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$\mathscr{L}_2$-gain of Port-Hamiltonian systems and application to a biochemical fermenter model

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***Abstract*— We consider the $\mathscr{L}_2$-gain of nonlinear Port-Hamiltonian systems. Using the Hamiltonian and an additional scaling matrix, we show that an upper bound on the $\mathscr{L}_2$-gain can be computed by solving a matrix inequality. The $\mathscr{L}_2$-gain is typically used in combination with the small-gain theorem. In particular it can be used to guarantee robust stability with respect to gain-bounded model uncertainties. This application of the $\mathscr{L}_2$-gain is demonstrated with a biochemical fermentation process where the specific cell growth rate is unknown but contained in a parameter interval.**

I. INTRODUCTION

Port-based network modeling of physical systems leads to a geometrically defined class of nonlinear systems called Port-Hamiltonian systems. Port-Hamiltonian systems have the advantage that when a set of subsystems is interconnected to form a network, structural information about the network is readily available from the model description. The total energy of the network is contained in the Hamiltonian, which should be seen as a storage function for the system, while the interconnection pattern is contained explicitly in an interconnection matrix. The energy dissipation, or the damping present in the network, is contained in a damping matrix. In short, Port-Hamiltonian systems give a transparent view of the underlying network structure. For more on Port-Hamiltonian systems and their origin, see e.g. [1], [2].

The class of Port-Hamiltonian systems has received an increasing amount of interest from the control community in recent years. Several controller design methods based on energy concepts and passivity have been developed for Port-Hamiltonian systems. A controller design method especially emphasizing and utilizing structural information is Interconnection and Damping Assignment Passivity-Based Control (IDA-PBC), which has proven successful for a range of applications (see e.g [3]–[6]).

The input-to-output behavior of Port-Hamiltonian systems has also been investigated, but then mainly considering passivity and passivity-based output-feedback control schemes (see e.g. [7] and references therein). A different tool for investigating the input-to-output behavior is the system gain. The input-to-output gain of a system is used for instance when applying the small-gain theorem, which can be used to guarantee robust stability of a nominal system with respect to norm bounded uncertainties [8], [9]. In this paper it is shown that an upper bound on the $\mathscr{L}_2$-gain of a nonlinear Port-Hamiltonian system can be found by solving a matrix inequality. This upper bound can be conservative, but in certain cases, when the Hamiltonian is the sum of independent storage functions, it is possible to reduce the conservativeness by using a scaling matrix.

A second order biochemical fermenter model is used to illustrate a possible application of the $\mathscr{L}_2$-gain result presented in this paper. Passivity based control of a fermenter model has been presented in [10] but does not address robustness properties of the closed-loop system. The fermenter model considered here is described by a Port-Hamiltonian system and has an uncertainty in the specific cell growth rate. A controller is designed, which robustly stabilizes the fermenter with respect to this uncertainty.

The outline of the paper is as follows: In the next section we first define the class of Port-Hamiltonian systems we are considering and present the standard stability condition for Port-Hamiltonian systems. Also, the $\mathscr{L}_2$-gain for general nonlinear systems along with the corresponding dissipation inequality is briefly presented. In Section III we specialize this result to Port-Hamiltonian systems. In Section IV we present an example where robust control of a fermentation process is considered. Section V provides a short summary of the presented results and an outlook.

II. PRELIMINARIES

A. *Port-Hamiltonian systems*

A Port-Hamiltonian system is a dynamical system described by the equations

$$\dot{x} = Q(x)\nabla H(x) + G(x)u, \tag{1}$$

$$y = C(x)\nabla H(x), \tag{2}$$

with state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^p$ and output $y(t) \in \mathbb{R}^q$. The scalar and continuously differentiable function $H(x)$ is the *Hamiltonian* of the system, and serves as a storage function candidate. The interconnecting ports to the environment are $G(x) \in \mathbb{R}^{n\times p}$ and $C(x) \in \mathbb{R}^{q\times n}$. The *structure matrix* $Q(x) \in \mathbb{R}^{n\times n}$ represents the internal interconnections and the damping; that is, it can be split into a skew-symmetric part $J(x)$ and a symmetric part $R(x)$ as $Q(x) = J(x) - R(x)$, where $J(x)$ is called the *interconnection matrix* and $R(x)$ is called the *damping matrix*. We use the structure matrix $Q(x)$ for notational convenience. The ∇-operator is a column vector defined as $\nabla = [\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}]^T$.

Evaluating the time derivative of the Hamiltonian along the system trajectories yields

$$\dot{H}(x) = \frac{1}{2}\nabla H^T(x)\left(Q(x) + Q(x)^T\right)\nabla H(x) + \nabla H^T(x)G(x)u. \tag{3}$$

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The system with zero input will have an asymptotically stable equilibrium point x_d if *(i)* $H(x)$ has an isolated minimum at x_d and if *(ii)* $Q(x)+Q(x)^T$ is negative definite. Furthermore, if *(i)* and *(ii)* are satisfied, and if the output matrix is chosen $C(x) = G^T(x)$, then $\dot{H}(x) < y^T u$ and hence the system with input u and output $y = G^T(x)\nabla H(x)$ is passive. The output $y = G^T(x)\nabla H(x)$ is, for this reason, called the *passive output*.

The stability condition given by $Q(x)+Q(x)^T < 0$ can be conservative. In certain cases the condition can be relaxed for example to $Q(x)$ being diagonally stable by using a parameterized Hamiltonian as storage function. For more on diagonal stability and diagonally stable matrices, see e.g. [11], [12]. This approach will be outlined in Section III.

B. $\mathscr{L}_2$-stability and $\mathscr{L}_2$-gain

In this subsection we briefly present the definitions of $\mathscr{L}_2$-stability and $\mathscr{L}_2$-gain as stated in [8]. A mapping from an input u to an output y is said to be $\mathscr{L}_2$-stable if the output is $\mathscr{L}_2$-norm bounded for all $\mathscr{L}_2$-norm bounded inputs. The mapping is said to be $\mathscr{L}_2$-stable with $\mathscr{L}_2$-gain less than γ if $\|y(t)\|_{\mathscr{L}_2} \leq \gamma \|u(t)\|_{\mathscr{L}_2} + \beta$ for all u with bounded $\mathscr{L}_2$-norm. The bias term $\beta \geq 0$ is a constant that depends on the initial condition. An upper bound on the $\mathscr{L}_2$-gain of a time-invariant nonlinear system can by found by solving the Hamilton-Jacobi-Bellman inequality. For completeness, we state this theorem here:

Theorem 2.1 ([8], Theorem 5.5): Consider the time-invariant nonlinear system

$$\dot{x} = f(x) + G(x)u, \quad x(0) = x_0 \tag{4}$$

$$y = h(x) \tag{5}$$

where $f(x)$ is locally Lipschitz, and $G(x)$, $h(x)$ are continuous over $\mathbb{R}^n$. The matrix G is $n \times p$ and $h : \mathbb{R}^n \to \mathbb{R}^q$. The functions f and h vanish at the origin; that is, $f(0) = 0$ and $h(0) = 0$. Let γ be a positive number and suppose there exists a continuously differentiable, positive semidefinite function $V(x)$ that satisfies the inequality

$$\begin{aligned}\mathscr{H}(V, f, G, h, \gamma) &\overset{def}{=} \nabla^T V(x) f(x) \\ &+ \frac{1}{2\gamma^2} \nabla^T V(x) G(x) G^T(x) \nabla V(x) + \frac{1}{2} h^T(x) h(x) \leq 0 \end{aligned} \tag{6}$$

for all $x \in \mathbb{R}^n$. Then, for each $x_0 \in \mathbb{R}^n$, the system (4)-(5) *is finite-gain $\mathscr{L}_2$-stable and its $\mathscr{L}_2$-gain is less than or equal to γ.*

III. $\mathscr{L}_2$-GAIN OF A PORT-HAMILTONIAN SYSTEM

The dissipation inequality used in Theorem 2.1 is only a sufficient condition and requires a storage function $V(x)$. Furthermore, when minimizing the $\mathscr{L}_2$-gain subject to this dissipation inequality, the result will depend on the choice of storage function. It is well known that for a general nonlinear system, a storage function is not easy to find. For Port-Hamiltonian systems, one possibility is to use the Hamiltonian as a storage function. As shown in the previous section, the Hamiltonian is used as storage function to show stability and passivity of Port-Hamiltonian systems, and it thus seems a "natural" choice. It may however lead to conservative results. A storage function using the Hamiltonian and additional parameters gives more degrees of freedom and may lead to reduced conservativeness. Therefore, we propose to use as storage function a positive semidefinite function $V(x)$ satisfying

$$\nabla V(x) = P \nabla H(x), \tag{7}$$

where $P \in \mathbb{R}^{n \times n}$ is a symmetric scaling matrix. A necessary integrability condition for $V(x)$ is given by the Perron-Frobenius theorem (see e.g. [11]), which requires the symmetry of its Hessian, i.e.

$$\nabla^2 V(x) = \nabla \left(P \nabla H(x) \right) \tag{8}$$

has to be symmetric. Assuming that $H(x) \geq 0$, then the integrability condition is met and the storage function $V(x)$ is positive semidefinite (p.s.d.) if P is chosen as:

$$P = \begin{cases} \text{p.s.d.} & \text{if } H(x) = x^T x, \\ \text{diagonal and p.s.d.} & \text{if } H(x) = \sum_{i=1}^n H_i(x_i), \\ pI > 0 & \text{else.} \end{cases}$$

The first case, that $H(x) = x^T x$, holds for example for linear systems. In that case, the scaling matrix results in a new storage function $V(x) = x^T P x$. The second case, namely $H(x) = \sum_{i=1}^n H_i(x_i)$, is from a mathematical point of view not obvious. Nevertheless, in many physical examples, the overall Hamiltonian $H(x)$ is the sum of independent Hamiltonians of interconnected subsystems. This is indeed the case for the biochemical fermenter considered in the next section.

Using a storage function $V(x)$ satisfying $\nabla V(x) = P \nabla H(x)$, we get the following corollary of Theorem 2.1 for the $\mathscr{L}_2$-gain of Port-Hamiltonian systems:

Corollary 3.1: Consider the Port-Hamiltonian system (1)-(2) *where $H(x) \geq 0$, $Q(x)\nabla H(x)$ is locally Lipschitz and $G(x)$, $C(x)\nabla H(x)$ are continuous over $\mathbb{R}^n$. The gradient $\nabla H(x)$ vanish at the origin. Let γ be a positive number and suppose there exists a positive semidefinite matrix P such that $\nabla \left(P \nabla H(x) \right)$ is symmetric and that satisfies the inequality*

$$\begin{bmatrix} PQ(x) + Q(x)^T P + C(x)^T C(x) & PG(x) \\ G(x)^T P & -\gamma^2 I \end{bmatrix} \leq 0 \tag{9}$$

for all $x \in \mathbb{R}^n$. Then, for each $x_0 \in \mathbb{R}^n$, the system (1)-(2) *is finite-gain $\mathscr{L}_2$-stable and its $\mathscr{L}_2$-gain is less than or equal to γ.*

Proof: Using a storage function $V(x)$ satisfying (7) yields

$$\begin{aligned}\mathscr{H}(V, Q\nabla H, G, C\nabla H, \gamma) &= \frac{1}{2} \nabla^T H(x) \big(PQ(x) + Q(x)^T P \\ &+ \frac{1}{\gamma^2} PG(x) G(x)^T P + C(x)^T C(x) \big) \nabla H(x) \leq 0 \end{aligned} \tag{10}$$

where the inequality follows from taking the Schur complement of the lower diagonal block of (9). The rest follows from Theorem 2.1. ∎

Of course one could minimize γ over P subject to the condition (9) in order to tighten the upper bound on the $\mathscr{L}_2$-gain. Also, requiring both P and $H(x)$ to be strictly positive definite, the standard stability condition $Q(x)+Q(x)^T<0$ can be relaxed to requiring $PQ(x)+Q(x)^TP<0$. If P is any positive diagonal matrix, this is the same as requiring $Q(x)$ to be diagonally stable [11].

For constant system matrices Q, G, and C, the inequality condition (9) is a standard linear matrix inequality (LMI), and for linear systems it corresponds to the bounded-real lemma [13], [14]. In the most general case (9) is a state modulated matrix inequality, which is generally neither an LMI nor a convex problem. Nevertheless we are in some cases able to solve such a matrix inequality also for state modulated matrices, using the boundedness of the matrix elements and exploiting the freedom in the choice of the scaling matrix P.

To give a simple example of the application of Corollary 3.1 and how the scaling matrix P can be used to reduce conservativeness, consider the scalar system

$$\dot{x} = -h(x) + gu \tag{11}$$

$$y = ch(x), \tag{12}$$

where $h(x)$ satisfies the sector condition $xh(x)>0$, $\forall x \neq 0$ and $h(0)=0$. Using the Hamiltonian

$$H(x) = \int_0^x h(s)ds > 0, \tag{13}$$

the system can be written in Port-Hamiltonian form as

$$\dot{x} = -\nabla H(x) + gu \tag{14}$$

$$y = c\nabla H(x), \tag{15}$$

and an upper bound on the $\mathscr{L}_2$-gain is given by solving the LMI

$$\begin{bmatrix} -2p+c^2 & pg \\ pg & -\gamma^2 \end{bmatrix} \leq 0. \tag{16}$$

Using $p=1$, i.e. neglecting the degree of freedom in p, gives the somewhat "artificial" condition $c^2<2$, and if this holds then $\gamma^2 \leq g^2/(2-c^2)$. On the other hand, minimizing γ by varying $p \geq 0$ gives the minimizer $p_* = c^2$, and we thus get $\gamma^2 \leq \gamma_*^2 = c^2 g^2$ as an upper bound for the $\mathscr{L}_2$-gain. In words, the $\mathscr{L}_2$-gain of the system is smaller than or equal to the product of the input and the output gains.

IV. Robustness analysis of a biochemical fermenter model

In this section a PI-type controller for a second order fermentation process, as illustrated in Fig. 1, is derived. We then consider a model/plant mismatch by transforming the closed-loop system into a nominal Port-Hamiltonian system interconnected with a norm bounded uncertainty. The matrix inequality (9) is applied to find an upper bound on the $\mathscr{L}_2$-gain of the nominal system and by using the small-gain theorem, the robust stability of the overall uncertain system with respect to uncertainty in the specific cell growth rate is investigated.

A. Model description

The dynamic model of a second order continuous biochemical fermenter is taken from [15] and is given by the equations

$$\dot{c}_x = \mu(c_s)c_x - \frac{q}{V}c_x \tag{17}$$

$$\dot{c}_s = -\frac{\mu(c_s)}{Y}c_x + \left(S_f - c_s\right)\frac{q}{V}, \tag{18}$$

where c_x denotes the cell concentration and c_s the substrate concentration. The term $\mu=\mu(c_s)$ denotes the specific cell growth rate, q is the volumetric inflow rate of the reactor and is equal to the outflow rate, V is the total reactor volume and is assumed to be constant, S_f is the feed of substrate entering the reactor and Y is the biomass/substrate yield coefficient.

The specific cell growth rate $\mu(c_s)$ could for example be given by the Monod-kinetics with an additional substrate overshoot term

$$\mu(c_s) = \frac{\mu_{max}c_s}{d_1 + c_s + d_2 c_s^2}, \tag{19}$$

which for any positive choice of the parameters μ_{max}, d_1 and d_2, is bounded by $0 \leq \mu(c_s) \leq \mu_{max}$.

Rewriting the model (17)-(18) with state $x=[x_1,x_2]^T=[c_x,c_s]^T$ and dilution rate $u=\frac{q}{V}$ as input leads to the Port-Hamiltonian system description

$$\dot{x} = \underbrace{\begin{bmatrix} \mu(x_2) & 0 \\ -\frac{\mu(x_2)}{Y} & 0 \end{bmatrix}}_{Q(x)} x + \underbrace{\begin{bmatrix} -x_1 \\ S_f - x_2 \end{bmatrix}}_{G(x)} u, \tag{20}$$

with Hamiltonian $H(x)=\frac{1}{2}x^Tx$, structure matrix $Q(x)$ and input matrix $G(x)$.

The state-space model (20) is restricted to positive states, since concentrations are not negative. The cell concentration x_1 must be strictly positive, otherwise the cells are washed out and the model reduces to a continuous stream of substrate. The state space of the model is therefore given by

$$\mathscr{X} = \left\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 > 0, x_2 \geq 0\right\}. \tag{21}$$

Analyzing the equilibrium point of the model leads, besides the trivial solution, to the relations

$$\mu_d = \mu(x_{2d}) = u_d \tag{22}$$

$$0 = S_f - x_{2d} - \frac{x_{1d}}{Y} \tag{23}$$

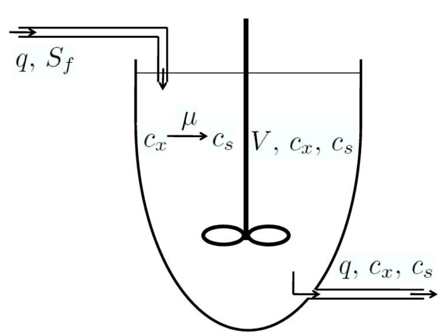

Fig. 1. Continuous biochemical fermenter

where the index d denotes the desired equilibrium point. The system (20) is a single-input system, and thus only one independent steady-state variable can be assigned, the other one results from equation (23).

The steady state input u_d directly assigns the steady state cell growth rate μ_d, which motivates the reformulation of (20) to

$$\dot{x} = \begin{bmatrix} \mu(x_2) - u & 0 \\ -\frac{\mu(x_2)}{Y} & -u \end{bmatrix} x + \begin{bmatrix} 0 \\ S_f \end{bmatrix} u, \tag{24}$$

where it can be seen that the input u can be used to influence the effect of the specific cell growth rate $\mu(x_2)$ in the first state. Therefore a goal for the controller is to cancel or to dominate the undesired influence of $\mu(x_2)$ on the system, thereby directly assigning a new closed-loop structure matrix.

B. Controller design

A feedback control law $u = \beta(x)$ can be used to shape the closed-loop structure matrix negative definite and to assign a closed-loop Hamiltonian with a global minimum at the desired equilibrium point x_d. As mentioned above, the input directly influences the effect of $\mu(x_2)$, and we cancel out this term by choosing the control law

$$\beta(x) = \mu(x_2) + v(x), \tag{25}$$

where $v(x)$ is a scalar function vanishing at the desired equilibrium point; i.e. $\beta(x_d) = \mu_d = u_d$. With (25) and using the steady-state relationship (23), the closed loop takes the Port-Hamiltonian form

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 0 \\ -\frac{\mu(x_2)}{Y} & -\mu(x_2) \end{bmatrix} (x - x_d) + \begin{bmatrix} -x_1 \\ S_f - x_2 \end{bmatrix} v(x) & (26) \\ &= Q_0(x)\nabla H_0(x) + G(x)v(x), & (27) \end{aligned}$$

with Hamiltonian $H_0(x) = \frac{1}{2}(x-x_d)^T(x-x_d)$. Because of the zero in the upper left element of the structure matrix $Q_0(x)$, the closed loop (27) is only marginally stable for $v(x) = 0$.

Energy shaping and passivity based feedback is used as a design tool for determining an asymptotically stabilizing control $v(x)$. To this end, the closed-loop Hamiltonian is chosen as

$$H_d(x) = \frac{1}{2}(x - x_d)^T P^{-1}(x - x_d), \tag{28}$$

where P is a positive definite matrix

$$P = \begin{bmatrix} 1 & -\frac{1}{Y} \\ -\frac{1}{Y} & \frac{1}{Y^2} + \frac{1}{p} \end{bmatrix}, \tag{29}$$

with $p > 0$. The parameter p is used to shape the closed-loop Hamiltonian. The desired Hamiltonian $H_d(x)$ clearly has a minimum at the desired equilibrium point. The term $v(x)$ in the control law is now chosen proportional to the passive output with the desired Hamiltonian, that is

$$\beta(x) = \mu(x_2) - k_p y_p, \tag{30}$$

where the passive output is defined as

$$y_p = G^T(x)\nabla H_d(x). \tag{31}$$

The closed loop using this control is given by

$$\dot{x} = \left(Q_0(x)P - k_p G(x)G(x)^T\right)\nabla H_d(x) = Q_d(x)\nabla H_d(x) \tag{32}$$

where $Q_d(x)$ is the closed-loop structure matrix. Multiplying out yields

$$Q_d = k_p \begin{bmatrix} -x_1^2 & x_1(S_f - x_2) \\ x_1(S_f - x_2) & -(S_f - x_2)^2 - \frac{1}{pk_p}\mu(x_2) \end{bmatrix}, \tag{33}$$

which is symmetric and negative definite for all $x_1 > 0$. Hence, the equilibrium point $x = x_d$ is asymptotically stable for any choice of controller parameters $k_p > 0$ and $p > 0$.

Remark 1: Note that the equilibrium point is not globally asymptotically stable, even though $H_d(x)$ is globally positive definite. The system description is only valid for positive states, and the level curves of $H_d(x)$ eventually cross the axis $x_1 = 0$ or $x_2 = 0$. It is not possible to guarantee, with this particular choice of Hamiltonian, that all trajectories starting within some level curve which crosses one of the axis $x_i = 0$, stay in the positive orthant of the state-space. What can be guaranteed is that for all initial conditions in the set defined by $H_d(x) < c$, where $H_d(x) = c$ defines the smallest level curve crossing either $x_1 = 0$ or $x_2 = 0$, all states will converge to their desired values.

C. Extension to PI-control

Although we do not exactly know all influences on the cell population, such as the specific growth rate, we still want to make sure that the biomass concentration will converge to the desired steady state. Therefore an integral term is added to the controller, integrating any deviation from the steady state. Integrating the passive output, and adding this to the control law, yields the PI-type controller

$$\beta(x) = \mu(x_2) - k_p y_p - k_I^2 \int_0^t y_p(\tau)\, d\tau. \tag{34}$$

Adding the integral state defined by

$$\dot{x}_3 = k_I y_P, \quad x_3(0) = 0, \tag{35}$$

to the system description leads to the extended Port-Hamiltonian system

$$\dot{x}_e = \begin{bmatrix} Q_d(x) & -k_I G(x) \\ k_I G(x)^T & 0 \end{bmatrix} \begin{bmatrix} \nabla H_d(x) \\ x_3 \end{bmatrix}, \tag{36}$$

with the extended state vector $x_e = [x_1, x_2, x_3]^T$ and Hamiltonian $H_d^e(x_e) = H_d(x) + \frac{1}{2}x_3^2$. The extended structure matrix can be split into the structure matrix of the stable system (32) and a skewsymmetric matrix containing the integral extension. When considering the time-derivative of $H_d^e(x_e)$, see (3), the skewsymmetric matrix cancels out and thus the equilibrium point $[x_{1d}, x_{2d}]^T$ is asymptotically stable.

D. Robustness Analysis

It is now assumed that the fermenter process is uncertain in the specific cell growth rate $\mu(x_2)$. Typically $\mu(x_2)$ is approximated by a positive rational function which is upper bounded by a maximum growth rate μ_{max}, for example the one given in (19). Using the $\mathscr{L}_2$-gain and small-gain arguments, we now show that by letting the controller dominate the uncertainty, robust stability of the closed-loop system with respect to a norm bounded uncertainty in $\mu(x_2)$ can be guaranteed.

Assume that the actual growth rate of the system is a function $\mu(x_2)$, bounded within $\mu \in [0, \mu_{max}]$, given by

$$\mu(x_2) = \frac{1}{2}\mu_{max}(1+\delta(x_2)), \tag{37}$$

where $\delta(x_2)$ is a scalar, static function of x_2 satisfying

$$||\delta(x_2)||_\infty = |\delta(x_2)| \le 1. \tag{38}$$

Applying the small-gain theorem, the uncertainty $\delta(\cdot)$ can be enlarged to a wider uncertainty class, namely any stable dynamic system Δ with $\mathscr{L}_2$-induced norm less than or equal to one, which contains the uncertainty $\delta(\cdot)$ as a special case. It is also not necessary that δ is a function of only x_2. For instance, μ may depend on both states, which can be handled in this setup without changes.

In the previous section, the controller directly canceled out the specific growth rate. Obviously, as the exact value of $\mu(x_2)$ is unknown, this is no longer possible. Instead, we seek to dominate the effect of $\mu(x_2)$ by replacing the term $\mu(x_2)$ with μ_{max} in the control law defined by (34). The fermenter model (20) controlled by (34), and with the additional state x_3 defined in (35), is given by

$$\begin{aligned}
\dot{x}_e &= \begin{bmatrix} \mu(x_2) & 0 & 0 \\ -\frac{\mu(x_2)}{Y} & 0 & 0 \\ -k_I x_1 & k_I(S_f - x_2) & 0 \end{bmatrix} x_e + G(x)\beta(x) \\
&= \begin{bmatrix} \mu_{max} & 0 & 0 \\ -\frac{\mu_{max}}{Y} & 0 & 0 \\ -k_I x_1 & k_I(S_f - x_2) & 0 \end{bmatrix} x_e + G(x)\beta(x) \\
&\quad + \begin{bmatrix} \left(\frac{1}{2}\mu_{max}(1+\Delta) - \mu_{max}\right) x_1 \\ -\frac{1}{Y}\left(\frac{1}{2}\mu_{max}(1+\Delta) - \mu_{max}\right) x_1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} Q_d(x) & -k_I G(x) \\ k_I G(x)^T & 0 \end{bmatrix} \nabla H_d^e(x_e) \\
&\quad + \begin{bmatrix} -\frac{1}{2}\mu_{max}x_1 + \frac{1}{2}\Delta\mu_{max}x_1 \\ -\frac{1}{Y}\left(-\frac{1}{2}\mu_{max}x_1 + \frac{1}{2}\Delta\mu_{max}x_1\right) \\ 0 \end{bmatrix},
\end{aligned} \tag{39}$$

which is the desired closed loop (36), where $\mu(x_2)$ is replaced by μ_{max}, plus an additional uncertainty term. Setting the input to the uncertainty Δ to z and its output to $w = \Delta(z)$, the system can be seen as an interconnection of a nominal system $\mathscr{N}$ and the uncertainty Δ, which contains $\delta(x_2)$ as a special case, as illustrated in Fig. 2, and (39) can be reformulated to

$$\begin{aligned}
\dot{x}_e &= \begin{bmatrix} Q_d(x) & -k_I G(x) \\ k_I G(x)^T & 0 \end{bmatrix} \nabla H_d^e(x_e) + \begin{bmatrix} -\frac{1}{2}\mu_{max}x_1 + w \\ \frac{1}{2Y}\mu_{max}x_1 - \frac{1}{Y}w \\ 0 \end{bmatrix} \\
z &= \frac{1}{2}\mu_{max}x_1.
\end{aligned} \tag{40}$$

In order to apply the small-gain theorem as stated in e.g. [8], bounded values can be added to the inputs and outputs of the dynamical system Δ, as this does not change the $\mathscr{L}_2$-gains. The input/output transformation

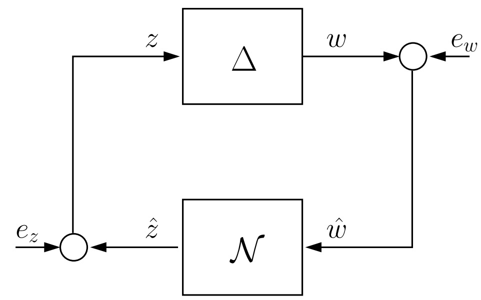

Fig. 2. Interconnection of nominal plant $\mathscr{N}$ and uncertainty Δ

$$\hat{w} = w + e_w = w - \frac{1}{2}\mu_{max}x_{1d} \tag{41}$$

$$\hat{z} = z - e_z = z - \frac{1}{2}\mu_{max}x_{1d} \tag{42}$$

is applied such that the inputs and the outputs at the equilibrium point are zero, that is for $x = x_d$ we have $\hat{w}_d = \hat{z}_d = 0$. From this transformation we finally get the nominal system $\mathscr{N}$ in the desired Port-Hamiltonian form

$$\dot{x}_e = \begin{bmatrix} Q_N(x) & -k_I G(x) \\ k_I G(x)^T & 0 \end{bmatrix} \nabla H_d^e(x_e) + \begin{bmatrix} G_N \\ 0 \end{bmatrix} \hat{w} \tag{43}$$

$$\hat{z} = \frac{1}{2}\mu_{max}[G_N^T, 0]\nabla H_d^e(x_e), \tag{44}$$

where the nominal structure matrix is

$$Q_N(x) = Q_d(x) - \frac{1}{2}\mu_{max}G_N G_N^T \tag{45}$$

and the input matrix is given by $G_N = [1, -\frac{1}{Y}]^T$.

From the small-gain theorem it is clear that the nominal system $\mathscr{N}$ is robustly stable with respect to Δ if it has $\mathscr{L}_2$-gain less than one. Applying Corollary 3.1 to $\mathscr{N}$ described by (43)-(44) with the diagonal positive definite scaling matrix

$$D = \frac{1}{2}\mu_{max}\gamma^2 I > 0 \tag{46}$$

yields, after some calculations, that the resulting matrix inequality (9) holds if

$$Q_d(x) + Q_d(x)^T + \frac{\mu_{max}}{2}\frac{1-\gamma^2}{\gamma^2}G_N G_N^T \le 0. \tag{47}$$

Because $Q_d(x) + Q_d(x)^T < 0$ for $x_1 > 0$, plugging $\gamma = 1$ into (47) yields a strict inequality. By continuity, (47) also holds for γ approaching one from below, and thus the $\mathscr{L}_2$-gain of $\mathscr{N}$ is strictly less than one. Using the small-gain theorem, robust stability of the nominal system $\mathscr{N}$ with respect to the uncertainty Δ in (37) can be concluded.

E. Simulation results

The fermenter system (17)-(18) with the PI-controller (34) is simulated starting from a non steady-state initial condition. The model parameter values are taken from [15] and are given in Table I. In the simulations, the cells have the specific growth rate $\mu(x_2)$ given by (19), while the controller is using the upper bound, namely

$$\beta(x) = \mu_{max} - k_p y_p - k_I \int_0^t k_I y_p(\tau)\, d\tau. \tag{48}$$

The controller parameters $k_p = 0.125$ and $k_I = 0.14$, and the Hamiltonian shaping parameter $p = 0.1$, were found to give an acceptable transient response. In Fig. 3 it is shown how the system states converge to the desired steady state solution x_d. Simulations using the proportional control (30) without the integral part yields, as expected, a shifted steady state (not shown).

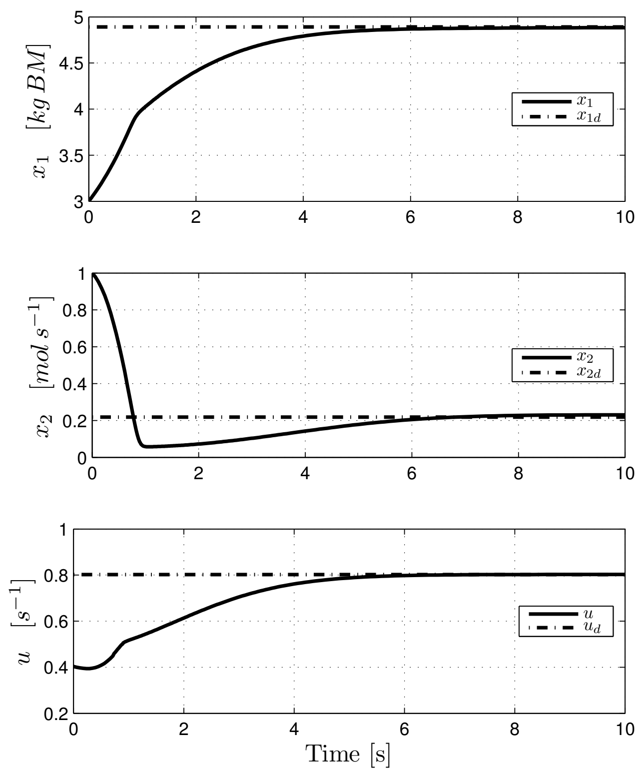

Fig. 3. PI-control of the fermenter model. The tuning parameters are $k_p = 0.125$, $k_I = 0.14$, and $p = 0.1$.

TABLE I

PARAMETER VALUES FOR THE BIOCHEMICAL FERMENTER

parameter	value	unit
μ_{max}	1	$[s^{-1}]$
d_1	0.03	$[mol\ m^{-3}]$
d_2	0.5	$[m^3\ mol^{-1}]$
Y	0.5	$[mol/kgBM]$
S_f	10	$[mol\ m^{-3}\ s^{-1}]$
c_{xd}	4.8907	$[kgBM\ m^{-3}]$
c_{sd}	0.2187	$[mol\ m^{-3}]$

V. SUMMARY

We have considered the $\mathscr{L}_2$-gain of nonlinear Port-Hamiltonian systems. Using the Hamiltonian together with a scaling matrix, an upper bound for the gain was given in terms of a matrix inequality, which in the linear case corresponds to the bounded-real lemma. An application of this result was shown by doing a robustness analysis for a controlled fermentation process. A PI-type controller was derived for this process assuming all parameters to be known. Requiring the $\mathscr{L}_2$-gain of the controlled nominal system to be less than one for a certain input/output pair, we showed that the overall system was robustly stable with respect to uncertainties in the specific cell growth rate.

It should be noted that what has been presented in the paper is an analysis result; given a controller, the $\mathscr{L}_2$-gain of the closed-loop system can be calculated. In certain cases, namely when considering a control of the type $u = K(x)\nabla H(x)$, the results can easily be extended to solve the synthesis problem via matrix inequalities. This corresponds in the linear case to the robust controller synthesis problem with static state feedback.

REFERENCES

[1] B. Maschke, R. Ortega, and A. J. van der Schaft, "Modeling and control of physical systems: an approach based on energy and interconnection," in *Proc.14th MTNS, Perpignan, France*, June 19–23 2000, cd-rom.

[2] J. Cervera, A. J. van der Schaft, and A. Baños, "Interconnection of port-Hamiltonian systems and composition of Dirac structures," *Automatica*, vol. 43, no. 2, pp. 212–225, 2007.

[3] R. Ortega, A. J. van der Schaft, I. Mareels, and B. Maschke, "Putting energy back in control," *IEEE Control Systems Magazine*, vol. 21, pp. 18–33, April 2001.

[4] R. Ortega, A. J. van der Schaft, B. Maschke, and G. Escobar, "Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems," *Automatica*, vol. 38, no. 4, pp. 585–596, 2002.

[5] R. Ortega and E. Garcia-Canseco, "Interconnection and damping assignment passivity-based control: A survey," *European Journal of Control*, vol. 10, no. 5, pp. 432–450, 2004.

[6] J. K. Johnsen and F. Allgöwer, "Interconnection and damping assignment passivity-based control of a four-tank system," in *Lagrangian and Hamiltonian Methods for Nonlinear Control 2006*, ser. LNCIS, F. Bullo and K. Fujimoto, Eds., vol. 366. Springer, 2007, pp. 111–122.

[7] R. Ortega, A. Loria, P. J. Nicklasson, and H. Sira-Ramírez, *Passivity–based Control of Euler–Lagrange Systems*, ser. Communications and Control Engineering. Springer, 1998.

[8] H. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2002.

[9] A. J. van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control*, 2nd ed. Springer, 2000.

[10] E. Fossas, R. M. Ros, and H. Sira-Ramírez, "Passivity-based control of a bioreactor system," *Journal of Mathematical Chemistry*, vol. 36, no. 4, pp. 347–360, 2004.

[11] E. Kaszkurewics and A. Bhaya, *Matrix Diagonal Stability in Systems and Computations*. Boston: Birkhäuser, 2000.

[12] M. Arcak and E. D. Sontag, "Diagonal stability of a class of cyclic systems and its connection with the secant criterion," *Automatica*, vol. 42, no. 9, pp. 1531–1537, September 2007.

[13] C. Scherer and S. Weiland, *Linear Matrix Inequalities in Control*, 2005, lecture notes, Dutch Institute of Systems and Control, Delft, The Netherlands.

[14] P. Gahinet and P. Apkarian, "A linear matrix inequality approach to H_∞ control," *Int. Journal of Robust and Nonlinear Control*, vol. 4, no. 4, pp. 421–448, 1994.

[15] K. M. Hangos, J.Bokor, and G. Szederkéngi, *Analysis and Control of Nonlinear Process Systems*. Springer, 2004.