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Rank constrained stabilization of complex networks

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***Abstract*— In this paper global stabilization of a complex network is attained by applying local decentralized output feedback control to a minimum number of nodes of the network. The stabilization of the network is treated as a rank constrained problem. Necessary conditions for stabilization of a complex network is derived as a convex LMI representation. Strict positive realness conditions on the node level dynamics allow nonlinearities/uncertainties which satisfy sector conditions to be considered. A randomly generated academic example with 20 nodes is used to demonstrate the efficacy of the approach.**

I. Introduction

The increasing number of control applications involving sensor arrays, cooperative unmanned air vehicles, formations of satellite systems, etc. have spotlighted the problems in the control of network systems. A significant research problem in such applications is answering how the multiple dynamical systems operating over a network can achieve global stabilization or performance. Many researchers have contributed to the research area of control of network systems/cooperative control (see [1] for an overview). In comparison to conventional control problems, the control of networks is much more demanding. The key issue is how the information topology of the network distribution can be exploited. The topology of the network and its associated connectivity plays an important role in determining the dynamical behaviour of the networked system. Making use of local controlling strategies is attractive from the perspective of limited computing power and sensing capability.

Early efforts in developing network models have appeared in [2]–[4]. A general scale-free dynamical network model was discussed in [5], and subsequently conditions for synchronization [6] and the V-stability concept [7] for such networks were derived. The results depend on establishing a common Lyapunov function for studying the 'pinning' of complex networks. In [7] a state feedback control structure was utilised to obtain the pinning result. State agreement, synchronisation, and consensus, can all be viewed from a similar point of view [8]. Central to these problems is the graph describing the topology of the network of dynamical systems. Algebraic graph theory has been employed in a variety of research dealing with network systems (e.g. [1], [3]–[16]).

An approach based upon graph and system theoretic methods has been investigated in [15]. The paper makes use of convexity properties to describe how state agreement is achieved. Provided the systems move toward the convex hull of a set of systems, then state agreement can be achieved. In [8], [14] this work was further investigated for state agreement/synchronization of continuous time coupled nonlinear systems and the rendezvous problem of mobile robots. The consensus problem was also studied in [13] based on tools from matrix, algebraic graph, and control theory. In [11], the consensus problem (which is also a state agreement problem [14]) over a network has been formulated as a semi-definite programming problem and solved.

Recently, the passivity concept has been used in [12] to study the coordination of dynamical systems in a group. In [12] the difference between the output variables of individual dynamical systems in a group is controlled to belong to a defined compact set, and studied as a set stability problem. The passivity concept is then employed to design the control law. The compact set is defined as a sphere in the case of a formation of vehicles and as the origin for consensus problems. Decentralized robust control has been studied for large scale interconnected nonlinear systems in [17].

In this paper, the stabilization of a class of systems operating over a network is considered. Algebraic graph theoretical tools, based on the connectivity of the graph, are used to represent the dynamical systems operating over the network. The individual node level dynamics are represented as a combination of linear and nonlinear parts. The objective is to stabilize the network in a locally decentralized manner using only a few of the nodes of the network. Such a requirement imposes a rank constraint in the stabilization problem. In this paper, the stabilization of the networked dynamical systems is formulated as a rank constrained linear matrix inequality problem. The paper also investigates the possibilities of exploiting positive realness in the closed-loop nodes so that the formulation can handle a certain class of nonlinearities/uncertainties satisfying sector conditions.

II. Preliminaries

The set of real numbers is denoted by $\mathbb{R}$. The set of real-valued vectors of length m is given by $\mathbb{R}^m$. The set of arbitrary real-valued $m \times n$ matrices are given by $\mathbb{R}^{m\times n}$. The expression col(.) defines a column vector and diag(.) defines a diagonal matrix. For a symmetric positive definite (s.p.d) matrix $P = P^{\mathrm{T}} > 0$, $\lambda_{min}(P)$ and $\lambda_{max}(P)$ are the minimum and maximum eigenvalues. The symbols $\mathscr{N}(\cdot)$ and $\mathscr{R}(\cdot)$ represent the null space and range space of a matrix.

The graph theoretic terminology employed is also quite standard. A network $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, represents a simple, finite graph consisting of N vertices and p edges. For the graph $\mathscr{G}$, the adjacency matrix $A(\mathscr{G}) = [a_{ij}]$, is defined by setting $a_{ij} = 1$ if i and j are adjacent nodes of the graph, and $a_{ij} = 0$ otherwise. This is a symmetric matrix. The symbol $\Delta(\mathscr{G}) = [\delta_{ij}]$ represents the degree matrix, and is an $N \times N$ diagonal matrix, where δ_{ii} is the degree of the vertex i. The Laplacian of $\mathscr{G}$, $L(\mathscr{G})$, is defined as $\Delta(\mathscr{G}) - A(\mathscr{G})$. The smallest eigenvalue of $L(\mathscr{G})$ is exactly zero and the corresponding eigenvector is given by $\mathbf{1}$. The Laplacian $L(\mathscr{G})$ is always rank deficient and positive semi-definite [16].

III. Network structure

A distributed dynamical system operated over a connected network, consisting of N identical dynamical elements in-

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dexed $1,2,...,N$ is considered in this paper. The system is viewed as a graph $\mathscr{G}$ with N labelled vertices, each representing a dynamical system. The state of the vertex i will be denoted as $x_i \in X_i$. The topology of the network interconnection of the dynamical systems is represented as a graph $\mathscr{G}$ and the connectivity between the systems is provided a-priori by the Laplacian of the graph $L(\mathscr{G})$, from here on denoted as L. The N identical dynamical systems represent the N nodes of the Graph $\mathscr{G}$. If there is an interconnection between any two dynamical systems, it constitutes an edge connecting those nodes. The dynamics of the i^{th} individual node of the graph $\mathscr{G}$ is given in equations (1) and (2)

$$\dot{X}_i = AX_i + BU_i - \sum_{j=1}^{N} cL_{ij}\Gamma X_j + f_i(X_i) \quad (1)$$

$$Y_i = CX_i \quad (2)$$

where, $X_i = [x_{i_1}, x_{i_2}, ..., x_{i_n}] \in \mathbb{R}^n$ represents the n-dimensional state vector of the i^{th} node of the network. The symbol X represents collective state $X = [X_i, X_2, ..., X_N]$. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ represent the nominal linear part of the system comprising the dynamics of the i^{th} node. Assume that the matrices B and C have full column and row rank respectively. The triplet (A,B,C) is assumed to be a minimal or irreducible realization of the i^{th} node of the network. The real constant $c > 0$ is the coupling strength between the i^{th} and j^{th} node. The coupling strength is assumed to be identical for all the connections between the nodes. As described earlier, $L \in \mathbb{R}^{N\times N}$ denotes the connectivity of the topology of the network being considered. If there is a connection between node i and node j, then $L_{ij} = L_{ji} = -1$; otherwise $L_{ij} = L_{ji} = 0$. The diagonal elements $L_{ii} = k_i,\ i = 1,2,..,N$ where k_i is the degree of the node defined as the number of connection incidents at the i^{th} node.

The matrix $\Gamma = \tau_{ij} \in \mathbb{R}^{n\times n}$ represents the local coupling configuration among the states of the nodes. All the entries of Γ are 1 or 0 and represent the existence or non-existence of coupling/distribution in the respective channels in the network. In the present study it is assumed that

$$\Gamma = diag[\tau_1, \tau_2, .., \tau_i, .., \tau_n]$$

is diagonal, implying the coupling is identical in each node of the network. The signals $U_i \in \mathbb{R}^m$ and $Y_i \in \mathbb{R}^p$ represent the control input and the measured outputs of the i^{th} node respectively. Here it is assumed that $p \geq m$. The functions $f_i(X_i)$, represent the nonlinear part of the dynamical system and are assumed to satisfy certain sector bounds which will be precisely defined later in the paper. *Note that the intention is to apply local decentralized output feedback control to a minimum number of nodes of the network such that stabilization is achieved globally.*

IV. Network Stabilisation

A. Linear Case

Initially, the theory will be developed for the linear case. The nonlinear part in (1) will not be considered and instead the linear system

$$\dot{X}_i = AX_i + BU_i - \sum_{j=1}^{N} cL_{ij}\Gamma X_j \quad (3)$$

$$Y_i = CX_i \quad (4)$$

is studied first. The theory is developed in such a way as to utilize the structure of the problem maximally, so that by looking at the node level dynamics, conditions for network stabilization are achieved. Some assumptions used in the paper will now be introduced and discussed.

A0) Assume that $\text{rank}(\Gamma) = m$

Since Γ is a diagonal matrix comprising entries 1 or 0, by rearrangement of the states of the dynamics of each node, it is possible to ensure without any loss of generality that Γ consists of the block diagonal matrices:

$$\Gamma = \begin{bmatrix} \mathbb{I}_m & 0 \\ 0 & 0 \end{bmatrix}$$

The following assumptions will be imposed on the linear system triples (A,B,C) throughout the paper:

A1) $\mathscr{N}(\Gamma) \cap \mathscr{R}(B) = \{0\}$

A2) There exists an $F \in \mathbb{R}^{m\times p}$ such that $\Gamma = BFC$

A3) The linear system (A,B,FC) is controllable, observable and minimum phase.

Some ramifications of these assumptions will be explored.

First notice that $rank(FCB) = rank(B) = m$. This can be shown as follows: first observe from the dimensions of the matrices Γ and B that $rank(\Gamma B) \leq m$. Now consider the homogeneous linear equation $\Gamma B\eta = 0$ where $\eta \in \mathbb{R}^m$. Considering this equation as $\Gamma(B\eta) = 0$, it is clear that the vector $B\eta \in \mathscr{N}(\Gamma)$ and $B\eta \in \mathscr{R}(B)$. By assumption A1, $\mathscr{N}(\Gamma)$ and $\mathscr{R}(B)$ are disjoint and hence $B\eta = 0$. However since by assumption B is full rank, $B\eta = 0$ implies $\eta = 0$. Hence the only solution to $\Gamma B\eta = 0$ is $\eta = 0$ which means ΓB has full column rank and therefore $rank(\Gamma B) = m$. According to assumption A2 there exists a design parameter matrix $F \in \mathbb{R}^{m\times p}$ such that $\Gamma = BFC$ holds. Multiplying both sides on the right by the matrix B and using the result that $rank(\Gamma B) = m$, it follows $rank(BFCB) = m$. Since

$$rank(\Gamma B) = rank(BFCB) \leq \min\{rank(B),\ rank(FCB)\}$$

it follows $rank(FCB) = m$ since if $rank(FCB) < m$ then the inequality above implies $rank(\Gamma B) < m$, which is a contradiction.

Remarks:

1) Assumptions A0 and A1 imply the range space of the coupling configuration matrix is constrained within a subspace of the range space of the control input. Intuitively this ensures that the control signals can be injected into the states of a particular node in the same channels that the interactions with the other nodes occur.

2) Having $rank(CB) = rank(FBC) = m$ can be interpreted as the dynamical mapping between the control signals and the measured outputs are relative degree one.

Using the fact that $rank(FCB) = m$, and from A3 that (A,B,FC) is minimum phase, as argued in [24] there exists a change of coordinates such that the triple (A,B,FC) has the special form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \quad FC = \begin{bmatrix} F_1 & 0 \end{bmatrix}$$

where $B_1, F_1 \in \mathbb{R}^{m\times m}$ and both matrices are nonsingular. Furthermore the matrix $A_{22} \in \mathbb{R}^{(n-m)\times(n-m)}$ is stable since the eigenvalues of A_{22} represents the invariant zeros of the triple (A,B,FC) (which are stable by assumption). Further details are given in [24]. In order that $BFC = \Gamma$ holds, it follows that by choice of F, the relationship $FB_1 = I_m$ or equivalently $F_1 = B_1^{-1}$ must hold.

The objective is now to develop conditions under which the closed loop node level system $(A-\gamma BFC)$ is Hurwitz for all $\gamma > \gamma_0$ for some scalar $\gamma_0 \geq 0$. From assumption A2, this is equivalent to showing $(A-\gamma\Gamma)$ is Hurwitz for all $\gamma > \gamma_0$. Here a slightly stronger condition will be required: namely, that there exists a s.p.d Lyapunov matrix P such that

$$\mathscr{L} = P(A-\gamma\Gamma) + (A-\gamma\Gamma)^{\mathrm{T}}P < 0 \tag{5}$$

for all $\gamma \geq \gamma_0$ for some $\gamma_0 \geq 0$. This of course is a sufficient condition for $(A-\gamma\Gamma)$ to be Hurwitz for all $\gamma > \gamma_0 \geq 0$.

First an observation will be made on the properties required of the symmetric matrix $(P\Gamma + \Gamma^{\mathrm{T}}P)$. In order for (5) to hold for all $\gamma > \gamma_0$ the matrix inequality

$$(P\Gamma + \Gamma^{\mathrm{T}}P) \geq 0$$

must hold. Suppose for a contradiction that $(P\Gamma + \Gamma^{\mathrm{T}}P)$ is not semi-positive definite. If this is the case, then the symmetric matrix $(P\Gamma + \Gamma^{\mathrm{T}}P)$ has a negative eigenvalue $\underline{\lambda} < 0$ and an associated eigenvector $v \neq 0$ such that

$$v^{\mathrm{T}}(P\Gamma + \Gamma^{\mathrm{T}}P)v = \underline{\lambda}\|v\|^2$$

Multiplying $\mathscr{L}$ from (5) on the left and right by v^{T} and v respectively, it follows

$$\begin{aligned} v^{\mathrm{T}}\mathscr{L}v &= v^{\mathrm{T}}(PA + A^{\mathrm{T}}P)v - \gamma v^{\mathrm{T}}(P\Gamma + \Gamma^{\mathrm{T}}P)v && (6)\\ &= v^{\mathrm{T}}(PA + A^{\mathrm{T}}P)v - \gamma\underline{\lambda}\|v\|^2 && (7)\end{aligned}$$

Since $-\underline{\lambda}\|v\|^2 > 0$, for a sufficiently large value of γ,

$$v^{\mathrm{T}}\mathscr{L}v = v^{\mathrm{T}}(PA + A^{\mathrm{T}}P)v - \gamma\underline{\lambda}\|v\|^2 > 0$$

which contradicts $P(A-\gamma\Gamma) + (A-\gamma\Gamma)^{\mathrm{T}}P < 0$ for all $\gamma > \gamma_0$. Consequently a necessary condition for (5) to hold is that

$$(P\Gamma + \Gamma^{\mathrm{T}}P) \geq 0$$

The s.p.d matrix P from (5) is partitioned into the 4-block matrix form

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^{\mathrm{T}} & P_3 \end{bmatrix}$$

where $P_1 \in \mathbb{R}^{m\times m}$. By direct computation

$$(P\Gamma + \Gamma^{\mathrm{T}}P) = \begin{bmatrix} 2P_1 & P_2 \\ P_2^{\mathrm{T}} & 0 \end{bmatrix}$$

Therefore necessary and sufficient conditions in order for $(P\Gamma + \Gamma^{\mathrm{T}}P) \geq 0$ are that $P_2 = 0$ and so $P = diag\{P_1, P_3\}$.

Notice if P_1 is chosen as $(B_1^{-1})^{\mathrm{T}}B_1^{-1}$ (which is quite legitimate since $(B_1^{-1})^{\mathrm{T}}B_1^{-1} > 0$ because $\det(B_1) \neq 0$) then by direct computation it can be shown that

$$PB = (FC)^{\mathrm{T}} \tag{8}$$

for any choice of P_3. The significance of (8) will be discussed later in the paper.

It will now be shown by direct construction that there does indeed exist a γ_0 such that (5) holds for all $\gamma > \gamma_0$. By direction computation, $\mathscr{L}$ from (5) has the form

$$\mathscr{L} = \begin{bmatrix} P_1A_{11} + A_{11}^{\mathrm{T}}P_1 - 2\gamma P_1 & P_1A_{12} + A_{21}^{\mathrm{T}}P_3 \\ P_3A_{21} + A_{12}^{\mathrm{T}}P_1 & P_3A_{22} + A_{22}^{\mathrm{T}}P_3 \end{bmatrix} \tag{9}$$

where $P_1 = (B_1^{-1})^{\mathrm{T}}B_1^{-1}$. Recall that from the minimum phase assumption A3, the matrix A_{22} is stable. Let P_3 be any symmetric positive definite matrix such that

$$Q_3 := P_3A_{22} + A_{22}^{\mathrm{T}}P_3 < 0$$

From classical Lyapunov theory the existence of such a P_3 is guaranteed to exist. From the Schur complement (see for example [23]), $\mathscr{L} < 0$ if and only if

$$2\gamma P_1 > P_1A_{11} + A_{11}^{\mathrm{T}}P_1 - (P_1A_{12} + A_{21}^{\mathrm{T}}P_3)Q_3^{-1}(P_1A_{12} + A_{21}^{\mathrm{T}}P_3)^{\mathrm{T}}$$

Since $P_1 > 0$ this can always be satisfied for a large enough γ. Let γ_0 be the minimum value of γ with respect to the choice of P_3 and $P_1 = (B_1^{-1})^{\mathrm{T}}B_1^{-1}$. Then by construction:

$$P(A-\gamma_0\Gamma) + (A-\gamma_0\Gamma)^{\mathrm{T}}P < 0$$

Furthermore since

$$\begin{aligned} \mathscr{L} &= P(A-\gamma\Gamma) + (A-\gamma\Gamma)^{\mathrm{T}}P \\ &= \underbrace{P(A-\gamma_0\Gamma) + (A-\gamma_0\Gamma)^{\mathrm{T}}P}_{<0} + (\gamma_0-\gamma)\underbrace{(P\Gamma + \Gamma^{\mathrm{T}}P)}_{\geq 0}\end{aligned}$$

If follows $\mathscr{L} < 0$ for all $\gamma \geq \gamma_0$ as required.

Formally the minimum value of γ_0 can be found via the LMI optimization [18]. Formally:

Min. w.r.t $\gamma > 0$ and $P_3 > 0$ the LMIs

$$\begin{bmatrix} (B_1^{-1})^{\mathrm{T}}B_1^{-1}A_{11} + A_{11}^{\mathrm{T}}(B_1^{-1})^{\mathrm{T}}B_1^{-1} - 2\gamma(B_1^{-1})^{\mathrm{T}}B_1^{-1} & (B_1^{-1})^{\mathrm{T}}B_1^{-1}A_{12} + A_{21}^{\mathrm{T}}P_3 \\ P_3A_{21} + A_{12}^{\mathrm{T}}(B_1^{-1})^{\mathrm{T}}B_1^{-1} & P_3A_{22} + A_{22}^{\mathrm{T}}P_3 \end{bmatrix} < 0 \tag{10}$$

$$0 < \gamma \tag{11}$$

This is a convex optimization problem and can be solved using standard LMI solvers.

Remark 2: This result can be interpreted as each node triple $((A-\gamma_iBFC), B, FC)$ is strictly positive real [27] for $\gamma > \gamma_0$ since there exists a P such that

$$P(A-\gamma_iBFC) + (A-\gamma_iBFC)^{\mathrm{T}}P < 0 \tag{12}$$

and $PB = (FC)^{\mathrm{T}}$. This is similar to the Constrained Lyapunov Problem studied in [24]–[26].

B. Output Feedback Control

The output feedback control algorithm, $U_i = -\gamma_iY_i$, will be employed to achieve synchronization. *Note that the intention is to apply decentralized output feedback control to the minimum number of nodes of the network such that stabilization is achieved globally.* This is beneficial from the point of view of minimising computational effort, and minimising sensory information. For ease of exposition the local feedback control law will be written as

$$U_i = -\gamma_iFY_i \qquad \text{for } i = 1\ldots N$$

where in the nodes in which no control signal is injected, $\gamma_i \equiv 0$. Substituting for Y_i, yields

$$U_i = -\gamma_iFCX_i \qquad \text{for } i = 1\ldots N$$

and (1) can be re-written as

$$\dot{X}_i = AX_i - \gamma_i \Gamma X_i - \sum_{j=1}^{N} cL_{ij}\Gamma X_j \quad (13)$$

$$Y_i = CX_i \quad (14)$$

since by construction $BFC = \Gamma$. With simple algebraic manipulation equation (13) can be conveniently written as

$$\dot{X}_i = AX_i - \sum_{j=1}^{N} c\tilde{L}_{ij}\Gamma X_j \quad (15)$$

where

$$\tilde{L} := L + D_r \quad (16)$$

and by definition

$$D_r = diag[\gamma_1, \gamma_2, .., \gamma_i, ..., \gamma_N] \quad (17)$$

When D_r is full rank, it implies that control signals are injected in every node of the network. The objective of the paper is to obtain global stabilization with most of the γ_i entries as zero. The solution to the problem is not trivial and necessarily this imposes a rank constraint on this matrix. The dynamics of the overall network can conveniently be written as

$$\dot{X} = \left(\mathbb{I}_N \otimes A - c(\tilde{L} \otimes \Gamma)\right) X \quad (18)$$

Note that by construction, $\tilde{L}$ is dependent on the control gains γ_i, $i = 1 \dots N$. Also by construction $\tilde{L}$ is a symmetric matrix since both L and D_r are symmetric. In [7] a similar separation of the topology matrix is achieved and a condition is introduced by incorporating the concept of a passivity degree. The methodology in this paper has origins in a more classical control approach. By spectral decomposition (see [22]), the symmetric matrix $\tilde{L}$ can be written as

$$\tilde{L} = VDV^{\mathrm{T}} \quad (19)$$

where the orthogonal matrix $V \in \mathbb{R}^{N \times N}$ is formed from the eigenvectors of $\tilde{L}$, and $D \in \mathbb{R}^{N \times N}$ is a diagonal matrix formed from the eigenvalues so that

$$D := diag(d_1, d_2, .., d_i, ..., d_N)$$

Define a co-ordinate transformation $T : X \mapsto Z := TX$, where

$$T = (V^{\mathrm{T}} \otimes \mathbb{I}_n) \quad (20)$$

and V is the orthogonal matrix from the spectral decomposition in (19). The transformation matrix T is an orthogonal transformation since using the properties of Kronecker products (see Appendix 1)

$$(V^{\mathrm{T}} \otimes \mathbb{I}_n)^{\mathrm{T}}(V^{\mathrm{T}} \otimes \mathbb{I}_n) = (V \otimes \mathbb{I}_n)(V^{\mathrm{T}} \otimes \mathbb{I}_n) = (VV^{\mathrm{T}} \otimes \mathbb{I}_n) = \mathbb{I}_{nN}$$

since V is orthogonal. Such a transformation will be shown to provide a decoupled structure for analysis, and restricts the analysis to the node level dynamics of the network.

On application of the transformation T to the state X of the graph, the first derivative of the new states are given by:

$$\dot{Z} = (V^{\mathrm{T}} \otimes \mathbb{I}_n)\dot{X} \quad (21)$$

The transformation given in (21) is applied to (18) and algebraic manipulations making using of the Kronecker identities discussed in Appendix 1 yields

$$\dot{Z} = ((\mathbb{I}_N \otimes A) - c(D \otimes \Gamma))Z \quad (22)$$

The structure which is obtained is simple and consistent with classical control systems theory. This structure enables the output feedback control problem for stabilization of the network to be investigated by considering the individual node level dynamics. The dynamics of an individual node in the transformed co-ordinates given in (22) can be written as

$$\dot{Z}_i = (A - cd_i\Gamma)Z_i \quad (23)$$

where $Z = col(Z_1, Z_2, \dots Z_N)$ because of the diagonal nature of D. It follows from the earlier results that if

$$cd_i > \gamma_0, \qquad \forall i = 1 \dots N$$

then $(A - cd_i\Gamma)$ is Hurwitz and furthermore there exists a s.p.d. P such that, for nodes $i = 1 \dots N$, the following strict matrix inequality holds.

$$P(A - cd_i\Gamma) + (A - cd_i\Gamma)^{\mathrm{T}}P < 0 \quad (24)$$

For stabilization of the network to a fixed point solution, the objective is to choose γ_i such that the stability condition is satisfied for the system at node level given in (23). Since the d_i are the eigenvalues of $\tilde{L}$ if

$$\tilde{L} > \frac{\gamma_0}{c}\mathbb{I}_N \quad (25)$$

then $d_i > \frac{\gamma_0}{c}$ for $i = 1 \dots N$. Thus the network stability problem is to choose the gains γ_i for all $i = 1 \dots N$ from (17) such that (25) holds. The LMI formulation of the problem is discussed in the next section.

C. Nonlinear Case

The results discussed so far pertain to the linear system in (3)-(4). This is now extended to the nonlinear case in (1)-(2). The dynamics of the network having nonlinearities/uncertainties can be represented as:

$$\dot{X} = (\mathbb{I}_N \otimes A)X - c(\tilde{L} \otimes \Gamma)X + f(X) \quad (26)$$

where the augmented state vector $X = \mathrm{col}(X_1, X_2, \dots X_N)$ and $f(X) = \mathrm{col}(f_1(X_1), \dots f_N(X_N))$ represents the vector of nonlinearities.

A4) Suppose that the nonlinearities satisfy

$$f_i(X_i) = B\xi_i(X_i) \quad i = 1 \dots N$$

for some functions of the states $\xi_i(X_i)$ where

$$(Fy_i)^{\mathrm{T}}(\xi_i) \leq 0 \quad (27)$$

is satisfied for all X_i where y_i is thought of as CX_i.

Equation (27) represents a sector condition on the nonlinearity $\xi_i(X_i)$. Define $\xi = \mathrm{col}(\xi_1, \dots, \xi_N)$. As argued earlier, the triples $((A - \gamma_i BFC), B, FC)$ are strictly positive real for $\gamma_i > \gamma_0$ i.e. there exists a P such that

$$P(A - \gamma_i BFC) + (A - \gamma_i BFC)^{\mathrm{T}}P < 0 \quad (28)$$

and $PB = (FC)^{\mathrm{T}}$. Define

$$\mathbb{B} := \mathbb{I}_N \otimes B \quad (29)$$

$$\mathbb{C} := \mathbb{I}_N \otimes C \quad (30)$$

$$\mathbb{F} := \mathbb{I}_N \otimes F \quad (31)$$

$$\mathbb{P} := \mathbb{I}_N \otimes P \quad (32)$$

then

$$(\mathbb{F}Y)^{\mathrm{T}}\xi = \sum_{i=1}^{N}(FY_i)^{\mathrm{T}}\xi_i \leq 0$$

since from A4, for $i = 1\ldots N$, the nonlinearities satisfy $(FY_i)^{\mathrm{T}}\xi_i \leq 0$. Also notice that

$$\mathbb{P}\mathbb{B} = (\mathbb{F}\mathbb{C})^{\mathrm{T}}$$

since $PB = (FC)^{\mathrm{T}}$ and

$$\mathbb{P}(\mathbb{I}_N \otimes A - c(\tilde{L}\otimes\Gamma)) + (\mathbb{I}_N \otimes A - c(\tilde{L}\otimes\Gamma))^{\mathrm{T}}\mathbb{P} < 0$$

It follows that $\mathbb{V}(X) = X^{\mathrm{T}}\mathbb{P}X$ is a Lyapunov function for the nonlinear system in (26) written as

$$\dot{X} = (\mathbb{I}_N \otimes A)X - c(\tilde{L}\otimes\Gamma)X + (\mathbb{I}_N \otimes B)\xi \tag{33}$$

V. Computational Algorithm

For a given network system, the problem of designing the local decentralized output feedback control laws will be tackled as a two stage LMI optimization problem: the first one as a convex LMI and the second one as a rank constrained LMI (and hence nonconvex).

For the node dynamics represented by the triple (A,B,C), assuming A0-A3 are met, the first the problem which is tackled is that of finding the minimum γ_0 such that

$$P(A-\gamma_i BFC) + (A-\gamma_i BFC)^{\mathrm{T}}P < 0$$

for some s.p.d matrix P. As argued earlier this can be cast as a convex optimization problem:

Minimize γ: subject to the LMIs (10) and (11).

This is can be solved using any LMI solver and is a well defined generalized eigenvalue problem (see [18]).

Once γ_0 has been computed, a second optimization problem can be solved involving the matrix D_r from (17) which represents the output feedback gains injected at each node. If the nodes at which the control is applied are decided upon a-priori (based on the designer's intuition), then the problem can be posed as:

Minimize Trace(D_r) subject to:

$$\begin{aligned} L + D_r &> \gamma_0/c\mathbb{I}_N \\ D_r &\geq 0 \end{aligned}$$

This also represents a convex optimization problem and so can be tackled using LMI solvers [18].

Remarks:

- The trace minimization attempts to minimize the use of control effort by choosing 'small' feedback gains.
- This approach is reliant on the designer choosing a-priori the nodes in which to inject control signals – which will be difficult for large networks.

Instead, a different (nonconvex) approach may be adopted to try to minimize the number of nodes at which control is applied. The numerical algorithm is required to find the solution to a rank constrained LMI problem: see [19] and the references therein. A new optimization problem can be cast as:

Minimize Trace(D_r) subject to:

$$\begin{aligned} L + D_r &> \gamma_0/c\mathbb{I}_N \\ D_r &\geq 0 \\ rank(D_r) &\leq r \end{aligned}$$

where the positive integer r is chosen by the designer.

Such a problem is generally hard to solve, however there exist available algorithms, such as LMIRank [20]. The LMIRank can be called using YALMIP [21], a MATLAB toolbox for rapid prototyping of optimization algorithms. However, LMIRank does not support objective functions, and only solves feasibility problems. However the objective function can be minimized using an outer loop bisection algorithm. The rank computations being inherently hard from a numerical view point, means occasionally the actual rank obtained is higher than r, even though LMIRank claims feasibility. Such situations require tuning of the tolerances of the solver. These are entirely known, reported issues associated with the solver [19], [20].

VI. Numerical example

To demonstrate the application of the theory developed in this paper, an academic example is provided. Consider the graph $\mathscr{G}(20,47)$. The graph represents a network of dynamical systems consisting of 20 identical nodes. Note that an arbitrary network has been constructed consisting of 20 nodes and 47 interconnections (edges of the graph). The individual node dynamics are given as follows:

$$\dot{x_{i1}} = x_{i1} + x_{i2} - x_{i1}^2 sign(x_{i1}) \tag{34}$$

$$\dot{x_{i2}} = -x_{i2} \tag{35}$$

$$y_i = x_{i1} \tag{36}$$

Compared with the usual state feedback policies, only output information will be utilised for stabilizing the network, which is realistic. The use of only output information is advantageous from the perspective of minimising sensor requirements. The objective is to achieve local decentralised output feedback stabilization of the network by injecting control signals at a minimum number of nodes. The two stage LMI optimization problem discussed in Section V is employed. Three different case studies are reported in this paper: solutions with rank 5, 7 and 9. In the LMI's, a stability margin has been selected as 0.005. By solving the necessary LMI conditions, γ_0 is obtained as 1.005. In the second stage of the optimization, the rank constraints are imposed.

Consider Case I with rank 5 (the minimum rank obtained for which the network is stabilisable). The LMI-Rank solver gives a solution which correspond to nodes $2,4,6,11$, and 18, having nonzero feedback. The respective output feedback control gains for these nodes are [112.693, 55.130, 40.864, 114.287, 108.130]. The nodes in which control signals have been injected and the respective control gains are provided in Table I. Note that all the nodes that were controlled in Case I, are also used in Case II; however the gains are of lower value in an average sense. However in Case III, a different set of nodes (apart from node 4 and 6) have been provided by the solver. Recall that the solver tries to minimize the trace and hence there might also exist other combinations of nodes that could be used to stabilise the network. In Table I the average value of the control gains in the case of rank 5 is larger than that of rank 7 and rank 9. When more nodes are used to inject the output feedback control signals, the smaller the gains. Fig. 1 shows the simulation results. The left subplots in figure 1 show the time history of the outputs for 200 seconds in the three different cases. The right subplots are enlarged views

TABLE I

RESULTS OF STABILISATION OF THE NETWORK

Case	Rank	Parameter	Values								
I	5	Node	2	4	6	11	18				
		Gain	112.693	55.130	40.864	114.287	108.130				
II	7	Node	2	4	5	6	9	11	18		
		Gain	55.815	13.628	7.610	17.392	13.833	53.543	53.999		
II	9	Node	1	3	4	6	12	13	14	16	20
		Gain	1.232	6.841	7.288	1.998	3.655	1.083	4.126	5.155	3.863

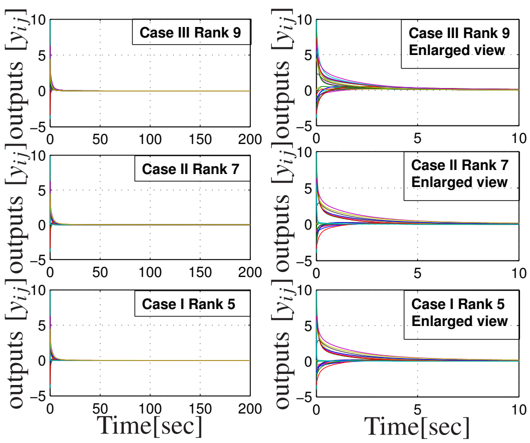

Fig. 1. Simulation results of stabilization of the network - cases

over the time window of 10 seconds. From figure 1, it can be seen that convergence is faster in case III compared to the other two cases.

VII. CONCLUSION

In this paper, the stabilization of the network dynamical system is achieved using a decentralised output feedback strategy. The four block representation and associated invariance structure of the networked dynamical systems have been exploited in the problem formulation. The stabilisation problem is formulated as a rank constrained Linear Matrix Inequality problem. The non-convex rank constraint emanating from the constraint on the minimum number of nodes for control signal injection is solved using LMIRank methods. This paper has further demonstrated how to exploit the notion of positive realness in the closed-loop node level dynamics so that the formulation can easily handle a wide class of nonlinearities/uncertainties satisfying sector conditions. The theory is applied to a relatively complex academic example and its efficacy is demonstrated.

APPENDIX 1

The Kronecker products of two matrices A and B, written $A\otimes B$, is a block matrix C with generic block entry $C_{ij} = A_{ij}B$. The following identities hold [28]:

1 $(A+B)\otimes C = (A\otimes C)+(B\otimes C)$,
2 $(A\otimes B)(C\otimes D) = AC\otimes BD$,
3 $(A\otimes B)^{\mathrm{T}} = A^{\mathrm{T}}\otimes B^{\mathrm{T}}$,
4 $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$.

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