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Semistability Theory for Differential Inclusions with Applications to Consensus Problems in Dynamical Networks with Switching Topology

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Abstract— **This paper focuses on semistability and finite-time semistability for discontinuous dynamical systems. Semistability is the property whereby the solutions of a dynamical system converge to Lyapunov stable equilibrium points determined by the system initial conditions. Using these results we develop a framework for designing semistable protocols in dynamical networks with switching topologies. Specifically, we present distributed nonlinear static and dynamic output feedback controller architectures for multiagent network consensus with dynamic communication topologies.**

I. INTRODUCTION

Modern complex dynamical systems are highly interconnected and mutually interdependent, both physically and through a multitude of information and communication networks. Distributed decision-making for coordination of networks of dynamic agents involving information flow can be naturally captured by graph-theoretic notions. These dynamical network systems cover a very broad spectrum of applications including cooperative control of unmanned air vehicles (UAV's), autonomous underwater vehicles (AUV's), distributed sensor networks, air and ground transportation systems, swarms of air and space vehicle formations, and congestion control in communication networks, to cite but a few examples. Hence, it is not surprising that a considerable research effort has been devoted to control of networks and control over networks in recent years.

Since communication links among multiagent systems are often unreliable due to multipath effects and exogenous disturbances, the information exchange topologies in network systems are often dynamic. In particular, link failures or creations in network multiagent systems result in switchings of the communication topology. This is the case, for example, if information between agents is exchanged by means of line-of-sight sensors that experience periodic communication dropouts due to agent motion. Variation in network topology introduces control input discontinuities, which in turn give rise to discontinuous dynamical systems. In this case, the vector field defining the dynamical system is a discontinuous function of the state, and hence, system stability can be analyzed using nonsmooth Lyapunov theory involving concepts such as weak and strong stability notions, differential inclusions, and generalized gradients of locally Lipschitz functions and proximal subdifferentials of lower semicontinuous functions [1].

In many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. In particular, it is important to develop information consensus protocols for networks of dynamic agents wherein a unique feature of the closed-loop dynamics under any control algorithm that achieves consensus is the existence of a continuum of equilibria representing a state of equipartitioning or *consensus*. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics, but depends on the initial system state as well. For such systems possessing a continuum of equilibria, *semistability* [2], and not asymptotic stability, is the relevant notion of stability. Semistability is the property whereby every trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. From a practical viewpoint, it is not sufficient to only guarantee that a network converges to a state of consensus since steady state convergence is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from a state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable, and consequently, semistable.

To address vector field discontinuities, in this paper we develop semistability and finite-time semistability [3] theory for differential inclusions. Using these results, we develop distributed control algorithms for addressing consensus problems for nonlinear multiagent dynamical systems with switching topologies. The proposed controller architectures are predicated on the recently developed notion of system thermodynamics [4] resulting in discontinuous controller architectures involving the exchange of information between agents that guarantee that the closed-loop dynamical network is consistent with basic thermodynamic principles.

II. LYAPUNOV-BASED SEMISTABILITY THEORY FOR DIFFERENTIAL INCLUSIONS

Consider the differential equation given by

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{1}$$

where $f : \mathbb{R}^q \to \mathbb{R}^q$ is Lebesgue measurable and locally essentially bounded [5]. Assume that the equilibrium set $f^{-1}(0) \triangleq \{x \in \mathbb{R}^q : f(x) = 0\}$ is closed. The *Filippov solution* [5] of (1) is defined by an absolutely continuous function $x : [0, \tau] \to \mathbb{R}^q$ such that

$$\dot{x}(t) \in \mathcal{K}[f](x(t)), \quad \text{a.a. } t \in [0, \tau], \tag{2}$$

where

$$\mathcal{K}[f](x) \triangleq \bigcap_{\delta>0} \bigcap_{\mu(\mathcal{S})=0} \mathrm{co}\, \{f(\mathcal{B}_\delta(x)\backslash\mathcal{S})\}, \tag{3}$$

$\mu(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^q$, $\mathcal{B}_\delta(x)$, $x \in \mathbb{R}^q$, denotes the open ball centered at x with radius $\delta > 0$, and "co" denotes the convex hull. Dynamical systems of the form given by (2) are called *differential inclusions* in the literature [6]. Note that it follows from 1) of Theorem 1 of [7] that there exists a set $\mathcal{N}_f \subset \mathbb{R}^q$ of measure zero such that

$$\mathcal{K}[f](x) = \mathrm{co}\left\{\lim_{i\to\infty} f(x_i) : x_i \to x,\, x_i \notin \mathcal{N}_f \cup \mathcal{W}\right\}, \tag{4}$$

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where $\mathcal{W} \subset \mathbb{R}^q$ is an arbitrary set of measure zero. Since the *set-valued map* given by (3) is upper semicontinuous with nonempty, convex, and compact values, and is also locally bounded, it follows that Filippov solutions to (1) exist [5]. We say that a set $\mathcal{M}$ is *weakly invariant* (resp., *strongly invariant*) with respect to (1) if for every $x_0 \in \mathcal{M}$, $\mathcal{M}$ contains a maximal solution (resp., all maximal solutions) of (1) [8], [9].

To develop Lyapunov theory for nonsmooth dynamical systems of the form given by (1), we need to introduce the notion of generalized derivatives and gradients. Here we focus on Clarke generalized derivatives and gradients [10].

Definition 2.1 ([10]): Let $V : \mathbb{R}^q \to \mathbb{R}$ be a locally Lipschitz continuous function. The *Clarke upper generalized derivative* of $V(x)$ at x in the direction of v is defined by

$$V^o(x,v) \triangleq \limsup_{y\to x, h\to 0^+} \frac{V(y+hv)-V(y)}{h}. \tag{5}$$

The *Clarke generalized gradient* of $V(x)$ at x is the set

$$\partial V(x) \triangleq \mathrm{co}\left\{\lim_{i\to\infty} \nabla V(x_i) : x_i \to x,\, x_i \notin \mathcal{N}\cup\mathcal{S}\right\}, \tag{6}$$

where ∇ denotes nabla operator, $\mathcal{N}$ is a set of measure zero points where ∇V does not exist, and $\mathcal{S}$ is an arbitrary set of measure zero in $\mathbb{R}^q$.

In order to state the main result of this section, we need some new notation and definitions. Given a locally Lipschitz continuous function $V : \mathbb{R}^q \to \mathbb{R}$, the *set-valued derivative* of V with respect to (1) [9], [11] is defined as

$$\mathcal{L}_f V(x) \triangleq \{a \in \mathbb{R} : \text{there exists } v \in \mathcal{K}[f](x) \text{ such that } p\cdot v = a \text{ for all } p \in \partial V(x)\}. \tag{7}$$

If $V(x)$ is continuously differentiable at x, then $\mathcal{L}_f V(x) = \{\nabla V(x)\cdot v, v \in \mathcal{K}[f](x)\}$. We use $\max \mathcal{L}_f V(x)$ to denote the largest nonempty element of $\mathcal{L}_f V(x)$.

Recall that a function $V : \mathbb{R}^q \to \mathbb{R}$ is *regular* at $x \in \mathbb{R}^q$ [10] if for all $v \in \mathbb{R}^q$, there exists the usual right directional derivative $V'_+(x,v)$ and $V'_+(x,v) = V^o(x,v)$. V is called *regular* on $\mathbb{R}^q$ if it is regular at every $x \in \mathbb{R}^q$. In this section, we assume that $f(\cdot)$ is locally Lipschitz continuous and regular. The next definition introduces the notion of semistability for Filippov dynamical systems. For this definition, Lyapunov stability for the solution $x(t) \equiv z$ to (1) can be found in [5] and [9].

Definition 2.2 ([3]): Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set with respect to the differential inclusion (1). An equilibrium point $z \in \mathcal{D}$ of (1) is *semistable* with respect to $\mathcal{D}$ if it is Lyapunov stable and there exists an open subset $\mathcal{D}_0$ of $\mathcal{D}$ containing z such that for all initial conditions in $\mathcal{D}_0$, the Filippov solutions of (1) converge to a Lyapunov stable equilibrium point. The system (1) is *semistable* with respect to $\mathcal{D}$ if every equilibrium point in $f^{-1}(0)$ is semistable with respect to $\mathcal{D}$. Finally, (1) is said to be globally semistable if (1) is semistable and $\mathcal{D} = \mathbb{R}^q$.

Next, we introduce the definition of finite-time semistability of (1).

Definition 2.3 ([3]): Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set with respect to the differential inclusion (1). An equilibrium point $x_\mathrm{e} \in f^{-1}(0)$ of (1) is said to be *finite-time-semistable* if there exist an open neighborhood $\mathcal{U} \subseteq \mathcal{D}$ of x_e and a function $T : \mathcal{U}\backslash f^{-1}(0) \to (0,\infty)$, called the *settling-time function*, such that the following statements hold:

i) For every $x \in \mathcal{U}\backslash f^{-1}(0)$ and any Filippov solution $\psi(t)$ of (1) with $\psi(0) = x$, $\psi(t) \in \mathcal{U}\backslash f^{-1}(0)$ for all $t \in [0, T(x))$, and $\lim_{t\to T(x)} \psi(t)$ exists and is contained in $\mathcal{U} \cap f^{-1}(0)$.

ii) x_e is semistable.

An equilibrium point $x_\mathrm{e} \in f^{-1}(0)$ of (1) is said to be *globally finite-time-semistable* if it is finite-time-semistable with $\mathcal{D} = \mathcal{U} = \mathbb{R}^n$. The system (1) is said to be *finite-time-semistable* if every equilibrium point in $f^{-1}(0)$ is finite-time-semistable. Finally, (1) is said to be *globally finite-time-semistable* if every equilibrium point in $f^{-1}(0)$ is globally finite-time-semistable.

Given a curve $\gamma : [0,\infty) \to \mathbb{R}^q$, *the positive limit set of* γ is the set $\Omega(\gamma)$ of points $y \in \mathbb{R}^q$ for which there exists an increasing sequence $\{t_i\}_{i=1}^\infty$ satisfying $\lim_{i\to\infty} \gamma(t_i) = y$. Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set with respect to the differential inclusion (1). For $x \in \mathcal{D}$, let $\psi(\cdot)$ denote the Filippov solution to (1) with $\psi(0) = x$ and let $\Omega(\psi)$ be the positive limit set of ψ.

Theorem 2.1: Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set with respect to (1) and let $V : \mathcal{D} \to \mathbb{R}$ be locally Lipschitz continuous and regular. Assume that for each $x \in \mathcal{D}$ and each Filippov solution $\psi(\cdot)$, $\psi(t)$ is bounded for all $t \geq 0$, and $\psi(0) = x$. Furthermore, assume that $\max \mathcal{L}_f V(x) \leq 0$ or $\mathcal{L}_f V(x) = \emptyset$ for all $x \in \mathcal{D}$. Let $\mathcal{Z} \triangleq \{x \in \mathbb{R}^q : 0 \in \mathcal{L}_f V(x)\}$. If every point in the largest weakly invariant subset $\mathcal{M}$ of $\overline{\mathcal{Z}} \cap \mathcal{D}$ is a Lyapunov stable equilibrium point with respect to $\mathcal{D}$, where $\overline{\mathcal{Z}}$ denotes the closure of $\mathcal{Z}$, then (1) is semistable with respect to $\mathcal{D}$.

Corollary 2.1: Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set with respect to (1) and let $V : \mathcal{D} \to \mathbb{R}$ be locally Lipschitz continuous and regular. Assume that $\max \mathcal{L}_f V(x) < 0$ or $\mathcal{L}_f V(x) = \emptyset$ for all $x \in \mathcal{D}\backslash f^{-1}(0)$. If (1) is Lyapunov stable with respect to $\mathcal{D}$, then (1) is semistable with respect to $\mathcal{D}$. If, in addition, $\max \mathcal{L}_f V(x) \leq -\varepsilon < 0$ or $\mathcal{L}_f V(x) = \emptyset$ for all $x \in \mathcal{D}\backslash f^{-1}(0)$, then (1) is finite-time-semistable with respect to $\mathcal{D}$.

A set $\mathcal{E} \subseteq \mathbb{R}^q$ is *connected* if and only if every pair of open sets $\mathcal{U}_i \subseteq \mathbb{R}^q$, $i = 1,2$, satisfying $\mathcal{E} \subseteq \mathcal{U}_1 \cup \mathcal{U}_2$ and $\mathcal{U}_i \cap \mathcal{E} \neq \emptyset$, $i = 1,2$, has a nonempty intersection. A *connected component* of the set $\mathcal{E} \subseteq \mathbb{R}^q$ is a connected subset of $\mathcal{E}$ that is not properly contained in any connected subset of $\mathcal{E}$ [2]. Given a set $\mathcal{E} \subseteq \mathbb{R}^q$, let $\mathrm{co}\,\mathcal{E}$ denote the union of the convex hulls of the connected components of $\mathcal{E}$, and let $\mathrm{coco}\,\mathcal{E}$ denote the cone generated by $\mathrm{co}\,\mathcal{E}$. Given $x \in \mathbb{R}^q$, the *direction cone* $\mathcal{F}_x$ of f at x relative to $\mathbb{R}^q$ is the intersection of all sets of the form $\overline{\mathrm{coco}\,(f(\mathcal{U})\backslash\{0\})}$, where $\mathcal{U} \subseteq \mathbb{R}^q$ is an open neighborhood of x. Let $z \in \mathcal{E} \subseteq \mathbb{R}^q$. A vector $v \in \mathbb{R}^q$ is *tangent* to $\mathcal{E}$ at $z \in \mathcal{E}$ if and only if there exist a sequence $\{z_i\}_{i=1}^\infty$ in $\mathcal{E}$ converging to z and a sequence $\{h_i\}_{i=1}^\infty$ of positive real numbers converging to zero such that $\lim_{i\to\infty} \frac{1}{h_i}(z_i - z) = v$. The *tangent cone* to $\mathcal{E}$ at z is the closed cone $T_z\mathcal{E}$ of all vectors tangent to $\mathcal{E}$ at z. Finally, the vector field f is *nontangent* to the set $\mathcal{E}$ at the point $z \in \mathcal{E}$ if and only if $T_z\mathcal{E} \cap \mathcal{F}_z \subseteq \{0\}$ [2].

Definition 2.4: Given a point $x \in \mathbb{R}^q$ and a bounded open neighborhood $\mathcal{U} \subset \mathbb{R}^q$ of x, the *restricted prolongation under Filippov solutions* of x with respect to $\mathcal{U}$ is the set $\mathcal{R}_x^{\mathcal{U}} \subseteq \overline{\mathcal{U}}$ of all subsequential limits of sequences of the form $\{\psi_i(t_i)\}$, where $\{t_i\}_{i=1}^\infty$ is a sequence in $[0,\infty)$, $\psi_i(\cdot)$ is a Filippov solution to (1) with $\psi_i(0) = x_i$, $i = 1,2,\ldots$, and $\{x_i\}_{i=1}^\infty$ is a sequence in $\mathcal{U}$ converging to x such that the set $\{z \in \mathbb{R}^q : z = \psi_i(t), t \in [0,t_i], \psi_i(0) = x_i\}$ is contained in $\overline{\mathcal{U}}$ for every $i = 1,2,\ldots$.

For the next result, we say a set $\mathcal{N} \subset \mathbb{R}^q$ is *weakly negatively invariant* if for every $x \in \mathcal{N}$, there exist $z \in \mathcal{N}$ and a Filippov solution $\psi(\cdot)$ to (1) with $\psi(0) = z$ such that

$\psi(t)=x$ and $\psi(\tau)\in\mathcal{N}$ for all $\tau\in[0,t]$, where $t>0$.

Theorem 2.2: Let $\mathcal{D}\subseteq\mathbb{R}^q$ be a strongly invariant set with respect to (1) and let $V:\mathcal{D}\to\mathbb{R}$ be locally Lipschitz continuous and regular. Assume that $V(x)\geq 0$, $x\in\mathcal{D}$, $V(z)=0$ for $z\in f^{-1}(0)$, and $\max\mathcal{L}_fV(x)\leq 0$ or $\mathcal{L}_fV(x)=\emptyset$ for all $x\in\mathcal{D}$. For every $z\in f^{-1}(0)$, let $\mathcal{N}_z$ denote the largest weakly negatively invariant connected subset of $\overline{\mathcal{Z}}\cap\mathcal{D}$ containing z, where $\mathcal{Z}=\{x\in\mathbb{R}^q:0\in\mathcal{L}_fV(x)\}$. If f is nontangent to $\mathcal{N}_z$ at the point $z\in f^{-1}(0)$, then (1) is semistable with respect to $\mathcal{D}$.

Example 2.1: Consider the nonsmooth dynamical system given by

$$\dot{x}_1(t)=\mathrm{sign}(x_2(t)-x_1(t)),\ x_1(0)=x_{10},\ t\geq 0, \quad (8)$$
$$\dot{x}_2(t)=\mathrm{sign}(x_1(t)-x_2(t)),\ x_2(0)=x_{20}, \quad (9)$$

where $x_1,x_2\in\mathbb{R}$, $\mathrm{sign}(x)\triangleq x/|x|$ for $x\neq 0$, and $\mathrm{sign}(0)\triangleq 0$. Let $f(x_1,x_2)\triangleq[\mathrm{sign}(x_2-x_1),\mathrm{sign}(x_1-x_2)]^{\mathrm{T}}$. Consider $V(x_1,x_2)=\frac{1}{2}(x_1-\alpha)^2+\frac{1}{2}(x_2-\alpha)^2$, where $\alpha\in\mathbb{R}$. Since $V(x_1,x_2)$ is differentiable at (x_1,x_2), it follows that $\mathcal{L}_fV(x_1,x_2)=[x_1-\alpha,x_2-\alpha]\mathcal{K}[f]$. Now, it follows from Theorem 1 of [7] that

$$\begin{aligned}[x_1-\alpha,x_2-\alpha]\mathcal{K}[f] &= \mathcal{K}[[x_1-\alpha,x_2-\alpha]f]\\ &= \mathcal{K}[-(x_1-x_2)\mathrm{sign}(x_1-x_2)]\\ &= -(x_1-x_2)\mathcal{K}[\mathrm{sign}(x_1-x_2)]\\ &= -(x_1-x_2)\mathrm{SGN}(x_1-x_2)\\ &= -|x_1-x_2|,\quad (x_1,x_2)\in\mathbb{R}^2,\end{aligned}$$

where $\mathrm{SGN}(\cdot)$ is defined by [7], [12]

$$\mathrm{SGN}(x)\triangleq\begin{cases}-1, & x<0,\\ [-1,1], & x=0,\\ 1, & x>0.\end{cases} \quad (10)$$

Hence, $\max\mathcal{L}_fV(x_1,x_2)\leq 0$ for all $(x_1,x_2)\in\mathbb{R}^2$. Now, it follows from Theorem 2 of [9] that $x_1=x_2=\alpha$ is Lyapunov stable. Finally, note that $0\in\mathcal{L}_fV(x_1,x_2)$ if and only if $x_1=x_2$, and hence, $\mathcal{Z}=\{(x_1,x_2)\in\mathbb{R}^2:x_1=x_2\}$. Since the largest weakly invariant subset $\mathcal{M}$ of $\mathcal{Z}$ is given by $\mathcal{M}=\{(x_1,x_2)\in\mathbb{R}^2:x_1=x_2=\alpha,\alpha\in\mathbb{R}\}$, it follows from Theorem 2.1 that (8) and (9) is semistable.

Finally, we show that (8) and (9) is finite-time-semistable. To see this, consider the nonnegative function $U(x_1,x_2)=|x_1-x_2|$. Note that

$$\partial U(x_1,x_2)=\begin{cases}\{\mathrm{sign}(x_1-x_2)\}\\ \times\{\mathrm{sign}(x_2-x_1)\}, & x_1\neq x_2,\\ [-1,1]\times[-1,1], & x_1=x_2.\end{cases}$$

Hence, it follows that

$$\mathcal{L}_fU(x_1,x_2)=\begin{cases}\{-2\}, & x_1\neq x_2,\\ \{0\}, & x_1=x_2,\end{cases} \quad (11)$$

which implies that $\max\mathcal{L}_fU(x_1,x_2)=-2<0$ for all $(x_1,x_2)\in\mathbb{R}^2\backslash\mathcal{Z}$. Now, it follows from Corollary 2.1 that (8) and (9) is globally finite-time-semistable. Figure 1 shows the solutions of (8) and (9) for $x_{10}=4$ and $x_{20}=-2$. $\triangle$

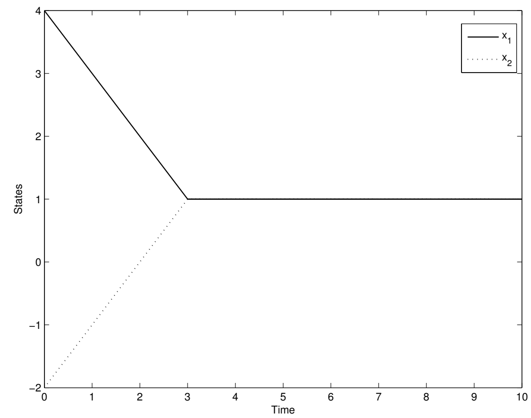

Fig. 1. Solutions for Example 2.1

Example 2.2: Consider the nonsmooth dynamical system given by

$$\dot{x}_1(t)=\mathrm{sign}(x_3(t)-x_4(t)),\ x_1(0)=x_{10},\ t\geq 0, \quad (12)$$
$$\dot{x}_2(t)=\mathrm{sign}(x_4(t)-x_3(t)),\ x_2(0)=x_{20}, \quad (13)$$
$$\dot{x}_3(t)=\mathrm{sign}(x_4(t)-x_3(t))+\mathrm{sign}(x_2(t)-x_1(t)),\quad x_3(0)=x_{30}, \quad (14)$$
$$\dot{x}_4(t)=\mathrm{sign}(x_3(t)-x_4(t))+\mathrm{sign}(x_1(t)-x_2(t)),\quad x_4(0)=x_{40}, \quad (15)$$

where $x_1,x_2,x_3,x_4\in\mathbb{R}$. Let $f:\mathbb{R}^4\to\mathbb{R}^4$ denote the vector field of (12)–(15) and $x\triangleq[x_1,x_2,x_3,x_4]\in\mathbb{R}^4$. Consider the function $V(x)=|x_1-x_2|+|x_3-x_4|$. Note that

$$\partial V(x)=\begin{cases}\{\mathrm{sign}(x_1-x_2)\}\times\{\mathrm{sign}(x_2-x_1)\}\\ \times\{\mathrm{sign}(x_3-x_4)\}\\ \times\{\mathrm{sign}(x_4-x_3)\}, & x_1\neq x_2,\ x_3\neq x_4,\\ [-1,1]\times[-1,1]\times\{\mathrm{sign}(x_3-x_4)\}\\ \times\{\mathrm{sign}(x_4-x_3)\}, & x_1=x_2,\ x_3\neq x_4,\\ \{\mathrm{sign}(x_1-x_2)\}\times\{\mathrm{sign}(x_2-x_1)\}\\ \times[-1,1]\times[-1,1], & x_1\neq x_2,\ x_3=x_4,\\ \overline{\mathrm{co}}\{(1,1),(-1,1),\\ (-1,-1),(1,-1)\}, & x_1=x_2,\ x_3=x_4.\end{cases}$$

Hence,

$$\mathcal{L}_fV(x)=\begin{cases}\{-2\}, & x_1\neq x_2,\ x_3\neq x_4,\\ \emptyset, & x_1=x_2,\ x_3\neq x_4,\\ \emptyset, & x_1\neq x_2,\ x_3=x_4,\\ \{0\}, & x_1=x_2,\ x_3=x_4,\end{cases} \quad (16)$$

which implies that $\max\mathcal{L}_fV(x)\leq 0$ for $x\in\mathbb{R}^4$ and $\mathcal{Z}=\{x\in\mathbb{R}^4:x_1=x_2,x_3=x_4\}$. Let $\mathcal{N}$ denote the largest weakly, negatively invariant subset contained in $\mathcal{Z}$. On $\mathcal{N}$, it follows from (12)–(15) that $\dot{x}_1=\dot{x}_2=0$ and $\dot{x}_3=\dot{x}_4=0$. Hence, $\mathcal{N}=\{x\in\mathbb{R}^4:x_1=x_2=a,x_3=x_4=b\}$, $a,b\in\mathbb{R}$, which implies that $\mathcal{N}$ is the set of equilibrium points.

Next, we show that f for (12)–(15) is nontangent to $\mathcal{N}$ at the point $z\in\mathcal{N}$. To see this, note that the tangent cone $T_z\mathcal{N}$ to the equilibrium set $\mathcal{N}$ is orthogonal to the vectors $\mathbf{u}_1\triangleq[1,-1,0,0]^{\mathrm{T}}$ and $\mathbf{u}_2\triangleq[0,0,1,-1]^{\mathrm{T}}$. On the other hand, since $f(z)\in\mathrm{span}\{\mathbf{u}_1,\mathbf{u}_2\}$ for all $z\in\mathbb{R}^4$, it follows that the direction cone $\mathcal{F}$ of f at $z\in\mathcal{N}$ relative to $\mathbb{R}^4$ satisfies $\mathcal{F}_z\subseteq\mathrm{span}\{\mathbf{u}_1,\mathbf{u}_2\}$. Hence, $T_z\mathcal{N}\cap\mathcal{F}_z=\{0\}$, which implies that the vector field f is nontangent to the set of equilibria $\mathcal{N}$ at the point $z\in\mathcal{N}$. Note that for every $z\in\mathcal{N}$,

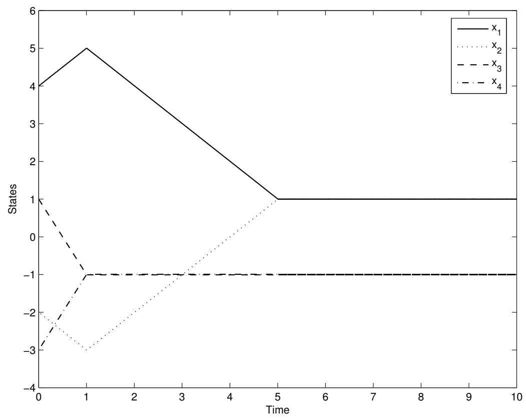

Fig. 2. Solutions for Example 2.2

the set $\mathcal{N}_z$ required by Theorem 2.2 is contained in $\mathcal{N}$. Since nontangency to $\mathcal{N}$ implies nontangency to $\mathcal{N}_z$ at the point $z \in \mathcal{N}$, it follows from Theorem 2.2 that the system (12)–(15) is semistable.

Finally, note that $\max \mathcal{L}_f V(x) \leq -2 < 0$ or $\mathcal{L}_f V(x) = \emptyset$ for all $x \in \mathbb{R}^4 \backslash \mathcal{Z}$, it follows from Corollary 2.1 that (12)–(15) is globally finite-time-semistable. Figure 2 shows the solutions of (12)–(15) for $x_{10} = 4$, $x_{20} = -2$, $x_{30} = 1$, and $x_{40} = -3$. △

III. Consensus Problems in Dynamical Networks

In this section, we develop a thermodynamically motivated information consensus framework for multiagent nonlinear systems that achieve semistability and state equipartition. Specifically, consider q continuous-time integrator agents with dynamics

$$\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad t \geq 0, \tag{17}$$

where for each $i \in \{1, \ldots, q\}$, $x_i(t) \in \mathbb{R}$ denotes the information state and $u_i(t) \in \mathbb{R}$ denotes the information control input for all $t \geq 0$. The general consensus protocol is given by

$$u_i(t) = \sum_{j=1, j\neq i}^{q} \phi_{ij}(x_i(t), x_j(t)), \tag{18}$$

where $\phi_{ij}(\cdot,\cdot)$, $i, j = 1, \ldots, q$, are locally Lipschitz continuous. Note that (17) and (18) describe an interconnected network with a graph topology $\mathfrak{G}$ where information states are updated using a distributed nonlinear controller involving neighbor-to-neighbor interaction between agents. The following assumptions are needed for the main results of the paper.

Assumption 1: For the *connectivity matrix* $\mathcal{C} \in \mathbb{R}^{q\times q}$ associated with the multiagent dynamical system (17) and (18) defined by

$$\mathcal{C}_{(i,j)} \triangleq \begin{cases} 0, & \text{if } \phi_{ij}(x_i, x_j) \equiv 0, \\ 1, & \text{otherwise,} \end{cases} \tag{19}$$

for $i \neq j$, $i, j = 1, \ldots, q$, and $\mathcal{C}_{(i,i)} \triangleq -\sum_{k=1,\, k\neq i}^{q} \mathcal{C}_{(i,k)}$, $i = 1, \ldots, q$, $\operatorname{rank} \mathcal{C} = q - 1$, and for $\mathcal{C}_{(i,j)} = 1$, $i \neq j$, $\phi_{ij}(x_i, x_j) = 0$ if and only if $x_i = x_j$.

Assumption 2: For $i, j = 1, \ldots, q$, $(x_i - x_j)\phi_{ij}(x_i, x_j) \leq 0$, $x_i, x_j \in \mathbb{R}$.

For further details on Assumptions 1 and 2, see [4].

Proposition 3.1 ([3]): Consider the multiagent dynamical system (17) and (18) and assume that Assumptions 1 and 2 hold. Then $f_i(x) = 0$ for all $i = 1, \ldots, q$ if and only if $x_1 = \cdots = x_q$. Furthermore, $\alpha\mathbf{e}$, $\alpha \in \mathbb{R}$, is an equilibrium state of (17) and (18).

IV. Network Consensus with Switching Topology and Finite-Time Semistability

Communication links among multiagent systems are often unreliable due to multipath effects and exogenous disturbances leading to dynamic information exchange topologies. In this section, we develop a switched consensus protocol to achieve agreement over a network with switching topology. First, we design a switching nonlinear consensus protocol for (17). Specifically, consider q mobile agents with the dynamics $\mathcal{G}_i$ given by (17). Furthermore, consider the switched controller $\mathcal{G}_{\mathrm{s}i}$ given by

$$u_i(t) = \sum_{j=1, j\neq i}^{q} \phi_{ij}^{\sigma(t)}(x_i(t), x_j(t)), \tag{20}$$

where $\sigma : [0, \infty) \to \mathcal{S}$ is a piecewise constant switching signal, $\mathcal{S}$ is a finite index set, and $\phi_{ij}^{\sigma} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and satisfies Assumptions 1 and 2 for every $\sigma \in \mathcal{S}$. Furthermore, we assume that $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$ in Assumption 1, where $\mathcal{C} = \mathcal{C}(t)$, $t \geq 0$.

Theorem 4.1: Consider the closed-loop system $\tilde{\mathcal{G}}$ given by the multiagent dynamical system (17) and the switched controller (20). Assume that Assumptions 1 and 2 hold for every $\sigma \in \mathcal{S}$. Furthermore, assume that $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$, where $\mathcal{C} = \mathcal{C}(t)$ in Assumption 1 and $t \geq 0$. Then for every $\alpha \in \mathbb{R}$, $x_1 = \cdots = x_q = \alpha$ is a semistable state of $\tilde{\mathcal{G}}$. Furthermore, $x_i(t) \to \frac{1}{q}\sum_{i=1}^{q} x_{i0}$ and $\frac{1}{q}\sum_{i=1}^{q} x_{i0}$ is a semistable equilibrium state.

Proof. Consider the Lyapunov function candidate

$$V(x) = \frac{1}{2}(x - \alpha\mathbf{e})^{\mathrm{T}}(x - \alpha\mathbf{e}), \tag{21}$$

where $x \triangleq [x_1, \ldots, x_q]^{\mathrm{T}} \in \mathbb{R}^q$ and $\alpha \in \mathbb{R}$. Then the Lyapunov derivative along the trajectories of the closed-loop system (17) and (20) is given by

$$\begin{aligned} \dot{V}(x) &= (x - \alpha\mathbf{e})^{\mathrm{T}}\dot{x} = \sum_{i=1}^{q} x_i \left[\sum_{j=1, j\neq i}^{q} \phi_{ij}^{\sigma}(x_i, x_j) \right] \\ &= \sum_{i=1}^{q} \sum_{j=i+1}^{q-1} (x_i - x_j)\phi_{ij}^{\sigma}(x_i, x_j) \leq 0, \quad x \in \mathbb{R}^q, \end{aligned} \tag{22}$$

which establishes Lyapunov stability of $x \equiv \alpha\mathbf{e}$.

Next, we rewrite the closed-loop system (17) and (20) as the differential inclusion (2). For any $v \in \mathcal{K}[f]$, let $V^o(x, v) \triangleq x^{\mathrm{T}}v$ and $\max V^o(x, v) \triangleq \max_{v\in\mathcal{K}[f]}\{x^{\mathrm{T}}v\}$. Now, it follows from Theorem 1 of [7] and (22) that

$$\begin{aligned} x^{\mathrm{T}}\mathcal{K}[f] &= \mathcal{K}[x^{\mathrm{T}}f] \\ &= \mathcal{K}\left[\sum_{i=1}^{q} \sum_{j=i+1}^{q-1} (x_i - x_j)\phi_{ij}^{\sigma}(x_i, x_j) \right], \end{aligned} \tag{23}$$

and hence, by definition of a differential inclusion, it follows that $\max V^o(x, v) = \max \overline{\mathrm{co}}\{\sum_{i=1}^{q}\sum_{j=i+1}^{q-1}(x_i -$

$x_j)\phi_{ij}^{\sigma}(x_i, x_j)\}$. Note that since, by (22), $\sum_{i=1}^{q}\sum_{j=i+1}^{q-1}(x_i - x_j)\phi_{ij}^{\sigma}(x_i, x_j) \le 0$, $x_i \in \mathbb{R}$, it follows that $\max V^o(x, v)$ cannot be positive, and hence, the largest value $\max V^o(x, v)$ can achieve is zero.

Finally, note that $0 \in \mathcal{L}_f V(x)$ if and only if $\sum_{i=1}^{q}\sum_{j=i+1}^{q-1}(x_i - x_j)\phi_{ij}^{\sigma}(x_i, x_j) = 0$, and hence, $\mathcal{Z} \triangleq \{x \in \mathbb{R}^q : \sum_{i=1}^{q}\sum_{j=i+1}^{q-1}(x_i - x_j)\phi_{ij}^{\sigma}(x_i, x_j) = 0\}$. Now, it follows from Proposition 3.1 that $\mathcal{Z} = \{x \in \mathbb{R}^q : x_1 = \cdots = x_q\}$. Since $\mathcal{Z}$ consists of equilibrium points, it follows that $\mathcal{M} = \mathcal{Z}$. Hence, it follows from Theorem 2.1 that $x = \alpha\mathbf{e}$ is semistable for all $\alpha \in \mathbb{R}$. □

Next, we extend Theorem 4.1 to the nonsmooth controllers $\mathcal{G}_{\mathrm{n}i}$ of the form

$$u_i = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} \mathrm{sign}(x_j - x_i). \tag{24}$$

It is important to note that the consensus protocol (24) is a logic-based, distributed decision-making protocol. Although a similar consensus protocol based on nonsmooth gradient flows is proposed in [13], the key difference between (24) and the one in [13] is that (24) is a *distributed* protocol while the consensus protocol in [13] is a *centralized* protocol.

In [3], the authors prove that the consensus protocol given by the form

$$u_i = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} \mathrm{sign}(x_j - x_i)|x_j - x_i|^{\alpha} \tag{25}$$

is a finite-time consensus protocol for $0 < \alpha < 1$. Next, we show that (25) is also a finite-time consensus protocol for $\alpha = 0$. Note that in this case, (25) reduces to (24).

Theorem 4.2: Consider the closed-loop system $\tilde{\mathcal{G}}$ given by the multiagent dynamical system (17) and the nonsmooth controller (24). Assume that Assumptions 1 and 2 hold. Furthermore, assume that $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$ in Assumption 1. Then for every $\alpha \in \mathbb{R}$, $x_1 = \cdots = x_q = \alpha$ is a finite-time-semistable state of $\tilde{\mathcal{G}}$. Furthermore, $x_i(t) = \frac{1}{q}\sum_{i=1}^{q} x_{i0}$ for $t \ge T(x_{10}, \ldots, x_{q0})$ and $\frac{1}{q}\sum_{i=1}^{q} x_{i0}$ is a semistable equilibrium state.

Proof. Consider the Lyapunov function candidate (21). Since $V(x)$ is differentiable at x, it follows that $\mathcal{L}_f V(x) = (x - \alpha\mathbf{e})^{\mathrm{T}}\mathcal{K}[f]$. Now, it follows from Theorem 1 of [7] that

$$\begin{aligned}(x - \alpha\mathbf{e})^{\mathrm{T}}\mathcal{K}[f] &= \mathcal{K}[(x - \alpha\mathbf{e})^{\mathrm{T}} f] \\ &= \mathcal{K}\Big[-\sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}(x_i - x_j)\mathrm{sign}(x_i - x_j)\Big], \\ &\subseteq -\sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}(x_i - x_j)\mathcal{K}[\mathrm{sign}(x_i - x_j)] \\ &= -\sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}(x_i - x_j)\mathrm{SGN}(x_i - x_j) \\ &= -\sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}|x_i - x_j|, \quad x \in \mathbb{R}^q,\end{aligned}$$

which implies that $\max \mathcal{L}_f V(x) \le 0$ for all $x \in \mathbb{R}^q$. Hence, it follows from Theorem 2 of [9] that $x_1 = \cdots = x_q = \alpha$ is Lyapunov stable. Finally, note that since

$$\mathcal{L}_f V(x) = \mathcal{K}\left[-\sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}|x_i - x_j|\right], \tag{26}$$

it follows that $0 \in \mathcal{L}_f V(x)$ if and only if $x_1 = \cdots = x_q$, and hence, $\mathcal{Z} = \{x \in \mathbb{R}^q : x_1 = \cdots = x_q\}$. Since the largest weakly invariant subset $\mathcal{M}$ of $\mathcal{Z}$ is given by $\mathcal{M} = \{x \in \mathbb{R}^q : x_1 = \cdots = x_q = \alpha, \alpha \in \mathbb{R}\}$, it follows from Theorem 2.1 that (17) and (24) is semistable.

Finally, we show that (17) and (24) is finite-time-semistable. To see this, consider the nonnegative function $U(x) = \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}|x_i - x_j|$. In this case, it follows, using similar arguments as in Example 2.1, that

$$\mathcal{L}_f U(x) = \begin{cases} \left\{-2\sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}\right\}, & \\ \qquad x_i \neq x_j,\ i, j = 1, \ldots, q,\ i \neq j, & \\ \emptyset, \quad x_k = x_l \text{ for some } k, l \in \{1, \ldots, q\}, & \\ \qquad k \neq l, & \\ \{0\}, \quad x_1 = \cdots = x_q, & \end{cases}$$

which implies that $\max \mathcal{L}_f U(x) \le -2\sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} < 0$ or $\mathcal{L}_f U(x) = \emptyset$ for all $x \in \mathbb{R}^q \backslash \mathcal{Z}$, and hence, it follows from Corollary 2.1 that (8) and (9) is globally finite-time-semistable. □

Finally, we design a nonsmooth *dynamic* consensus protocol for (17). In contrast to the static controllers addressed in [14] and [15], the proposed controller is a dynamic compensator. This controller architecture allows us to design finite-time consensus protocols via quantized feedback in a dynamical network. Specifically, consider q mobile agents with the dynamics $\mathcal{G}_i$ given by (17). Furthermore, consider the nonsmooth dynamic compensators $\mathcal{G}_{\mathrm{c}i}$ given by

$$\begin{aligned}\dot{x}_{\mathrm{c}i}(t) &= \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}\mathrm{sign}(x_{\mathrm{c}j}(t) - x_{\mathrm{c}i}(t)) \\ &\quad + \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}\mathrm{sign}(x_i(t) - x_j(t)), \quad (27) \\ u_i(t) &= \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(j,i)}\mathrm{sign}(x_{\mathrm{c}j}(t) - x_{\mathrm{c}i}(t)), \quad (28)\end{aligned}$$

where $x_{\mathrm{c}i}(t) \in \mathbb{R}$, $t \ge 0$, and $x_{\mathrm{c}i}(0) = x_{\mathrm{c}i0}$. Here, we assume that Assumption 1 holds and $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$.

Theorem 4.3: Consider the closed-loop system $\tilde{\mathcal{G}}$ given by the multiagent dynamical system (17) and the nonsmooth dynamic controller (27) and (28). Assume that Assumption 1 holds and $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$. Then for every $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, $x_1 = \cdots = x_q = \alpha$ and $x_{\mathrm{c}1} = \cdots = x_{\mathrm{c}q} = \beta$ is a finite-time-semistable state of $\tilde{\mathcal{G}}$. Furthermore, $x_i(t) = \frac{1}{q}\sum_{i=1}^{q} x_{i0}$ and $x_{\mathrm{c}i}(t) = \frac{1}{q}\sum_{i=1}^{q} x_{\mathrm{c}i0}$ for all $t \ge T(x_{10}, \ldots, x_{q0}, x_{\mathrm{c}10}, \ldots, x_{\mathrm{c}q0})$ and $(\frac{1}{q}\sum_{i=1}^{q} x_{i0}, \frac{1}{q}\sum_{i=1}^{q} x_{\mathrm{c}i0})$ is a semistable equilibrium state.

Proof. Note that for every $a, b \in \mathbb{R}$, $x(t) \equiv a\mathbf{e}$ and $x_{\mathrm{c}}(t) \equiv b\mathbf{e}$ are the equilibrium points for the closed-loop system. Consider the nonnegative function given by

$$V(\tilde{x}) = \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}|x_i - x_j| + \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}|x_{ci} - x_{cj}|, \quad (29)$$

where $\tilde{x} \triangleq [x^{\mathrm{T}}, x_{\mathrm{c}}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{2q}$. In this case, it follows using similar arguments as in Example 2.2 that

$$\mathcal{L}_f V(\tilde{x}) = \begin{cases} \left\{-2\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}\right\}, & \\ \quad x_i \neq x_j,\, x_{ci} \neq x_{cj},\, i,j = 1,\ldots,q,\, i \neq j, & \\ \emptyset, \quad x_k = x_l \text{ or } x_{ck} = x_{cl} & \\ \quad \text{for some}\, k,l \in \{1,\ldots,q\},\, k \neq l, & \\ \{0\}, \quad x_1 = \cdots = x_q,\, x_{c1} = \cdots = x_{cq}, & \end{cases}$$

which implies that $\max \mathcal{L}_f V(\tilde{x}) \leq 0$ or $\mathcal{L}_f V(\tilde{x}) = \emptyset$ for all $\tilde{x} \in \mathbb{R}^{2q}$. Next, define $\mathcal{Z} \triangleq \{\tilde{x} \in \mathbb{R}^{2q} : x_1 = \cdots = x_q, x_{c1} = \cdots = x_{cq}\}$ and let $\mathcal{N}$ denote the largest negatively invariant set of $\mathcal{Z}$. On $\mathcal{N}$, it follows from (17), (27), and (28) that $\dot{x}_i = 0$ and $\dot{x}_{ci} = 0$, $i = 1,\ldots,q$. Hence, $\mathcal{N} = \{\tilde{x} \in \mathbb{R}^{2q} : x = a\mathbf{e}, x_c = b\mathbf{e}\}$, $a, b \in \mathbb{R}$, which implies that $\mathcal{N}$ is the set of equilibrium points.

Since the connectivity matrix $\mathcal{C}$ of the closed-loop system is irreducible, assume, without loss of generality, that $\mathcal{C}_{(i,i+1)} = \mathcal{C}_{(q,1)} = 1$, where $i = 1,\ldots,q-1$. Now, for $q = 2$, it was shown in Example 2.2 that the vector field f of the closed-loop system given by (17), (27), and (28) is nontangent to $\mathcal{N}$ at a point $\tilde{x} \in \mathcal{N}$. Next, we show that for $q \geq 3$, the vector field f of the closed-loop system given by (17), (27), and (28) is nontangent to $\mathcal{N}$ at a point $\tilde{x} \in \mathcal{N}$. To see this, note that the tangent cone $T_{\tilde{x}}\mathcal{N}$ to the equilibrium set $\mathcal{N}$ is orthogonal to the $2q$ vectors $\mathbf{u}_i \triangleq [0_{1\times(i-1)}, \mathcal{C}_{(i,i+1)}, -\mathcal{C}_{(i,i+1)}, 0_{1\times(2q-i-1)}]^{\mathrm{T}} \in \mathbb{R}^{2q}$, $\mathbf{u}_q \triangleq [-\mathcal{C}_{(q,1)}, 0_{1\times(q-2)}, \mathcal{C}_{(q,1)}, 0_{1\times q}]^{\mathrm{T}} \in \mathbb{R}^{2q}$, $\mathbf{v}_i \triangleq [0_{1\times(q+i-1)}, -\mathcal{C}_{(i,i+1)}, \mathcal{C}_{(i,i+1)}, 0_{1\times(q-i-1)}]^{\mathrm{T}} \in \mathbb{R}^{2q}$, and $\mathbf{v}_q \triangleq [0_{1\times q}, \mathcal{C}_{(q,1)}, 0_{1\times(q-2)}, -\mathcal{C}_{(q,1)}]^{\mathrm{T}} \in \mathbb{R}^{2q}$, $i = 1,\ldots,q-1$, $q \geq 3$. On the other hand, since $f(\tilde{x}) \in \mathrm{span}\{\mathbf{u}_1,\ldots,\mathbf{u}_q,\mathbf{v}_1,\ldots,\mathbf{v}_q\}$ for all $\tilde{x} \in \mathbb{R}^{2q}$, it follows that the direction cone $\mathcal{F}_{\tilde{x}}$ of f at $\tilde{x} \in \mathcal{N}$ relative to $\mathbb{R}^{2q}$ satisfies $\mathcal{F}_{\tilde{x}} \subseteq \mathrm{span}\{\mathbf{u}_1,\ldots,\mathbf{u}_q,\mathbf{v}_1,\ldots,\mathbf{v}_q\}$. Hence, $T_{\tilde{x}}\mathcal{N} \cap \mathcal{F}_{\tilde{x}} = \{0\}$, which implies that the vector field f is nontangent to the set of equilibria $\mathcal{N}$ at the point $\tilde{x} \in \mathcal{N}$. Note that for every $z \in \mathcal{N}$, the set $\mathcal{N}_z$ required by Theorem 2.2 is contained in $\mathcal{N}$. Since nontangency to $\mathcal{N}$ implies nontangency to $\mathcal{N}_z$ at the point $z \in \mathcal{N}$, it follows from Theorem 2.2 that the closed-loop system (17), (27), and (28) is semistable.

Finally, note that $\max \mathcal{L}_f V(x) \leq -2\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)} < 0$ or $\mathcal{L}_f V(\tilde{x}) = \emptyset$ for all $x \in \mathbb{R}^4\backslash\mathcal{Z}$, it follows from Corollary 2.1 that (17), (27), and (28) is globally finite-time-semistable. □

The dynamic compensator (27) and (28) is a state feedback controller. A natural question regarding (17) is how can one design finite-time consensus protocols for multiagent coordination via output feedback. To address this question, we consider q continuous-time integrator agents given by (17) with the output y_i given by

$$y_i = \sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}(x_j - x_i), \quad i = 1,\ldots,q. \quad (30)$$

Next, consider the dynamic output feedback compensator given by

$$\dot{x}_{ci}(t) = \sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}\mathrm{sign}(x_{cj}(t) - x_{ci}(t)) + y_i(t), \quad (31)$$

$$u_i(t) = \sum_{j=1,j\neq i}^{q}\mathcal{C}_{(j,i)}\mathrm{sign}(x_{cj}(t) - x_{ci}(t)), \quad (32)$$

where $x_{ci}(t) \in \mathbb{R}$, $t \geq 0$, and $x_{ci}(0) = x_{ci0}$. Here, once again, we assume that Assumption 1 holds and $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$.

Theorem 4.4: Consider the closed-loop system $\tilde{\mathcal{G}}$ given by the multiagent dynamical system (17) and the nonsmooth dynamic controller (31) and (32) with (30). Assume that Assumption 1 holds and $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$. Then for every $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, $x_1 = \cdots = x_q = \alpha$ and $x_{c1} = \cdots = x_{cq} = \beta$ is a finite-time-semistable state of $\tilde{\mathcal{G}}$. Furthermore, $x_i(t) = \frac{1}{q}\sum_{i=1}^{q}x_{i0}$ and $x_{ci}(t) = \frac{1}{q}\sum_{i=1}^{q}x_{ci0}$ for all $t \geq T(x_{10},\ldots,x_{q0},x_{c10},\ldots,x_{cq0})$ and $(\frac{1}{q}\sum_{i=1}^{q}x_{i0}, \frac{1}{q}\sum_{i=1}^{q}x_{ci0})$ is a semistable equilibrium state.

Proof. The proof is similar to the proof of Theorem 4.3. □

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