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On the relation between risk sensitive control and indifference pricing

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Abstract— **In this paper the connection between the indifference price and risk sensitive control is explored for stochastic volatility models. It is proved that the indifference price of a European option can be written as the difference of the value functions of two different stochastic optimal control problems. The quasilinear PDEs involved in the solution of this problem are written and under suitable conditions a verification theorem is given.**

I. The model

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where a standard two dimensional Brownian motion (W^1, W^2) is defined. Throughout the standard augmented filtration $\{\mathcal{F}_t; \ ; t \geq 0\}$ generated by this Brownian motion is fixed. The securities market we will consider consists of a riskless bond, which can be used to lending or borrowing, paying a zero interest rate, as well as a risky asset, with dynamics

$$dS_t = S_t[\mu(Y_t)dt + \sigma(Y_t)dW_t^1].$$

The stochastic process Y_t appearing in the coefficients of the above equation represents the stochastic volatility in the market, and it is assumed that it satisfies the following SDE:

$$dY_t = g(Y_t)dt + \beta(Y_t)[\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2],$$

with initial condition $Y_0 = y$. The number $\rho \in (-1, 1)$ represents the correlation between the noises driving the risky asset and the volatility. Specific assumptions about the functions μ, σ, β and g will be given below. The assumption that the interest rate paid by the bond in zero can be dispensed by discounting in the appropriate way.

The previous market model is incomplete and hence, given a European option $h(S_T)$ with expiration date $T > 0$, an arbitrage pricing method analogous to the Black-Scholes is not available any more. In order to determine an arbitrage free price for derivatives in incomplete markets different approaches have been proposed in recent years. As it was explained in the Introduction, in this paper we are interested in the indifference price, which takes into account the risk preferences of the investor, who is willing to buy the option h.

Following the original idea of Hodges and Neuberger in [15] based in utility theory, the *indifference price* of option $h(\cdot)$ at time $t < T$ is introduced next. Consider an investor with initial capital $x > 0$ at time t and risk preferences defined by the utility function $U : \mathbb{R} \to \mathbb{R}$, a concave nondecreasing function. Let α_t be an $\mathcal{F}_t$-adapted process representing the amount of money invested in the risky asset at time t such that

$$\int_0^T \pi_t dt < \infty \quad \text{a.s.};$$

the class of such processes satisfying the following additional integrability condition is denoted by $\mathcal{A}$:

$$\sup_{t\in[0,T]} \mathbb{E}\exp\{\varepsilon|\alpha_t|\} < \infty, \qquad (1)$$

for some ε that may depend on the strategy α_t. Then, the dynamics for the wealth process are given by

$$dX_t = \alpha_t(\mu(Y_t)dt + \sigma(Y_t)dW_t^1), \quad X_0 = x.$$

Throughout the following assumptions on the coefficients of the model will be assumed.

Assumption A

- The functions $\mu(\cdot)$, $\sigma(\cdot)$ are bounded and of class $C_b^1(\mathbb{R})$, where $C_b^1(\mathbb{R})$ is the space of functions $C^1(\mathbb{R})$ which are bounded together with their first derivative.
- The functions $g(\cdot)$ and $\beta(\cdot)$ belong to $C^1(\mathbb{R})$, and are Lipschitz continuous.
- There exists a constant c such that $\sigma(\cdot),\ \beta(\cdot) \geq c > 0$
- The function $h(\cdot)$ is nonnegative, continuous and bounded.

Now consider the following two optimal investment problems, with value functions

$$\begin{aligned} W(t,x,y) &= \max_{\alpha\in\mathcal{A}} \mathbb{E}[U(X_T) \mid X_t = x,\ Y_t = y] \quad (2)\\ V(t,x,y,s) &= \max_{\alpha\in\mathcal{A}} \mathbb{E}[U(X_T + h(S_T)) \mid \\ &\qquad X_t = x, Y_t = y, S_t = s]. \quad (3) \end{aligned}$$

In the first case, it corresponds to the classical maximum expected utility problem studied originally by Merton, while in the second case the investor will receive at time T the value of his portfolio and the value of the option $h(S_T)$. So, at time $t \in (0, T]$ the investor has two alternatives, one consists in investing his money in the market and the second is to buy the option for price p, and to invest in the market the rest of his money; at time T he receives his capital and the value of the option at the exercise time T.

Definition 1.1: We say that p is the *buyer's indifference price* at time t if

$$W(t,x,y) = V(t,x-p,y,s). \qquad (4)$$

When the utility function is of exponential type the expression for p shall be given in terms of the value functions of two different risk sensitive optimal control problems.

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These kind of problems have been studied in an independent way for many authors; the interested reader is referred in particular to [5], [21], [9], [8] and references therein.

In order to derive the expression for p let us calculate the expectation $\mathbb{E}[e^{-\gamma(X_T+h(S_T))} \mid X_t = x, Y_t = y, S_t = s]$ for a given admissible strategy α, which is given by

$$\begin{aligned} & e^{-\gamma x}\,\mathbb{E}\exp\{-\gamma \int_t^T \alpha_u \mu(Y_u)du - \\ & \qquad \gamma \int_t^T \alpha_u \sigma(Y_u) dW_u^1 - \gamma h(S_T)\} \\ & = e^{-\gamma x}\mathbb{E}_Q \exp\{\int_t^T [\frac{1}{2}\gamma^2\alpha_u^2\sigma^2(Y_u) - \\ & \qquad \gamma\alpha_u\mu(Y_u)]du - \gamma h(S_T)\}. \end{aligned} \tag{5}$$

The second equality was obtained changing in $\mathcal{F}_T$ the original probability measure $\mathbb{P}$ by the absolutely continuous measure Q with Radon-Nikodym derivative

$$\frac{dQ}{d\mathbb{P}}|_{\mathcal{F}_T} = \exp\left\{\int_0^T [-\gamma\alpha_u\sigma(Y_u)dW_u^1 - \frac{1}{2}\gamma^2\alpha_u^2\sigma^2(Y_u)du]\right\}. \tag{6}$$

By Girsanov's theorem, under measure Q the process defined as

$$\tilde{W}_t^1 := W_t^1 + \int_0^t \gamma\alpha_u\sigma(Y_u)du, \quad \tilde{W}_t^2 = W_t^2$$

is a Brownian motion in $\mathbb{R}^2$ and, hence, the dynamics of S_t and Y_t can be written, respectively, as

$$\begin{aligned} dS_t &= S_t[(\mu(Y_t) - \gamma\alpha_t\sigma^2(Y_t))dt + \sigma(Y_t)d\tilde{W}_t^1] \\ dY_t &= [g(Y_t) - \gamma\rho\alpha_t\sigma(Y_t)\beta(Y_t)]dt + \beta(Y_t)[\rho d\tilde{W}_t^1 + \\ & \quad \sqrt{1-\rho^2}d\tilde{W}_t^2]. \end{aligned}$$

Multiplying by -1 in both sides of (5) and maximizing with respect to the set of admissible strategies $\mathcal{A}$, we write the value function $V(t,x,y,s)$ in (3) as

$$\begin{aligned} V(t,x,y,s) &= -e^{-\gamma x}\min_{\alpha\in\mathcal{A}}\mathbb{E}_Q\exp\{\int_t^T (\frac{1}{2}\gamma^2\alpha_u^2\sigma^2(Y_u) \\ & \quad -\gamma\mu(Y_u)\alpha_u)du - \gamma h(S_T)\} \\ &= -e^{-\gamma x - \gamma\tilde{\psi}(t,y,s)}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} \tilde{\psi}(t,y,s) &= \max_{\alpha\in\mathcal{A}}\{-\frac{1}{\gamma}\log\mathbb{E}_Q\exp\{-\gamma\int_t^T l(Y_u,\alpha_u)du - \\ & \quad \gamma h(S_T)\}\}, \end{aligned} \tag{8}$$

with $l(y,\alpha) := \alpha\mu(y) - \frac{1}{2}\gamma\alpha^2\sigma^2(y)$. Observe that defining $Q(y) = \frac{\mu(y)}{2\gamma\sigma^2}$ and $\tilde{l}(y,\alpha) := l(y,\alpha) - \|Q\|$ the value function $\tilde{\psi}$ can be written as

$$\tilde{\psi}(t,y,s) = \|Q\|(T-t) + \psi(t,y,s),$$

with

$$\begin{aligned} \psi(t,y,s) &= \max_{\alpha\in\mathcal{A}}\{-\frac{1}{\gamma}\log\mathbb{E}_Q\exp\{-\gamma\int_t^T \tilde{l}(Y_u,\alpha_u)du - \\ & \quad \gamma h(S_T)\}\} \end{aligned} \tag{9}$$

Analogously, for the value function $W(t,x,y)$ we have the expression

$$\begin{aligned} W(t,x,y) &= -e^{-\gamma x}\min_{\alpha\in\mathcal{A}}\mathbb{E}_Q\exp\{\int_t^T (\frac{1}{2}\gamma^2\alpha_u^2\sigma^2(Y_u) - \\ & \quad \gamma\mu(Y_u)\alpha_u)du\} \\ &= -e^{-\gamma x - \gamma\tilde{\phi}(t,y)}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} \tilde{\phi}(t,y) &= \max_{\alpha\in\mathcal{A}}\{-\frac{1}{\gamma}\log\mathbb{E}_Q\exp\{-\gamma\int_t^T l(Y_u,\alpha_u)du\}\} \\ &= \|Q\|(T-t) + \max_{\alpha\in\mathcal{A}}\{-\frac{1}{\gamma}\log\mathbb{E}_Q \\ & \quad \exp\{-\gamma\int_t^T \tilde{l}(Y_u,\alpha_u)du\}\} \\ &=: \|Q\|(T-t) + \phi(t,y). \end{aligned} \tag{11}$$

Remark 1.2: These representations of the value functions $\tilde{\psi}$ and $\tilde{\phi}$ are important because both $\psi(\cdot,y,s)$ and $\phi(\cdot,y)$ are increasing in t since $\tilde{l} \le 0$.

Therefore, going back to the definition of the indifference price of the option $h(S_T)$ in (4), (7) and (10) yield

$$p = \psi(t,y,s) - \phi(t,y). \tag{12}$$

In the next section we will write the Hamilton Jacobi Bellman equation associated with both risk sensitive control problems. Under suitable condition on the smoothness of the value functions $\psi(t,y,s)$ and $\phi(t,y)$ it is possible to derive a verification theorem. The results will be presented only for the value function ψ, since ϕ is a particular case of the former when $h(\cdot) \equiv 0$.

II. HAMILTON-JACOBI-BELLMAN EQUATION

We begin this section with the description of the HJB equations associated to the value functions (8) and (11). In the first case, the function ψ satisfies at least formally the parabolic semilinear PDE

$$\begin{aligned} 0 &= \psi_t + s\mu(y)\psi_s + g(y)\psi_y + \frac{1}{2}s^2\sigma^2(y)\psi_{ss} + \\ & \quad \frac{1}{2}\beta^2(y)\psi_{yy} + \rho s\sigma(y)\beta(y)\psi_{ys} \\ & \quad +\gamma(1-\rho)s\sigma(y)\beta(y)\psi_s\psi_y - \frac{\gamma}{2}(s\sigma(y)\psi_s + \beta(y)\psi_y)^2 \\ & \quad +\max_{\alpha\in\mathbb{R}}\{-\gamma\alpha\sigma^2(y)s\psi_s - \gamma\rho\alpha\sigma(y)\beta(y)\psi_y + \tilde{l}(y,\alpha)\}, \end{aligned} \tag{13}$$

with boundary condition $\psi(T,y,s) = h(s)$. The equation for the function ϕ is analogous to the previous one, removing all the terms involving s, i.e.

$$\begin{aligned} 0 &= \phi_t + g(y)\phi_y + \frac{1}{2}\beta^2(y)\phi_{yy} - \frac{\gamma}{2}\beta^2(y)\phi_y^2 \\ & \quad +\max_{\alpha\in\mathbb{R}}\{-\gamma\rho\alpha\sigma(y)\beta(y)\phi_y + \tilde{l}(y,\alpha)\}, \end{aligned} \tag{14}$$

with $\phi(T,y) = 0$.

The maximum in the r.h.s. of (13) and (14) is achieved, respectively, at

$$\alpha^\psi = -s\psi_s - \frac{\rho\beta(y)}{\sigma(y)}\psi_y + \frac{\mu(y)}{\gamma\sigma^2(y)} \tag{15}$$

and

$$\alpha^{\phi} = -\frac{\rho\beta(y)}{\sigma(y)}\phi_y + \frac{\mu(y)}{\gamma\sigma^2(y)}.$$

Note that these values are the candidates for being the optimal strategies for each risk sensitive control problem.

Substituting these values in their respective equations, and after some calculations, we obtain the equations

$$\begin{aligned} 0 &= \psi_t + \tilde{g}(y)\psi_y - \frac{1}{2}\tilde{\beta}^2(y)\psi_y^2 + \frac{1}{2}s^2\sigma^2(y)\psi_{ss} + \\ & \frac{1}{2}\beta^2(y)\psi_{yy} + \rho s\sigma(y)\beta(y)\psi_{sy} + \tilde{Q}(y) \end{aligned} \quad (16)$$

and

$$\phi_t + \tilde{g}(y)\phi_y + \frac{1}{2}\beta^2(y)\phi_{yy} - \frac{1}{2}\tilde{\beta}^2(y)\phi_y^2 + \tilde{Q}(y) = 0, \quad (17)$$

where $\tilde{g}(y) := g(y) - \rho\frac{\mu(y)\beta(y)}{\sigma(y)}$, $\tilde{\beta}(y) := \sqrt{\gamma(1-\rho^2)}\beta(y)$ and $\tilde{Q}(y) = Q(y) - \|Q\|$. The terminal condition for these equations is, respectively, $\psi(T, y, s) = h(s)$ and $\phi(T, y) = 0$.

III. VERIFICATION THEOREM

In this section we present a verification theorem for the optimal control problem (9), which states that if there exists a classical solution to the Hamilton Jacobi Bellman equation (13) between a suitable class of functions then it corresponds to the value function $\psi(t, x, s)$; a straightforward corollary of this result is the uniqueness of classical solutions to the PDE (16). Similar results can be stated for the value function $\phi(t, y)$, but we will present only the proof for the first one to avoid unnecessary repetitions.

Theorem 3.1: Let $\varphi \in C^{1,2}((0,T)\times\mathbb{R}\times\mathbb{R}) \cap C((0,T]\times\mathbb{R}\times\mathbb{R})$ be a smooth solution of equation (13). Then, for all $(t, x, s) \in [0,T]\times\mathbb{R}^+\times\mathbb{R}^+$:

1) $\varphi(t, x, s) \geq \psi(t, x, s)$.
2) If the Markov control α^{ψ} defined in (15) is admissible (i.e. it belongs to $\mathcal{A}$)) then is is optimal, i.e.

$$\begin{aligned} \varphi(t, y, s) &= -\frac{1}{\gamma}\log \mathbb{E}_Q \exp\{-\gamma \int_t^T \tilde{l}(Y_u, \alpha_u^{\psi})du - \\ & \gamma h(S_T)\}. \end{aligned}$$

Proof. Let α_t be an arbitrary strategy in $\mathcal{A}$. Then, in view of (1) and [19] p.220, the change of measure in (6) associated with α_t is well defined. Let φ be as in the statement of the theorem, and define $\tau_n := \inf\{u \geq t \mid |Y_u| > n\}$. Then, by Ito's formula and (16),

$$\begin{aligned} & \varphi(T\wedge\tau_n, Y_{T\wedge\tau_n}, S_{T\wedge\tau_n}) - \varphi(t, y, s) = \\ & \int_t^{T\wedge\tau_n} [\varphi_t + S_u\varphi_y(\mu(Y_u) - \gamma\alpha_u\sigma^2(Y_u)) + \\ & \varphi_y(g(Y_u) - \gamma\rho\alpha_u\sigma(Y_u)\beta(Y_u)) + \frac{1}{2}S_u^2\sigma^2(Y_u)\varphi_{ss} + \\ & \frac{1}{2}\beta^2(Y_u)\varphi_{yy} + \rho\sigma(Y_u)\beta(Y_u)\varphi_{ys}]du + \\ & \int_t^{T\wedge\tau_n} S_u\sigma(Y_u)\varphi_s d\tilde{W}_u^1 + \beta(Y_u)\varphi_y(\rho d\tilde{W}_u^1 + \bar{\rho}d\tilde{W}_u^2) \\ \leq \; & -\int_t^{T\wedge\tau_n} \tilde{l}(Y_u, S_u)du + \int_t^{T\wedge\tau_n} [\frac{\gamma}{2}(S_u\sigma(Y_u)\varphi_s + \\ & \beta(Y_u)\varphi_y)^2 - \gamma\bar{\rho}S_u\sigma(Y_u)\beta(Y_u)\varphi_s\varphi_y]du + \\ & \int_t^{T\wedge\tau_n} S_u\sigma(Y_u)\varphi_s d\tilde{W}_u^1 + \beta(Y_u)\varphi_y(\rho d\tilde{W}_u^1 + \bar{\rho}d\tilde{W}_u^2). \end{aligned}$$

Then,

$$\begin{aligned} & \mathbb{E}_Q \exp\{-\gamma \int_t^{T\wedge\tau_n} \tilde{l}(Y_u, S_u)du - \\ & \quad \gamma\varphi(T\wedge\tau_n, Y_{T\wedge\tau_n}, S_{T\wedge\tau_n})\} \geq e^{-\gamma\varphi(t,y,s)} . \\ & \quad \mathbb{E}_Q \exp \int_t^{T\wedge\tau_n} -\{\frac{\gamma^2}{2}(S_u\sigma(Y_u)\varphi_s + \beta(Y_u)\varphi_y)^2 - \\ & \quad \gamma^2\bar{\rho}S_u\sigma(Y_u)\beta(Y_u)\varphi_s\varphi_y\}du \cdot \\ & \quad \exp\{-\gamma \int_t^{T\wedge\tau_n} S_u\sigma(Y_u)\varphi_s d\tilde{W}_u^1 + \\ & \quad \beta(Y_u)\varphi_y(\rho d\tilde{W}_u^1 + \bar{\rho}d\tilde{W}_u^2)\} \\ = \; & e^{-\gamma\varphi(t,y,s)}\mathbb{E}_Q \exp\{-\gamma \int_t^{T\wedge\tau_n} S_u\sigma(Y_u)\varphi_s d\tilde{W}_u^1 + \\ & \quad \beta(Y_u)\varphi_y(\rho d\tilde{W}_u^1 + \bar{\rho}d\tilde{W}_u^2)\} \cdot \\ & \quad \exp\{-\frac{\gamma^2}{2}\int_t^{T\wedge\tau_n} (S_u^2\sigma^2\varphi_s^2 + \beta\varphi_y^2 + \\ & \quad 2\rho S_u\sigma\beta\varphi_s\varphi_y)du\}. \end{aligned}$$

The process inside the expectation in the r.h.s. is a Q-local martingale and a super-martingale. Then, using the terminal condition $\varphi(T, Y_T, S_T) = h(S_T)$, and taking the limit when n goes to infinity, we get that

$$-\frac{1}{\gamma}\log \mathbb{E}_Q \exp\{-\gamma \int_t^T \tilde{l}(Y_u, S_u)du - \gamma h(S_T)\} \leq \varphi(t, y, s).$$

Since α_t was chosen arbitrarily, it follows that

$$\varphi(t, y, s) \geq \psi(t, y, s).$$

To prove the reverse inequality, we observe that when we use the admissible strategy defined through the Markov strategy (15) we obtain equalities instead of inequalities in the above displays.

REFERENCES

[1] G. Barles and H.M. Soner, Option pricing with transaction costs and a nonlinear Black-Scholes equation. Finance and Stoch. **2**, 369-397 (1998).

[2] T.R. Bielecki and S.R. Pliska, Risk sensitive dynamic asset management, Applied Math. Optim. **39**, 337-360 (1999).
[3] P. Briand, Y. Hu, BSDE with quadratic growth and unbounded terminal value. Probab. Theory Relat. Fields (2006).
[4] M.H.A. Davis, Optimal hedging with basis risk, From stochastic calculus to mathematical finance, 169–187, Springer, Berlin, 2006.
[5] R. Cavazos-Cadena and D. Hernández-Hernández, Characterization of the optimal risk sensitive average cost in finite controlled Markov chains, Ann. Applied Probab. **15**, 175-212 (2005).
[6] N. El Karoui and M.C. Quenez, Dynamic programming and pricing of contingent claims in a incomplete market, SIAM J. Control Optim., **33** 29-69 (1993).
[7] N. El Karoui and Rouge, Pricing by utility maximization, Math. Finance, (2000).
[8] W.H. Fleming and D. Hernández-Hernández, The tradeoff between consumption and investment in incomplete financial markets. Appl. Math. Optim. **52**, 219-235 (2005).
[9] W.H. Fleming and S.J. Sheu, Risk sensitive and an optimal investment model II, Ann. Appl. Probab. **12**, 730-767 (2002).
[10] W.H. Fleming and H.M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer Verlag, New York, 1993.
[11] J-P Fouque, G. Papanicolaou, K.R. Sircar, *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press (2000).
[12] M. Frittelli, Introduction to a theory of value coherent with the no-arbitrage principle, Finance and Stoch. **4**, 275-297 (2000).
[13] V. Henderson, Valuation of claims on nontraded assets using utility maximization. Math. Finance **12**, 351-373, (2002).
[14] V. Henderson and D. Hobson, Utility indifference pricing: an overview. In Volume on Indifference Pricing (Ed. R. Carmona), Princeton University Press, 2004.
[15] S.D. Hodges and A. Neuberger, Optimal replication of contingent claims under transaction costs. Rev. Future Markets **8**, 222-239 (1989).
[16] H. Kaise and S.-J. Sheu, On the structure of solutions of ergodic type Bellman type equation related to risk sensitive control, preprint (2004).
[17] M. Kobilanski, Backward stochastic differential equations with quadratic growth. Ann. Probab. 28, 558-602 (2000).
[18] O.A. Ladyzenskaya, V.A. Solonikov and N.N. Uralseva, *Linear and Quasilinear Equations of Parabolic Type*, AMS Transl. of Math. Monographs, Providence, RI, 1968.
[19] R.S. Lipster and A.N. Shiryayev, *Statistics of Random Processes I: General Theory*, Springer-Verlag, New York, 1977.
[20] M. Musiela and T. Zariphopoulou, A valuation algorithm for indifference prices in incomplete markets. Finance and Stoch. **8**, 339-414 (2004).
[21] H. Nagai, Bellman equations of risk sensitive control. SIAM J. Control Optim. **34**, 74-101 (1996).
[22] H. Pham, Smooth solutions to optimal investment models with stochastic volatilities and portfolio constraints. Appl. Math. Optim. **46**, 55-78 (2002).