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# Set Stability of Phase-Coupled Agents in Discrete Time

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*Abstract*— **The work in this paper addresses the stability of a discretized version of the well-known phase-coupled oscillator model from the physics community. The main contribution is a pair of stability proofs for a system of $N$ phase-coupled agents. The first proof establishes asymptotic stability to the balanced set for a range $K\Delta T$, where $K$ is a coupling gain and $\Delta T$ is the time discretization. In the second proof, a reference vector in the unit ball is introduced and asymptotic stability of the phase centroid to the reference vector is guaranteed, again for a range of $K\Delta T$. These results are of particular interest to researchers looking to apply phase coupling to systems in which continuous communication is not possible. Possible applications of this work include cooperative target tracking and modeling of neurological processes and of biological aggregations.**

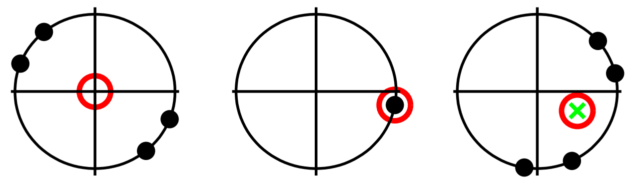


Fig. 1. Example headings are plotted on the unit circle as dark dots for (a) balanced, (b) aligned, and (c) reference matched group states with $N = 4$. The large red circle denotes the phase centroid. In (c), the reference vector is denoted by a green "x" located at $[0.75, -0.25]$. The work in this paper establishes stability to both balanced and reference matched sets.

## I. Introduction

First introduced in the physics community by Kuramoto and later studied by Crawford, Strogatz, and many others, phase-coupled oscillator models were developed to explore how loosely coupled distributed systems are able to achieve synchrony or anti-synchrony. Recently, phase coupling has been adopted by control theorists seeking a solid platform on which to build distributed algorithms. Some examples include stabilization of collective motion [1], [2] and coordinated target tracking [3], [4].

In the classical model of phase-coupled oscillators, the rate of each oscillator is equal to a natural frequency minus a scaled mean of sines of the angle differences between the local state and all others' states. Synchrony is said to be achieved when all oscillators cycle in unison and is only possible when the coupling strength is sufficiently large. Smaller coupling gains result in an incoherent state. If the sign of the coupling gain is switched, and the gain is sufficiently large, the oscillators will end up in a balanced (i.e. anti-synchronized) state in which the phase centroid is at the origin. Please refer to [5], [6], [7], [8], [9], [10] and references therein for further details.

For engineering purposes, the objective is usually to drive the agents to a desired group state, and thus the incoherent state is less desirable. By choosing all natural frequencies at zero, the incoherent state can be avoided entirely. The sign of the coupling gain then determines whether the equilibrium state will be balanced (Fig. 1a) or aligned (Fig. 1b). With zero natural frequencies, each "oscillator" no longer oscillates, so we simply refer to the individuals as phase-coupled agents.

The phase-coupled agent model has been employed quite often in the multi-agent systems literature. Much of this literature has used this model as a steering controller for a group of unicycle-type vehicles. Early work here was done by Justh and Krishnaprasad [11], [1] and connections to the Kuramoto model were established in [2], [12]. In each of these works, the vehicles' headings will stabilize to a state in either the aligned set or the balanced set, depending on the sign of the control gain. Related work in this area has focused on splay state stabilization [12], [13], interconnection topology [14], [15], [16], and trajectory tracking [17], [18].

In order to apply the ideas of phase-coupled oscillator models to the problem of cooperative target tracking, a reference vector was introduced in [3], [19]. While the standard model of phase coupling can only result in balanced or aligned group states, the reference-augmented model drives the phase centroid to the reference vector, which can be anywhere in the unit ball, see Fig. 1c. For target tracking with a group of unit-speed unicycles, the phase centroid corresponds to the velocity of the spatial centroid. Thus, stability of the phase centroid to the reference vector enables the spatial centroid to track a target using steering inputs alone. Extensions of cooperative target tracking have been made to three dimensions [4].

One main drawback of applying the classical phase-coupled agent model to engineered systems is that the inter-vehicle communication takes place in continuous time. In other words, each agent needs to receive the state of every other agent at every time instant. Clearly, this communication scheme is infeasible in practice because the required bandwidth is infinite. In order to address this drawback, researchers have recently proposed a discrete time reformulation of the phase-coupled agent model [20]. This reformulation permits inter-vehicle communication to occur every $\Delta T$ seconds, as opposed to continually. In [20], linearization and Monte Carlo simulations were used to show

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that the discretized model preserves many of the interesting properties of the classical model for a certain range of system parameters.

The work in this paper focuses on the stability of phase coupling in discrete time and has two main contributions. The first is a proof showing that a system of $N$ phase-coupled agents is asymptotically stable to the balanced set for a conservative range of parameters. The second is a related proof showing asymptotic stability of the phase centroid to an arbitrary reference vector in the unit ball. While sufficient only, these analytical results are the first to prove stability to these sets, as previous results used simulation and local linearization techniques.

The presentation is organized as follows. The discrete time phase-coupled agent model is presented in Section II. Stability to the balanced and aligned sets is established in Sections III and IV, respectively. Simulation results demonstrating the theory are available in Section V, and concluding remarks are in Section VI. Proofs of lemmas can be found in the Appendix.

## II. Phase Coupling in Discrete Time

The classical continuous time model of phase-coupled oscillators studied by Kuramoto and others is:

$$\dot{\theta}_i(t) = \omega_i(t) - \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j(t) - \theta_i(t)). \tag{1}$$

Here $\theta_i$ and $\omega_i$ are the phase and natural frequency of the $i^{th}$ oscillator, and $N$ is the number of agents. The gain parameter $K$ models how tightly the group is coupled. To arrive at the discrete time phase-coupled agent model from [20], set $\omega_i = 0$ for all individuals (to avoid undesirable incoherent states), and apply a zero order hold:

$$\theta_i(h+1) = \theta_i(h) - \frac{K\Delta T}{N}\sum_{j=1}^{N}\sin(\theta_j(h) - \theta_i(h)). \tag{2}$$

Here $h$ is a time index and $\Delta T$ is the discretization period, which has been used as a strict deadline on communication [21], [22].

A useful parameter in the study of phase-coupled oscillator models is the phase centroid,

$$\bar{\mathbf{x}}(h) = \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i(h) = \bar{\rho}(h)\angle\bar{\theta}(h). \tag{3}$$

where

$$\mathbf{x}_i(h) = \begin{bmatrix}\cos\theta_i(h)\\ \sin\theta_i(h)\end{bmatrix}. \tag{4}$$

The magnitude of the phase centroid, $\bar{\rho}(h) = \|\bar{\mathbf{x}}(h)\|$, is called the order parameter. Kuramoto showed [5] that the phase centroid can be used to express (2) in mean field coupling form,

$$\theta_i(h+1) = \theta_i(h) - K\Delta T\bar{\rho}(h)\sin(\bar{\theta}(h) - \theta_i(h)). \tag{5}$$

The equilibria of (2) and (5), for non-zero $K$, occur whenever

$$\frac{1}{N}\sum_{i=1}^{N}\sin(\theta_j - \theta_i) = 0. \tag{6}$$

States, $\boldsymbol{\theta}$, for which (6) holds can be partitioned into aligned, balanced, and unstable sets, defined respectively as

$$\mathcal{A} = \{\boldsymbol{\theta} \in \mathbb{T}^N \mid \bar{\rho} = 1\} \tag{7}$$
$$\mathcal{B} = \{\boldsymbol{\theta} \in \mathbb{T}^N \mid \bar{\rho} = 0\} \tag{8}$$
$$\mathcal{U} = \{\boldsymbol{\theta} \in \mathbb{T}^N \setminus \mathcal{A}\cup\mathcal{B} \mid \sin(\theta_j - \theta_i) = 0\ \forall ij\}. \tag{9}$$

In the unstable set, all headings are parallel, but the state is neither aligned nor balanced.

As observed in previous work, the stability of the discrete time phase-coupled oscillator model to either aligned or balanced sets is related to $K\Delta T$. Conjecture 1 of [20] suggested without proof that the $N$ oscillator system (2), (5) is asymptotically stable to the aligned set for $-2 < K\Delta T < 0$ and to a balanced set for $0 < K\Delta T < 2$. Local linearization and Monte Carlo simulations for $N = 7$ were used to support this conjecture. Aligned set asymptotic stability for $-2 < K\Delta T < 0$ was later proved in [21], [22] by showing that the order parameter increases on each step.

## III. Balanced Set Stability

In this section, almost global asymptotic stability of $N$ phase-coupled agents to the balanced set is proven for $0 < K\Delta T < 1$. The main theorem is built upon two lemmas which will be presented before the theorem. Throughout this section, Fig. 2 may be used to give some graphical significance to the approach. Proofs of the lemmas can be found in the Appendix. Also, the following assumption will hold for the remainder of this section.

*Assumption 1:* Assume without loss of generality that the state $\boldsymbol{\theta}(h)$ has been rotated so that $\bar{\theta}(h) = 0$. Note then that

$$\bar{\mathbf{x}}(h) = [\bar{\rho}(h), 0]^T, \tag{10}$$

and $\boldsymbol{\theta}(h) \in \mathcal{A}\cup\mathcal{U} \implies \theta_i \in \{0, \pi\},\ i = 1, \ldots, N$. Also, note that $\bar{\mathbf{x}}(h+1)$ will not typically lie on the x-axis.

*Lemma 1:* Denote by $\mathbb{B}_r(\mathbf{c})$ the *open* ball of radius $r$ centered at the point $\mathbf{c}$. If there exists a nonempty $\mathcal{I} \subseteq \{1, \ldots, N\}$ such that

$$\mathbf{x}_i(h+1) \in \mathbb{B}_{\bar{\rho}(h)}\left(\mathbf{x}_i(h) - \bar{\mathbf{x}}(h)\right) \text{ for } i \in \mathcal{I}, \tag{11}$$

and

$$\mathbf{x}_i(h+1) = \mathbf{x}_i(h) \quad \text{otherwise}, \tag{12}$$

then

$$\bar{\mathbf{x}}(h+1) \in \mathbb{B}_{\bar{\rho}(h)}\left(\mathbf{0}\right). \tag{13}$$

Lemma 1 will be used to show that the phase centroid at $h+1$ will be closer to the origin than at $h$ (13), provided that individual state changes satisfy (11).

*Lemma 2:* Starting from any point $\mathbf{x}_i(h) = [\cos(\theta_i(h)), \sin(\theta_i(h))]^T$ on the unit circle, the heading update (2), (5) results in a new point

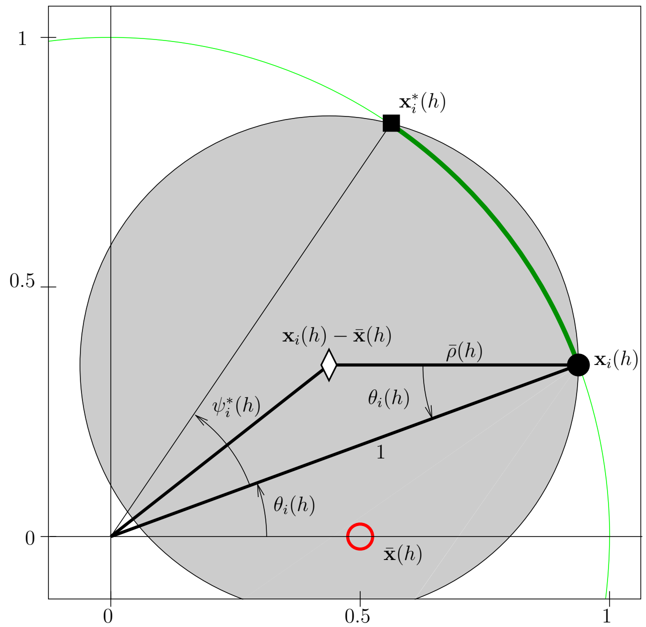


Fig. 2. This figure shows the general setup of the work in this paper. The unit circle is shown in green (gray). The phase centroid is drawn at $\bar{\mathbf{x}} = [0.5, 0]$ and is marked with a red "o", the dot is $\mathbf{x}_i$, and the diamond is located at $\mathbf{x}_i - \bar{\mathbf{x}}$. The square denoted $\mathbf{x}^*$ is the farthest point $\mathbf{x}_i$ could move around the unit circle before leaving the shaded ball, $\mathbb{B}_{\bar{\rho}}(\mathbf{x}_i - \bar{\mathbf{x}})$.

$\mathbf{x}_i(h+1) = [\cos(\theta_i(h+1)), \sin(\theta_i(h+1))]^T$ on the unit circle that satisfies

$$\mathbf{x}_i(h+1) \in \mathbb{B}_{\bar{\rho}(h)}(\mathbf{x}_i(h) - \bar{\mathbf{x}}(h)), \quad (14)$$

provided $0 < K\Delta T < 1$, $\bar{\rho}(h) > 0$, and $\theta_i(h) \neq k\pi$, $k \in \mathbb{Z}$. Referring to Fig. 2, Lemma 2 establishes that phase coupling will move an agent from $\mathbf{x}_i(h)$ to some point on the bold portion of the green unit circle.

The following theorem is a main result of this paper.

*Theorem 1:* For $N \geq 2$, the discrete time system (2), (5) is asymptotically stable to the balanced set for $0 < K\Delta T < 1$ from all initial conditions $\boldsymbol{\theta}(0) \in \mathbb{T}^N \backslash \mathcal{U} \cup \mathcal{A}$.

*Proof:* Take as a candidate Lyapunov function

$$V(h) = \bar{\rho}(h), \quad (15)$$

which is non-negative and equals zero for all $\boldsymbol{\theta} \in \mathcal{B}$. We will show that for $\boldsymbol{\theta}(h) \notin \mathcal{B}$, $V(h)$ is monotonically decreasing in $h$ by showing that $\bar{\mathbf{x}}(h+1) \in \mathbb{B}_{\bar{\rho}}(\mathbf{0})$.

First, assume that the state $\boldsymbol{\theta}(h) \notin \mathcal{U} \cup \mathcal{A}$, and define $\mathcal{J}$ as

$$\mathcal{J} = \{j \in 1, \ldots, N \mid \theta_j(h) \neq k\pi, k \in \mathbb{Z}\}. \quad (16)$$

Either the state is balanced, $\boldsymbol{\theta}(h) \in \mathcal{B}$, or the index set is nonempty, $|\mathcal{J}| > 0$. If $i \in \mathcal{J}$, $\mathbf{x}_i(h+1) \in \mathbb{B}_{\bar{\rho}(h)}(\mathbf{x}_i(h) - \bar{\mathbf{x}}(h))$, from Lemma 2, and otherwise $\mathbf{x}_i(h+1) = \mathbf{x}_i(h)$ because $\sin(\bar{\theta}(h) - \theta_i(h)) = 0$ in (5). Lemma 1 can then be used to conclude that $\bar{\mathbf{x}}(h+1) \in \mathbb{B}_{\bar{\rho}(h)}(\mathbf{0})$, so $V(h+1) < V(h)$.

Now, assume that the state $\boldsymbol{\theta}(h) \in \mathcal{U} \cup \mathcal{A}$. These points are invariant, but we will show by linearization that they are unstable. Without loss of generality, from Assumption 1 and the definitions of $\mathcal{U}$ and $\mathcal{A}$, reorder the state so that

$$\theta_i = \begin{cases} 0 & \text{for } i = 1, \ldots, \eta \\ \pi & \text{for } i = \eta + 1, \ldots, N, \end{cases} \quad (17)$$

and also note that $\eta > N/2 > 0$ because $\bar{\rho}(h)$ lies on the positive x-axis ($\boldsymbol{\theta}(h) \notin \mathcal{B}$). The linearized state transition matrix has $\eta$ eigenvalues at $1 + K\Delta T\bar{\rho}(h)/N$, which are unstable for $\bar{\rho}(h) > 0$ and $K\Delta T > 0$, as is the case here. Thus, because $\eta > 0$, $\boldsymbol{\theta}$ is unstable for all $\boldsymbol{\theta} \in \mathcal{U} \cup \mathcal{A}$. Note, therefore, that the state will never reach $\mathcal{U} \cup \mathcal{A}$ because all points in this union are unstable and $\boldsymbol{\theta}(0) \notin \mathcal{U} \cup \mathcal{A}$.

Thus, $V(h+1) < V(h)$ provided $V(h) \neq 0$, in which case the state is already balanced. LaSalle's invariance principle for discrete time systems can be used to conclude that the state will converge asymptotically to the largest invariant set, which here is the balanced set alone. ■

Note that Theorem 1 establishes a sufficient condition only. If $K\Delta T$ is larger than 1, Lemma 2 fails to hold. From the simulation study in [20], the Lyapunov function used here appears to work for $K\Delta T$ up to two. Ongoing work is aimed at formalizing this result.

## IV. Reference Set Stability

In this section, we modify the phase-coupled agent model to include a reference vector. The objective is to drive the phase centroid to this reference. The inclusion of a reference vector has been studied in continuous time for the application of coordinated target tracking using a group of $N$ planar or 3D unicycles. The main contribution presented in this section is a proof that the reference-augmented system is asymptotically stable in discrete time, for a range of $K\Delta T$.

Let the reference vector be $\mathbf{x}_{ref} = \rho_{ref}\angle\theta_{ref} \in \mathbb{B}_1(\mathbf{0})$, and consider the reference-augmented phase-coupled agent system,

$$\theta_i(h+1) = \theta_i(h) - \frac{K\Delta T}{N}\sum_{j=1}^{N}\sin(\theta_j(h) - \theta_i(h)) + K\Delta T\rho_{ref}\sin(\theta_{ref} - \theta_i(h)). \quad (18)$$

As with the standard phase-coupled oscillator model, the reference-augmented model can be written in mean field coupling form,

$$\theta_i(h+1) = \theta_i(h) - K\Delta T\tilde{\rho}(h)\sin(\tilde{\theta}(h) - \theta_i(h)), \quad (19)$$

where $\tilde{\rho}(h)$ and $\tilde{\theta}(h)$ are the magnitude and phase of centroid error, $\tilde{\mathbf{x}}(h) = \bar{\mathbf{x}}(h) - \mathbf{x}_{ref}$. One main difference between (19) and (5) is that the $\tilde{\rho} \leq 1 + \rho_{ref} \leq 2$ whereas $\bar{\rho} \leq 1$. The following assumption will be made throughout the remainder of this section.

*Assumption 2:* Assume without loss of generality that the state, $\boldsymbol{\theta}(h)$, has been rotated so that $\tilde{\theta}(h) = 0$. Note $\tilde{\theta}(h+1) \neq 0$ in general.

Let the reference set $\mathcal{R}$ and reference-augmented unstable set $\widetilde{\mathcal{U}}$ be defined as

$$\mathcal{R} = \{\boldsymbol{\theta} \in \mathbb{T}^N \mid \bar{\mathbf{x}} = \mathbf{x}_{ref}\} \tag{20}$$

$$\widetilde{\mathcal{U}} = \{\boldsymbol{\theta} \in \mathbb{T}^N \mid \sin(\tilde{\theta} - \theta_i) = 0 \ \forall i,\ \boldsymbol{\theta} \notin \mathcal{R}\}. \tag{21}$$

The following theorem establishes asymptotic stability to state in which the phase centroid matches the reference vector.

*Theorem 2 (Reference Set Stability):* The discrete time reference-augmented phase-coupled system (18), (19) is asymptotically stable to the reference set for $0 < K\Delta T < 2/(2+\rho_{ref})$, provided $\boldsymbol{\theta}(0) \notin \widetilde{\mathcal{U}}$ and $N \geq 2$.

*Proof:* The proof is similar to the one presented for Theorem 1 with the main difference being that $\tilde{\rho}(h)$ replaces $\bar{\rho}(h)$ and $\tilde{\theta}(h)$ replaces $\bar{\theta}(h)$. Due to space limitations, only differences from the proof of Theorem 1 will be described here.

Lemma 1 remains unchanged other than the notation change. Lemma 2 requires a slight modification to account for the new range of $K\Delta T$, however the result will remain unchanged. Specifically, for $0 < K\Delta T < 2/(2+\rho_{ref})$, (35) becomes

$$\psi_i(h) < \frac{2\tilde{\rho}(h)}{2+\rho_{ref}} \sin\left(\theta_i(h)\right). \tag{22}$$

Using the triangle inequality, $\tilde{\rho}(h) = \|\tilde{\mathbf{x}}(h)\| \leq 2 + \rho_{ref}$, so

$$\psi_i(h) < 2\sin(\psi_i^*/2), \tag{23}$$

as in (37), and the conclusion remains $\psi_i(h) < \psi_i^*$.

The only change required to the text of Theorem 1 is that $\widetilde{\mathcal{U}}$ replaces $\mathcal{A} \cup \mathcal{U}$. All states $\boldsymbol{\theta} \in \widetilde{\mathcal{U}}$ are unstable. Then, as before, LaSalle's invariance principle can be used to conclude asymptotic stability to $\mathcal{R}$. ■

Note that if the reference vector is unknown, $0 < K\Delta T < 2/3$ is always sufficient. Also, if the reference vector is at the origin, the result of Theorem 1 is recovered.

## V. SIMULATION RESULTS

In this section, simulations are presented to support and demonstrate the technical results of this paper. Two simulation scenarios will be presented. The first demonstrates balanced set stability for various values for $K\Delta T$ in a seven agent system. The second scenario examines stability to a reference vector for a group of ten agents.

### A. Balanced Set Stability

Theorem 1 guarantees that the balanced set is stable for values of $K\Delta T$ between zero and one for $N \geq 2$. However, other work [20] suggests but does not prove that the same Lyapunov candidate (15) will hold for $K\Delta T$ up to two. No known necessary conditions on $K\Delta T$ exist. Here, we take $\Delta T = 1$ and consider $K \in \{0.75, 1.75, 3.5\}$. The theory presented here guarantees only that $K = 0.75$ will be stable. Note that negative values of $K\Delta T$, down to $-2$, will drive the system to an aligned state [21]. The simulation results are presented in Fig. 3.

From the figure, it is apparent that the balanced set is approached for both $K\Delta T = 0.75$ and $K\Delta T = 1.75$. For the case of $K\Delta T = 3.5$, the Lyapunov candidate is clearly invalid, yet a balanced state may eventually be approached.

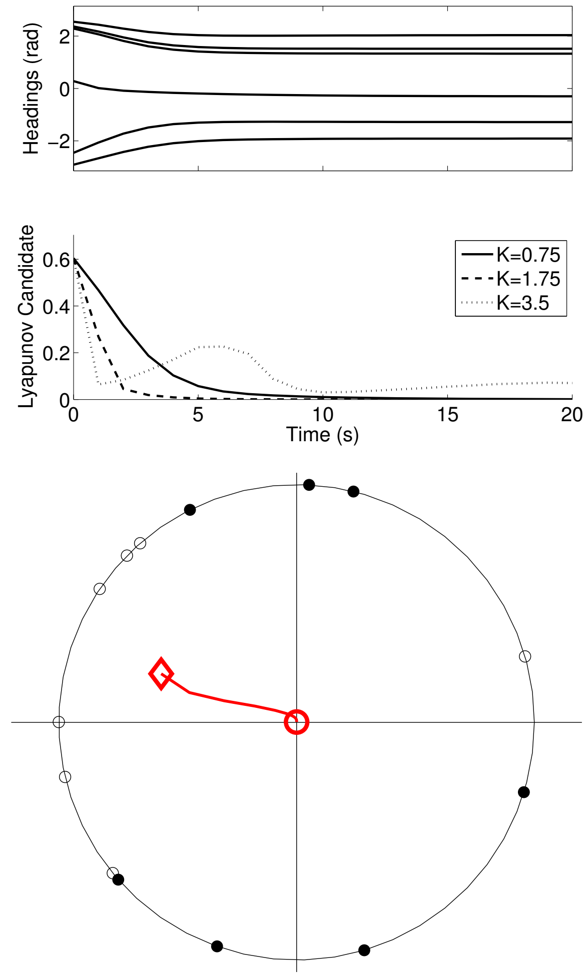


Fig. 3. A simulation of phase-coupling in discrete time (2),(5) is shown for $N = 7$. The upper plot shows the general trend of heading vs. time for $K = 0.75$ and $\Delta T = 1$. The middle plot shows the Lyapunov candidate (15), which, when zero, indicates that the state is balanced. The bottom plot shows the same simulation as the top plot, but plotted on a unit circle phase diagram. The initial state is marked by circles, the final state marked with dark dots, and the centroid path is the red trace from the red diamond to the large red circle. The same randomly selected initial condition was used for each value of $K$.

### B. Reference Set Stability

Theorem 2 guarantees that the phase centroid will approach any reference vector in the unit ball for $0 < K\Delta T < 2/3$. Here, we test this theorem in simulation, and see what happens as $K\Delta T$ becomes larger. The reference vector is $\mathbf{x}_{ref} = [.75, -.25]'$, $\Delta T = 1$, and $K \in \{0.5, 2.0, 3.0\}$. The simulation results are shown in Fig. 4.

As expected, the system is stable for $K\Delta T = 0.5$. For $K\Delta T = 2.0$, the system is not only stable, but also converges faster. With $K\Delta T$ at 3.0, the Lyapunov candidate is no longer valid, but the reference set may have been approached if the system was allowed to run longer.

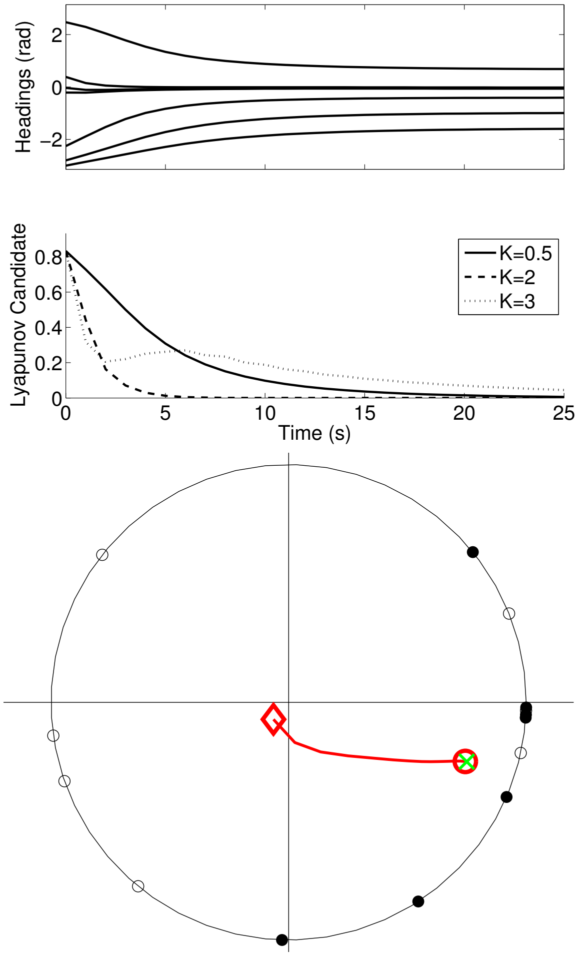


Fig. 4. A simulation of phase coupling in discrete time with a reference (18) is shown for $\Delta T = 1$ and $N = 10$. The upper plot shows the general trend of heading vs. time for $K = 0.5$. The middle plot shows $\tilde{\rho}$ (the Lyapunov candidate), which, when zero, indicates that the phase centroid equals the reference. The bottom plot shows the initial state as circles and the final state as dark dots. The centroid started at the red diamond and ended at the large red circle, which is on top of the reference, marked as a green "x".

## VI. Conclusion

The theory developed in this paper guarantees asymptotic stability of phase-coupled agents in discrete time to the balanced set for $K\Delta T$ between zero and one. Further, asymptotic stability of the phase centroid to an arbitrary vector in the unit ball was guaranteed for $K\Delta T$ up to $2/3$. While sufficient only, these analytical results are the first to prove stability to these sets, as previous results used simulation and linearization techniques.

In preparation is a paper applying the reference set stability developed here to the problem of target tracking. Previous work used the continuous time model which has the problem of being physically unrealizable. The theory developed in the present paper will allow control gains to be selected so that target tracking can be guaranteed with discrete time communications. The resulting controller will be demonstrated on the University of Washington's fin-actuated autonomous underwater multi-vehicle testbed.

In addition, future work will branch out in several directions. First, it will be interesting to see how large $K\Delta T$ can be pushed while still guaranteeing stability. We suspect that as $K\Delta T$ increases, the system behavior will become more chaotic, and the region of attraction about $\mathcal{B}$ and $\mathcal{R}$ will decrease in size. However, it may be possible to guarantee non-asymptotic stability because the state cannot escape the $N$-torus.

Another direction for future work is to examine limited communication topologies. In typical distributed systems, each agent can only communicate with a subset of the group. Work has been done here with the continuous time model, and it will be interesting to see how these results carry over to the discretized model.

Finally, time delay is a consideration in many engineered distributed systems. Time delay has been studied in the context of the continuous time model [23], [24] and preliminary simulation results [20] indicate that the discrete time phase-coupled agent model is robust to delay, but this result needs to be formalized.

## Appendix

*Proof:* [Lemma 1] To begin, add and subtract $\bar{\mathbf{x}}(h)$ from the definition of $\bar{\mathbf{x}}(h+1)$:

$$\bar{\mathbf{x}}(h+1) = \bar{\mathbf{x}}(h) + \frac{1}{N}\sum_{i=1}^{N}\left(\mathbf{x}_i(h+1) - \mathbf{x}_i(h)\right) \tag{24}$$

$$= \bar{\mathbf{x}}(h) + \frac{1}{N}\sum_{i=\mathcal{I}}\left(\mathbf{x}_i(h+1) - \mathbf{x}_i(h)\right). \tag{25}$$

Then, from (11),

$$\mathbf{x}_i(h+1) - \mathbf{x}_i(h) \in \mathbb{B}_{\bar{\rho}}\left(-\bar{\mathbf{x}}(h)\right) \text{ for } i \in \mathcal{I}. \tag{26}$$

Using this result, (25) can be rewritten as

$$\bar{\mathbf{x}}(h+1) = \bar{\mathbf{x}}(h) + \frac{1}{N}\sum_{i=\mathcal{I}} p_i, \tag{27}$$

where $p_i \in \mathbb{B}_{\bar{\rho}}\left(-\bar{\mathbf{x}}(h)\right)$. Now, note that

$$\frac{1}{N}\sum_{i=\mathcal{I}} p_i = \frac{\gamma}{|\mathcal{I}|}\sum_{i=\mathcal{I}} p_i = \gamma\bar{\mathbf{p}} \tag{28}$$

with $\gamma = |\mathcal{I}|/N \in (0, 1]$. Further, $\gamma p_i \in \mathbb{B}_{\bar{\rho}(h)}\left(-\bar{\mathbf{x}}(h)\right)$, which guarantees

$$\gamma\bar{\mathbf{p}} \in \mathbb{B}_{\bar{\rho}(h)}\left(-\bar{\mathbf{x}}(h)\right), \tag{29}$$

because the mean of a set of points must lie within the (open) convex hull of those points. Returning now to (27) and using (28),

$$\bar{\mathbf{x}}(h+1) = \bar{\mathbf{x}}(h) + \gamma\bar{\mathbf{p}} \tag{30}$$

which, when combined with (29) reveals

$$\bar{\mathbf{x}}(h+1) \in \mathbb{B}_{\bar{\rho}(h)}\left(\bar{\mathbf{x}}(h) - \bar{\mathbf{x}}(h)\right) = \mathbb{B}_{\bar{\rho}(h)}\left(\mathbf{0}\right). \tag{31}$$

■

*Proof:* [Lemma 2] Assume by symmetry that $\theta_i(h) \in (0, \pi)$ and note that $\theta_i(h+1) \in (\theta_i(h), \pi)$. The magnitude of the phase centroid, $\bar{\rho}(h)$, must lie in the open interval between zero and one because $\bar{\rho}(h) > 0$ from the lemma statement, and $\bar{\rho}(h) = 1$ implies $\theta_i(h) = 2k\pi$, which contradicts the lemma statement.

A circle of radius $\bar{\rho}(h) \in (0,1)$ centered at $\mathbf{x}_i(h) - \bar{\mathbf{x}}(h)$ intersects the unit circle exactly twice, for $\theta_i(h) \neq 0$. One of these points must lie at $\mathbf{x}_i(h)$ and the other at a point $\mathbf{x}_i^*(h) = 1\angle\theta_i^*(h)$ such that $\theta_i^*(h) \in (\theta_i(h), \pi)$, see Fig. 2. Define $\psi_i^*(h) = \theta_i^*(h) - \theta_i(h) \in (0, \pi - \theta_i(h))$ and note that

$$\begin{bmatrix} \cos(\theta_i(h) + \alpha) \\ \sin(\theta_i(h) + \alpha) \end{bmatrix} \in \mathbb{B}_{\bar{\rho}(h)}(-\bar{\mathbf{x}}(h)) \tag{32}$$

holds for all $\alpha \in (0, \psi_i^*(h))$. In what follows, we show that $\psi_i(h) = \theta_i(h+1) - \theta_i(h) < \psi_i(h)^*$, thereby guaranteeing that $\mathbf{x}_i(h+1) \in \mathbb{B}_{\bar{\rho}(h)}(\mathbf{x}_i(h) - \bar{\mathbf{x}}(h)$. In other words, $\mathbf{x}_i(h+1)$ will lie on the bold portion of the green unit circle shown in Fig. 2.

Applying the law of sines to the triangle highlighted in Fig. 2,

$$\bar{\rho}(h)\sin(\theta_i(h)) = \|\mathbf{x}_i(h) - \bar{\mathbf{x}}(h)\| \sin(\psi_i^*(h)/2). \tag{33}$$

Recalling (5) with $\bar{\theta} = 0$ from Assumption 1,

$$\begin{aligned} \psi_i(h) &= K\Delta T \bar{\rho}(h)\sin(\theta_i(h)) && (34) \\ &< \bar{\rho}(h)\sin(\theta_i(h)), && (35) \end{aligned}$$

for $0 < K\Delta T < 1$. Using (33) with (35),

$$\begin{aligned} \psi(h) &< \|\mathbf{x}_i(h) - \bar{\mathbf{x}}(h)\| \sin(\psi_i^*(h)/2) && (36) \\ &\leq 2\sin(\psi_i^*(h)/2) && (37) \\ &< \psi_i^*(h), && (38) \end{aligned}$$

because $\|\mathbf{x}_i(h) - \bar{\mathbf{x}}(h)\| \leq 2$ by the triangle inequality and because $\sin(x/2) \leq x/2$ for $x > 0$. ■

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