conditions as in Theorem 2, the sequence of value functions $\{V_k(z)\}_{k=0}^\infty$ is equicontinuous on the unit sphere.

Proof: Denote by B_u the unit sphere in $\mathbb{R}^n$. Obviously, each value function $V_k(z)$ is continuous on B_u. By theorem 2, $V_k(\cdot)$ converges uniformly on B_u. Since B_u is a compact set, the desired result follows directly from Theorem 7.24 in [10]. ∎

With all the results developed so far, a general procedure for solving the DLQRS problem is described in Algorithm 1.

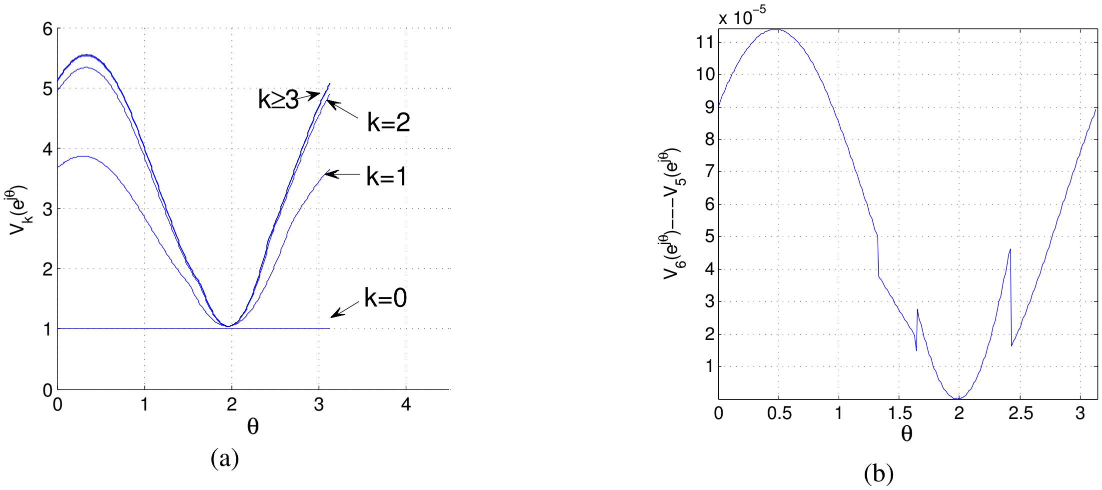

Fig. 1. Convergence results for Ex1. (a) Convergence of the Value function. (b) Difference between the last two iterations.

Algorithm 1

1) Step 1: Set $\mathcal{H}_0^* = Q_f$ and specify a tolerance ϵ for the minimum cost.
2) Step 2: For each step $k \geq 1$, update the MES according to equation (13) and compute the value function at certain sampling points on the unit sphere in $\mathbb{R}^n$ using Corollary 1.
3) Step 3: If $|V_k(z) - V_{k-1}(z)| > \epsilon$ for some sampling points, then go back to step 2. Otherwise continue.
4) Step 4: Define $\mathcal{H}_k^* = \mathcal{H}_{k_\epsilon}^*$ for $k_\epsilon \leq k \leq N$, where k_ϵ is the number of steps taken for the convergence.
5) Step 5: The suboptimal trajectory can now be obtained by
$$x(t+1) = A_{v^*(t)}x(t) + B_{v^*(t)}u^*(t), \text{ with } x(0) = x_0,$$
where $v^*(t)$ and $u^*(t)$ are determined using Corollary 1 based on the set $\mathcal{H}_{N-(t+1)}^*$.

V. Examples

A. Example 1

First consider a simple DLQRS problem, referred to as Ex1, with control horizon $N = 100$ and two second-order subsystems:

$$A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 2 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

with state and control weights $Q_1 = Q_2 = I_{2\times 2}$ and $R_1 = R_2 = 1$. Both subsystems are unstable but controllable. Algorithm 1 is applied to solve this DLQRS problem. It turns out that with the error tolerance $\epsilon = 10^{-3}$ the value function of Ex1 converges in 6 steps. By (12), it suffices to plot $V_k(\cdot)$ on the unit circle. If we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, the points on the unit circle are of the form $e^{j\theta}$. It can be easily verified that for second-order system, $V_k(e^{j\theta}) = V_k(e^{j(\theta+\pi)})$, i.e., the value function is periodic along the unit circle with period π. Therefore, in Fig. 1-(a), the value function is plotted only at points $e^{j\theta}$ for $\theta \in [0, \pi]$ at each time step. The difference between the value functions in the last two iterations are shown in Fig. 1-(b). The number of elements in $\mathcal{H}_k^*$ at each step is listed in Table I. It can be seen that N_k^* is indeed very small, and will stabilize at the maximum value 5 as opposed to growing exponentially as k increases.

Furthermore, the optimal switching strategy is illustrated in Fig. 2. At each time step, the whole space is divided into several conic regions. The regions with the same gray scale share the same optimal mode. However, the points with the same optimal mode may correspond to different optimal feedback gains. The radial lines in Fig 2 further divide the optimal-mode regions into smaller conic regions each with a different optimal-feedback gain. In this way, the proposed approach actually characterizes the optimal control strategies for the entire state space.

TABLE I

N_k^* FOR S1

k	1	2	3	4	5	6
N_k^*	2	4	5	5	5	5

B. Example 2

Consider a more complex DLQRS example, referred to as Ex2, with 4 subsystems. The first two subsystems are the same as Ex1 and the other two are defined as:

$$A_3 = \begin{bmatrix} 3 & 1 \\ 0 & 0.2 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 \\ 0 & 0.8 \end{bmatrix},$$
$$B_3 = B_1, \quad \text{and} \quad B_4 = B_2.$$

With the same tolerance, the value function of Ex2 converges in 5 steps. This indicates that under the same tolerance, the speed of the convergence of the value function may not necessarily increase with the number of subsystems. However,

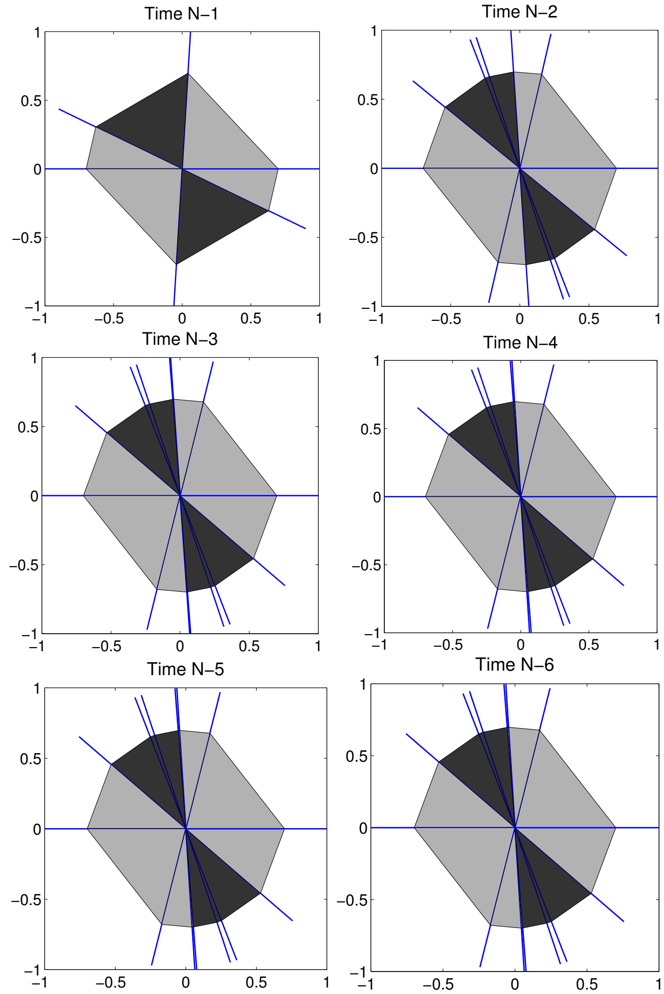

Fig. 2. Switching Regions for Ex1: Gray Region – mode 1 optimal; Black Region – mode 2 optimal.

with more subsystems, N_k^* grows more rapidly as listed in Table II. The switching regions at the final step is shown in Fig. 3, which can be interpreted in the same way as Fig. 2. Compared with Ex1, the optimal state feedback gain in this example has more distinct values. It is worth mentioning that the maximum N_k^* for this example is only 15 (as opposed to the nominal size of $\mathcal{H}_k$, $N_k = 4^5 = 1024$). Therefore, the proposed method has dramatically simplified the problem, making an NP hard problem numerically tractable.

TABLE II

N_k^* FOR EX2

k	1	2	3	4	5
N_k^*	3	9	15	15	15

VI. CONCLUSION

This paper studies the DLQRS problem based on dynamic programming approach. Different from the traditional LQR problem, the value function of the DLQRS problem is no longer a single quadratic function; it is a pointwise minimum of a finite number of quadratic functions. In addition, instead of having a single Kalman feedback gain as in the LQR case, the optimal state-feedback gain in the DLQRS problem becomes state dependent. Analytical expressions have been derived for both the optimal switching strategy and optimal control inputs. The concept of minimum equivalent subsets is introduced to simplify the computation of the value function. An efficient algorithm is developed to compute the optimal strategies with guaranteed accuracy of the optimal cost. Simulation results indicate that the proposed algorithm can solve the second-order DLQRS problems with fairly low computational complexity. Future research will focus on how to extend proposed approach to solve the continuous time LQR problem for switched linear systems.

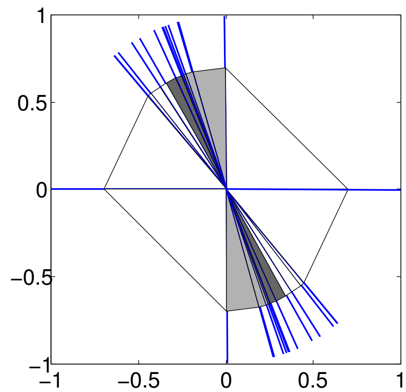

Fig. 3. Converged switching regions of Ex2

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