2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

On Optimal Preview Control across Erasure Communication Links

Vijay Gupta and Nuno C. Martins

***Abstract*— Consider two identical discrete-time, finite-dimensional, linear and time-invariant systems denoted as leader and follower. The leader system is driven by a deterministic control input and by a zero-mean white Gaussian process noise. In this paper, we address the problem of designing a networked control scheme for the follower system that guarantees that the state of the follower system tracks the state of the leader system optimally, according to a mean squared cost. We adopt a networked control scheme featuring two erasure links and three design blocks: a controller acting on the follower system and two encoders, one at the output of each system. The controller has remote access to noisy measurements of the state of both systems, via two erasure links that are used to connect each encoder to the controller. We consider erasure links whose erasure events occur according to a Bernoulli process. If an erasure occurs then no information is transmitted, otherwise a finite vector of real numbers is conveyed through the link. The purpose of the encoders is to process noisy measurements of the output of each system prior to transmission over the corresponding erasure link. While the encoder of the follower system has access to current measurements, the encoder of the leader system has access to measurements that are advanced in time by a finite time interval (also denoted as preview). This paper describes a methodology for the design of controller and encoders that are jointly optimal, with respect to optimal tracking of the leader system by the follower system. We also obtain explicit necessary and sufficient conditions for the existence of a scheme that guarantees that the tracking error has finite second moment.**

I. INTRODUCTION

Networked control systems have emerged as an important research area in recent years, see, e.g., [1], [21]. The estimation and control performance in such systems is severely affected by the properties of the communication channels. Many different models of channels have become popular, depending on the particular effect considered most important. Thus, digital noiseless channels have been studied to account for finite number of bits being transmitted, erasure links have been popular to model stochastic data loss, channels that introduce delay have been analyzed to study the effect of retransmission and routing delays, and so on.

In this work, we are specifically interested in the problem of estimation and control across communication links that exhibit data loss. Preliminary work in this area has largely concentrated on the case when only one sensor is present. A good overview of work dealing with the effect of the erasure links on estimation and control, as well as designing a compensator at the observer end to estimate the data when links drop packets, has been provided in [21], [8], [22], [10]. This paper takes a more general view by allowing the sensors to encode or pre-process information prior to transmission and by utilizing the fact that a typical network / communication data packet allows transmission of extra data apart from that required inside a traditional control loop. As has been shown in [9], [8], [10], this approach can yield significant improvements in terms of stability and performance. Moreover, for a given performance level, it can also lead to a reduced amount of communication.

In this paper, we extend the principle to the preview control case. We adopt a framework comprising two identical linear and time-invariant systems, denoted as the leader and the follower. We aim to design a control scheme for steering the follower so that its state tracks the state of the leader system optimally, according to a mean-squared cost. In this paper, we consider control schemes comprising one controller, for driving the follower system, and two encoders, one at the output of each system. The encoders are connected to the controller via erasure links that suffer stochastic information-loss events in a Bernouli fashion. Time advanced measurements, or preview measurements, of the state of the leader system are available at the respective encoder.

Our paradigm is closely related to preview control, which is an important problem with a rich history. A classical approach in (deterministic) servomechanism design is the internal model principle [6] that ensures asymptotic tracking of periodic reference signals. In the late eighties, techniques based on operator theory were used to derive control laws for linear and time-invariant systems that guarantee optimal reference tracking, under the assumption of finite horizon and infinite horizon preview [20], [11]. The papers [4], [5], [27], [24], [23] are also relevant contributions. More recently, for deterministic systems, more general performance metrics, such as H_2 and H_∞, were considered [3], [13], [14], [17], [25], [26]. The design of controllers that achieve optimal reference tracking, as measured by the H_∞ metrics, has also been addressed in the presence of sampling effects [18], for linear and time-invariant plants. We would also like to mention the important works of [15], [16], [19]. All of these results, in one way or another, conclude that reference preview can lead to a substantial increase in the tracking performance. However, the case featuring preview information for stochastic systems across communication links remains an open problem. This problem cannot be viewed as a simple extension of the traditional preview

Department of Electrical Engineering, University of Notre Dame. Email: vgupta2@nd.edu

Institute for Systems Research, University of Maryland, College Park. Email: nmartins@isr.umd.edu.

978-1-4244-2079-7/08/$25.00 ©2008 AACC.

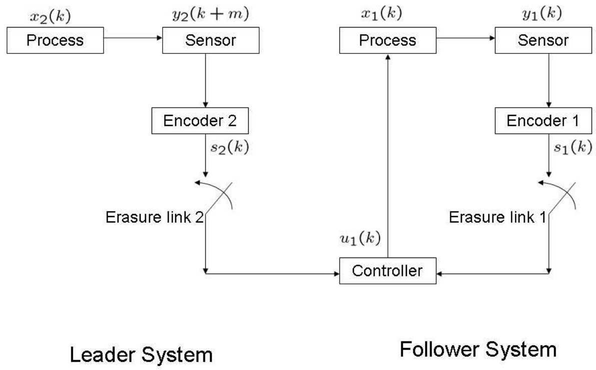

Fig. 1. Basic framework for leader-follower tracking control with preview, in the presence of erasure links.

control problem, just as the control of an LTI plant across communication links is not a simple extension of traditional LQR design problem. In this paper, we fill this gap.

The paper is organized as follows. We begin in the next section by describing the problem set-up and our notation. In Section III, we present a re-parameterization that allows us to map our paradigm into the problem of controlling a process using multiple sensors across erasure links. In Section IV, we take advantage of the mapping to use existing machinery and tools to design the optimal control input. The mapping is not perfect, and in particular, the system does not retain controllability assumptions. However, in Section V, we give explicit necessary and sufficient conditions for the existence of a controller and encoders that stabilize the tracking error, in a mean squared sense.

II. Framework Description and Problem Formulation

Consider the set-up of Fig 1, and the following associated assumptions. The follower system is described by the following discrete-time state-space representation:

$$\begin{aligned} \mathbf{x}_1(k+1) &= F\mathbf{x}_1(k) + G\mathbf{u}_1(k) + \mathbf{w}_1(k), k \geq 0 \quad (1) \\ \mathbf{y}_1(k) &= H\mathbf{x}_1(k) + \mathbf{v}_1(k), \end{aligned}$$

where the state $\mathbf{x}_1(k)$ takes values in $\mathbf{R}^n$ and the control input $\mathbf{u}_1(k)$ takes values in $\mathbf{R}^p$. The process noice $\mathbf{w}_1(k)$ is assumed to be white, Gaussian, zero mean with covariance $\Sigma_{w1} > 0$. The initial state $\mathbf{x}_1(0)$ is a zero mean Gaussian random variable with covariance matrix Σ_{01}. The state is observed by a sensor that generates the measurements $\mathbf{y}_1(k)$ taking values in $\mathbf{R}^q$. The measurement noise $\mathbf{v}_1(k)$ is also assumed to be white, Gaussian, zero mean with covariance $\Sigma_{v1} > 0$. The pair (F, H) is assumed to be observable and the pair (F, G) is assumed to be controllable. We also assume that the matrices G and H are full rank.

The leader system is given by the following state-space representation:

$$\begin{aligned} \mathbf{x}_2(k+1) &= F\mathbf{x}_2(k) + Gu_2(k) + \mathbf{w}_2(k), k \geq 0 \quad (2) \\ \mathbf{y}_2(k) &= H\mathbf{x}_2(k) + \mathbf{v}_2(k), \end{aligned}$$

where the state $\mathbf{x}_1(k)$ once again takes values in $\mathbf{R}^n$, while $u_2(k)$ is a deterministic (known a-priori) control input taking values in $\mathbf{R}^p$ and $\mathbf{w}_2(k)$ is the reference process noise assumed to be white, Gaussian, zero mean with covariance $\Sigma_{w2} > 0$. The initial state $\mathbf{x}_2(0)$ is a zero mean Gaussian random variable with covariance matrix Σ_{02}. The direct feedthrough noise $\mathbf{v}_2(k)$ is also assumed to be white, Gaussian, zero mean with covariance $\Sigma_{v2} > 0$.

We assume that all the sources of randomness $\{\mathbf{w}_1(l)\}_{l=0}^{\infty}$, $\{\mathbf{w}_2(l)\}_{l=0}^{\infty}$, $\{\mathbf{v}_1(l)\}_{l=0}^{\infty}$, $\{\mathbf{v}_2(l)\}_{l=0}^{\infty}$, $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ are mutually independent.

The output of the follower system $\mathbf{y}_1(k)$ and the output of the leader system, advanced by m time steps ($m \geq 0$), $\mathbf{y}_2(k+m)$ are made available to encoders $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. At every time step k, encoder $\mathcal{E}_1$ calculates and transmits the vector $\mathbf{s}_1(k)$, according to the following functional structure:

$$\mathbf{s}_1(k) = \mathcal{E}_1\left(k, \mathbf{y}_1(0), \ldots, \mathbf{y}_1(k)\right) \qquad k \geq 0, \qquad (3)$$

where $s_1(k)$ takes values in $\mathbf{R}^{j_1}$ and j_1 is a given positive integer. Similarly, the encoder $\mathcal{E}_2$ calculates and transmits the vector $s_2(k)$ taking values in $\mathbf{R}^{j_2}$, for some positive integer j_2, according to the following functional structure:

$$\mathbf{s}_2(k) = \mathcal{E}_2\left(k, \mathbf{y}_2(0), \ldots, \mathbf{y}_2(k+m)\right) \qquad k \geq 0. \qquad (4)$$

In the sequel, we will also refer to the encoder maps as encoding algorithms or information processing algorithms.

A. *Description of the information structure*

The encoders are connected to the controller via links featuring erasure events that are governed by Bernouli processes $\{\mathbf{r}_1(k)\}_{k=0}^{\infty}$ and $\{\mathbf{r}_2(k)\}_{k=0}^{\infty}$, taking values in the set $\{\mathcal{T}, \emptyset\}$ and characterized by a probability mass function of the type:

$$p_{i,j} \stackrel{def}{=} Pr\left(\mathbf{r}(k) = (i,j)\right), \ (i,j) \in \{\mathcal{T}, \emptyset\}^2$$

where $\mathbf{r}(k) \stackrel{def}{=} (\mathbf{r}_1(k), \mathbf{r}_2(k))$. More specifically, let us denote erasure link 1 as being the one across which the encoder $\mathcal{E}_1$ transmits the vector $\mathbf{s}_1(k)$ and erasure link 2 as being the one across which the encoder $\mathcal{E}_2$ transmits the vector $s_2(k)$. Let us also denote the outputs of the links by $\mathbf{z}_1(k)$ and $\mathbf{z}_2(k)$, respectively. Then the relationship between the input of the each erasure link and its output is specified by

$$\mathbf{z}_i(k) = \begin{cases} \emptyset & \text{if } \mathbf{r}_i(k) = \emptyset \\ \mathbf{s}_i(k) & \text{if } \mathbf{r}_i(k) = \mathcal{T} \end{cases}, \ i \in \{1, 2\}$$

where we adopt the symbol $\emptyset$ to represent erasure, i.e., it indicates that no information is received at the output of the link. Note that, in general, we do *not* assume that the erasure events in the links are uncorrelated. However, we presuppose that $\{\mathbf{r}(l)\}_{l=0}^{\infty}$ is independent of all the other sources of randomness, namely $\{\mathbf{w}_1(l), \mathbf{w}_2(l), \mathbf{v}_1(l), \mathbf{v}_2(l)\}_{l=0}^{\infty}$ and $(\mathbf{x}_1(0), \mathbf{x}_2(0))$. If the information about the erasure events till time $k-1$ is available to the encoder at time k, we say that the link has acknowledgements; otherwise, it does not.

The controller has access to the outputs of links 1 and 2. In particular, we consider that the controller, denoted by $\mathcal{K}$, is of the following form:

$$\mathbf{u}_1(k) = \mathcal{K}(k, \mathbf{z}_1(0), \mathbf{z}_2(0), \ldots, \mathbf{z}_1(k-1), \mathbf{z}_2(k-1)). \quad (5)$$

It may be noted that the the system parameters, such as the matrices F, G and H among others, as well as of the leader control input sequence $\{u_2(k)\}_{k=0}^{\infty}$ are assumed known a-priori.

B. Problem statement

Given the erasure link statistics, specified by the probability mass function $p_{i,j}$, the description of the follower and leader systems, given by (1) and (2), respectively, we want to jointly design the controller and the encoders that minimize the following cost function:

$$\begin{aligned} J_T = E\Big[\sum_{k=0}^{T}\Big(&(\mathbf{x}_1(k)-\mathbf{x}_2(k))^T Q(\mathbf{x}_1(k)-\mathbf{x}_2(k)) \\ &+ (\mathbf{u}_1(k)-u_2(k))^T R(\mathbf{u}_1(k)-u_2(k))\Big) \\ +(\mathbf{x}_1(T+1)-&\mathbf{x}_2(T+1))^T Q(\mathbf{x}_1(T+1)-\mathbf{x}_2(T+1))\Big], \end{aligned} \quad (6)$$

where R is a symmetric positive definite matrix and Q is a symmetric positive semi-definite matrix for which the pair $(A, Q^{1/2})$ is detectable.

We also wish to investigate conditions for the existence of a controller $\mathcal{K}$, and encoders $\mathcal{E}_1$ and $\mathcal{E}_2$ such that the infinite horizon cost is bounded in the following sense:

$$J_\infty = \lim_{T\to\infty} \frac{1}{T} J_T < c,$$

for some positive real constant c. The expectation in the above cost function is taken with respect to all the independent sources of randomness defined so far.

III. Multi-sensor re-parameterization

In this section, we re-parameterize our problem so as to establish a comparison and make use of the existing results in [10]. We start by giving the following augmented state space representation whose state will include both $\mathbf{x}_1(k)$ and $\mathbf{x}_2(k)$:

$$\begin{aligned} \mathbf{x}(k+1) &= A\mathbf{x}(k) + B_1\mathbf{u}_1(k) \\ &\quad + B_2 u_2(k+m-1) + \mathbf{w}(k) && (7) \\ \mathbf{g}_1(k) &\overset{def}{=} \mathbf{y}_1(k) = C_1\mathbf{x}(k) + \mathbf{d}_1(k) && (8) \\ \mathbf{g}_2(k) &\overset{def}{=} \mathbf{y}_2(k+m) = C_2\mathbf{x}(k) + \mathbf{d}_2(k) && (9) \end{aligned}$$

where $\mathbf{g}_1(k)$ and $\mathbf{g}_2(k)$ represent the *n*ew output, while $\mathbf{x}(k)$, $\mathbf{w}(k)$ and $\mathbf{d}_i(k)$ are defined below and they denote the new state, process noise and measurement noises, respectively.

$$\mathbf{x}(k) \overset{def}{=} \begin{bmatrix} \mathbf{x}_1(k) \\ \mathbf{x}_2(k+m) \\ \mathbf{x}_2(k+m-1) \\ \vdots \\ \mathbf{x}_2(k) \end{bmatrix} \quad (10)$$

$$\mathbf{w}(k) = \begin{bmatrix} \mathbf{w}_1(k) \\ \mathbf{w}_2(k+m-1) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \begin{aligned} \mathbf{d}_1(k) &= \mathbf{v}_1(k) \\ \mathbf{d}_2(k) &= \mathbf{v}_2(k+m) \end{aligned}$$

The following are the state-space matrices that correspond to the re-parameterization (7):

$$\begin{aligned} A &= \begin{bmatrix} F & 0 \\ 0 & A_{2,2} \end{bmatrix} && (11) \\ A_{2,2} &= \begin{bmatrix} F & 0 & 0 & \cdots & 0 \\ I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & I & 0 \end{bmatrix} && (12) \\ B_1 &= \begin{bmatrix} G^T & 0 & 0 & \cdots & 0 \end{bmatrix}^T && (13) \\ B_2 &= \begin{bmatrix} 0 & G^T & 0 & \cdots & 0 \end{bmatrix}^T && (14) \\ C_1 &= \begin{bmatrix} H & 0 & 0 & \cdots & 0 \end{bmatrix} && (15) \\ C_2 &= \begin{bmatrix} 0 & H & 0 & \cdots & 0 \end{bmatrix} && (16) \end{aligned}$$

The cost function (6) can be recast in terms of $\mathbf{x}(k)$ as

$$\begin{aligned} J_T = E\Big[\sum_{k=0}^{T}\Big(&\mathbf{x}(k)^T Q_s \mathbf{x}(k) + f(\Sigma_{v1}, \Sigma_{v2}) \\ &+ (\mathbf{u}_1(k)-u_2(k))^T R(\mathbf{u}_1(k)-u_2(k))\Big) \\ &+ \mathbf{x}(T+1)^T Q_s \mathbf{x}(T+1)\Big], \end{aligned} \quad (17)$$

where

$$Q_s = \begin{bmatrix} Q & 0 & \cdots & 0 & -Q \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & 0 \\ -Q & 0 & \cdots & 0 & Q \end{bmatrix},$$

and $f(\Sigma_{v1}, \Sigma_{v2})$ is a term depending only on constant matrices, such as Σ_{v1} and Σ_{v2}, but not on problem variables $\mathbf{x}(k)$ or $\mathbf{u}_1(k)$. It is clear by simple inspection that if, at time k, the encoder $\mathcal{E}_i$ has access to measurements $\mathbf{g}_i(j)$ from time 0 to time k, then the description in (7) is exactly the description provided in Section II.

Minimization of the cost function (17) for the system (10) is similar to the problem of LQG control of a process being observed by two sensors that transmit over two erasure links, as solved in [10]. However, there are three main differences between the problem definitions given here and in [10]:

1) A pre-defined (known) input term $u_2(k)$ is present in the evolution of the system as defined in (7).
2) A pre-defined (known) term $u_2(k)$ is present in the cost function (17).
3) The system (10) is not controllable from $\mathbf{u}_1(k)$, which is the only control input that can be designed.

While the first two differences alter the form of the optimal control input, the third difference means that the stabilization of the tracking error needs to be examined closely, even in the absence of any erasure links.

IV. Optimal Control Input

We begin by presenting a separation principle for our problem. To state the principle, we need to introduce some terminology. Similar to [10], at any time k, define the time-stamp corresponding to sensor i as the latest time at which transmission was possible through link i:

$$\mathbf{t}_i(k) = \max\{j \in \{0,1,2,\ldots\} \mid j \leq k, \quad \mathbf{r}_i(j) = \mathcal{T}\}.$$

Further, define the maximal information set $\mathcal{I}_i^{\max}(k)$ for each sensor as

$$\begin{aligned}\mathcal{I}_1^{\max}(k) &= \{\mathbf{y}_1(0), \mathbf{y}_1(1), \cdots, \mathbf{y}_1(\mathbf{t}_1(k)),\\ &\quad \mathbf{r}_1(0), \mathbf{r}_1(1), \cdots, \mathbf{r}_1(k)\}\\ \mathcal{I}_2^{\max}(k) &= \{\mathbf{y}_2(0), \mathbf{y}_2(1), \cdots, \mathbf{y}_2(\mathbf{t}_2(k)+m),\\ &\quad \mathbf{r}_2(0), \mathbf{r}_2(1), \cdots, \mathbf{r}_2(k)\}.\end{aligned}$$

The maximal information set includes the largest set of measurements from sensor i that the controller can possibly have access to at time k. For any encoding algorithm $\mathcal{A}$ followed by the sensors, we define the information set corresponding to sensor i at time k as

$$\mathcal{I}_i^{\mathcal{A}}(k) = \{\mathbf{s}_i(0), \mathbf{s}_i(\mathbf{t}_i(1)), \mathbf{s}_i(\mathbf{t}_i(2)), \ldots, \mathbf{s}_i(\mathbf{t}_i(k))\},$$

where $\mathbf{s}_i(k)$ is the output of the sensor i at time k, when the algorithm $\mathcal{A}$ is followed at sensor i. From the definition of the time stamp $\mathbf{t}_i(k)$ it follows that $\mathcal{I}_i^{\mathcal{A}}(k)$ comprises the time-samples of $\mathbf{s}_i(k)$ which can be recovered at the controller via the output of the erasure link from sensor i. For any encoding strategy followed by the sensors, the inclusion $\mathbb{I}_i^{\mathcal{A}}(k) \subseteq \mathbb{I}_i^{\max}(k)$ holds, where $\mathbb{I}_i^{\mathcal{A}}(k)$ and $\mathbb{I}_i^{\max}(k)$ are the smallest sigma algebras (filtrations) generated by $\mathcal{I}_i^{\mathcal{A}}(k)$ and $\mathcal{I}_i^{\max}(k)$, respectively. Further, for any two encoding algorithms $\mathcal{A}_1$ and $\mathcal{A}_2$, if $\mathbb{I}_i^{\mathcal{A}_1}(k) \subseteq \mathbb{I}_i^{\mathcal{A}_2}(k)$ holds at every time step, then the inequality $J_K^{\star,\mathcal{A}_2} \leq J_K^{\star,\mathcal{A}_1}$ is true, where $\mathbb{I}_i^{\max}(k)$ is as defined above and $J_K^{\star,\mathcal{A}_i}$ is the optimal value of the cost function J_K realized when the encoding algorithm $\mathcal{A}_i$ is used along with a controller that optimally utilizes the information available to it. Now consider an algorithm $\bar{\mathcal{A}}$ under which, at every time step k the encoder for sensor i transmits the set

$$\mathbf{S}_i(k) = \{\mathbf{z}_i(0), \mathbf{z}_i(1), \cdots, \mathbf{z}_i(k), \mathbf{r}_i(0), \mathbf{r}_i(1), \cdots, \mathbf{r}_i(k)\}.$$

At any time step k, the decoder (and the controller) would thus have access to the maximal information sets $\mathcal{I}_1^{\max}(k)$ and $\mathcal{I}_2^{\max}(k)$. This implies that for any other encoding algorithm $\mathcal{A}$, the inequality $J_K^{\star,\bar{\mathcal{A}}} \leq J_K^{\star,\mathcal{A}}$ holds.

We state the following separation principle when algorithm $\bar{\mathcal{A}}$ is used. (Note: For any stochastic process $\mathbf{h}(k)$ and random variable $\mathbf{e}$, denote by $\hat{\mathbf{h}}(k|\mathbf{e})$ the minimum mean squared error (mmse) estimate of $\mathbf{h}(k)$ given $\mathbf{e}$.)

Proposition 4.1 (Separation Principle): Consider the problem set-up for the system (7) as defined in section III. Suppose that each encoder transmits all the current and past measurements it has access to at every time step, so that the controller has access to the maximal information sets $\mathcal{I}_1^{\max}(k)$ and $\mathcal{I}_2^{\max}(k)$ at every time step k. Then, for an optimizing choice of the control, the control and estimation costs decouple. Specifically, the optimal control input at time k is calculated to be

$$\mathbf{u}_1(k) = \hat{\mathbf{u}}_{1,LQ}\left(k|\mathcal{I}_1^{\max}(k), \mathcal{I}_2^{\max}(k), \{\mathbf{u}_1(t)\}_{t=0}^{k-1}\right)$$

where $\mathbf{u}_{1,LQ}(k)$ is the optimal LQ control law and $\hat{\mathbf{u}}_{1,LQ}\left(k|\mathcal{I}_1^{\max}(k), \mathcal{I}_2^{\max}(k), \{\mathbf{u}_1(t)\}_{t=0}^{k-1}\right)$ denotes its minimum mean squared error estimate given the information sets $\mathcal{I}_1^{\max}(k)$ and $\mathcal{I}_2^{\max}(k)$, and the previous control inputs $\mathbf{u}_1(0), \cdots, \mathbf{u}_1(k-1)$.

Proof: The proof is along the lines of the standard separation principle (see [10, Proposition IV.1]; see also [12, Chapter 9]) and is omitted for space constraints. ■

The LQ optimal control input refers to the following problem. Consider the system

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B_1\mathbf{u}_1(k) + B_2u_2(k+m-1),$$

where, at time k, the controller has access to $\{\mathbf{x}(l)\}_{l=0}^{k}$. The control input $\mathbf{u}_1(k)$ has to be designed to minimize the cost function

$$\begin{aligned}\bar{J}_T = \sum_{k=0}^{T}\Big(&\mathbf{x}(k)^TQ_s\mathbf{x}(k)\\ &+(\mathbf{u}_1(k)-u_2(k))^TR(\mathbf{u}_1(k)-u_2(k))\Big)\\ &\qquad + \mathbf{x}(T+1)^TQ_s\mathbf{x}(T+1),\end{aligned} \quad (18)$$

with the tracking error being stable if $\bar{J}_\infty = \lim_{T\to\infty}\bar{J}_T$ is bounded.

Due to the presence of the additional variable $u_2(.)$ in the system evolution equation and the cost function, the LQ optimal control input becomes affine (rather than linear) in the state value. We have the following result.

Proposition 4.2 (Optimal Control Input): Consider the problem formulation for system (7) as stated in Section III, except that the controller has access to the state value $\mathbf{x}(k)$ while calculating the control input at any time step k. Then the optimal LQ control input at time step k is given by

$$\begin{aligned}\mathbf{u}_1(k) = -S^{-1}(k)\big(&B_1^TP(k+1)A\mathbf{x}(k) + Ru_2(k)\\ &- B_1^T\Lambda_{k+1} - B_1^TP(k+1)B_2u_2(k+m-1)\big),\end{aligned} \quad (19)$$

where

$$\begin{aligned}
S(k) &= B_1^T P(k+1) B_1 + R \\
P(k) &= Q_s + A^T P(k+1) A \\
&\quad - A^T P(k+1) B_1 S^{-1}(k) B_1^T P(k+1) A \\
\Lambda(k) &= A^T P(k+1) B_2 u_2(k+m-1) + A^T \Big(\Lambda(k+1) \\
&\quad - P(k+1) B_1 S^{-1}(k) \big(R u_2(k) + B_1^T \Lambda(k+1) \\
&\quad + B_1^T P(k+1) B_2 u_2(k+m-1) \big) \Big),
\end{aligned}$$

with $P(T+1) = Q_s$ and $\Lambda(T+1) = 0$.

Proof: Proof follows by standard dynamic programming arguments and is omitted for space constraints. The difference from the standard LQ control derivation is the presence of a constant term in the system evolution equation and the cost function. Including an affine term in the cost function compensates for those terms. ■

From the form of the optimal LQ control and the separation principle, it is apparent that the encoders need to calculate the minimum mean squared error estimate of $\mathbf{x}(k)$ based on the maximal information sets and the previous control inputs available to the controller. For the two sensor case, this problem has been solved in [10]. Specifically, optimal yet recursive designs for the encoders are presented for the case when acknowledgements are available. When acknowledgements are not available, or are subject to erasures themselves, the optimal designs, in general, are unknown. However, sub-optimal designs that are stabilizing under the same conditions are identified. Owing to space constraints, we do not present the algorithms here. An important feature of the algorithms is that the encoders do not need access to the previous control inputs.

V. Stability Conditions

Because of the separation principle, the estimation and control costs decouple. Hence, to analyze stability of the tracking error, we have to obtain conditions that determine when the estimation error covariance and the average LQR cost for the system diverge, as the horizon is increased. This analysis is given in the following Proposition:

Proposition 5.1 (Stability Conditions): Consider the problem set-up for the system (7), as defined in section III. Assume that the statistics of the erasure links are specified by a given probability mass function $Pr(r(k) = (i,j))$, with $(i,j) \in \{\mathcal{T}, \emptyset\}^2$ that is independent of the time index k. Further, assume that control input is calculated by the controller using the LQ optimal control law and the estimate of the state using encoder design and the estimator presented in [10]. Finally, assume that the matrix pair (F,G) is controllable. Then, the cost $\bar{J}_\infty$ is bounded if the following inequalities hold:

$$\begin{aligned}
\varrho(F)^2 Pr(\mathbf{r}_2(k) = \emptyset) &< 1 \\
\varrho(A_{2,2})^2 Pr(\mathbf{r}_1(k) = \emptyset) &< 1
\end{aligned} \tag{20}$$

where F is the dynamic matrix of the systems (1) and (2), while $A_{2,2}$ is defined in (12). Moreover, if at least one of the inequalities does not hold, then there does not exist causal encoders and a controller that will result in a bounded cost $\bar{J}_\infty$.

Proof: Due to the separation principle, we need to prove the following two statements:

1) *Stability of Estimation Error:* We need to ensure that there exists a causal observer at the controller, as well as encoders $\mathcal{E}_1$ and $\mathcal{E}_2$ that lead to a stable state estimation error at the controller(in the mean squared sense) if and only if the inequalities (20) hold. This statement follows directly from Theorem III.3 in [10] on using the re-parameterization of Section III.
2) *Stability of the LQR Solution:* We need to ensure that the cost $\bar{J}_\infty$ is bounded if the LQR optimal control input is used at every time step (i.e., when the controller has access to the state $\mathbf{x}_k$ at every time step). Since the matrix pair (A, B_1) is not controllable, this is not obvious. From Proposition 4.2, we need to study the convergence of terms $P(0)$ and $\Lambda(0)$ as the horizon increases. However, using the optimality result, we have an easier way to ensure a bounded $\bar{J}_\infty$ when the LQR optimal control law is used. We only need to ensure that there exists a control input that achieves bounded infinite horizon cost. Since the cost for the LQR optimal control input is upper bounded by this cost, we will have proved the desired result. Consider the input

$$\mathbf{u}_1(k) = u_2(k) + K_1(\mathbf{x}_1(k) - \mathbf{x}_2(k)),$$

where K_1 is selected such that the matrix $F + GK_1$ is stable. Note that such a matrix K_1 always exists under the assumption of controllability of the pair (F,G). With this control law, the following evolution for the difference $\mathbf{x}_1(k) - \mathbf{x}_2(k)$ occurs:

$$\mathbf{x}_1(k+1) - \mathbf{x}_2(k+1) = (F + GK_1)(\mathbf{x}_1(k) - \mathbf{x}_2(k)).$$

Since the matrix $F + GK_1$ is stable, the cost

$$\begin{aligned}
\bar{J}_\infty = \lim_{T\to\infty} \sum_{k=0}^{T} \Big(&(\mathbf{x}_1(k) - \mathbf{x}_2(k))^T Q (\mathbf{x}_1(k) - \mathbf{x}_2(k)) \\
&+ (\mathbf{u}_1(k) - u_2(k))^T R (\mathbf{u}_1(k) - u_2(k)) \Big),
\end{aligned}$$

is bounded. Thus the cost for the LQR optimal control law is bounded as well.

The above two results, together with the separation principle, show that under the assumption of controllability of the pair (F,G) and the satisfaction of the inequalities (21), the system is stable with the optimal design identified in the previous section. ■

The above result does not rely on the presence of acknowledgements at the encoders. In our problem, $\varrho(A_{2,2})$ can be explicitly calculated as follows:

$$\varrho(A_{2,2}) = \varrho\left(\begin{bmatrix} F & 0 & \cdots & 0 \\ I & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & I & 0 \end{bmatrix}\right) = \varrho(F)$$

Consequently, the necessary and sufficient conditions for the existence of a scheme, for which $\bar{J}_\infty$ is bounded, reduce to:

$$\varrho(F)^2 Pr(\mathbf{r}_i(k) = \emptyset) < 1, \ i \in \{1, 2\} \tag{21}$$

Note, in particular, that the condition (21) is independent of the amount of preview m that is available about the output of the leader system. Our results can be easily extended to more complicated models of erasure, such as Markovian models.

VI. Conclusions and Future Work

In this paper, we looked at the problem of optimally controlling a system (follower) so that its state tracks the state of another plant (leader) that is driven by exogenous inputs. The optimal control scheme must be implemented via erasure links that are used to convey state measurements to the controller. This is a three-block design problem in which a controller acting on the follower system and two encoders, associated with the leader and follower systems, must be designed. Using a re-parameterization, we were able to develop a method for optimally designing all the components, according to a mean-squared cost. We also provide explicit necessary and sufficient conditions for the existence of a solution leading to bounded infinite horizon cost.

The paper can be extended in many directions. We have not been able to obtain a clear characterization of the performance improvement as the preview length is increased. Similarly, the case of tracking when the reference signal is a given sequence of random variables remains open.

Acknowledgements

The second author would like to acknowledge partial support by NSF CAREER award ECCS 0644764.

References

[1] P. Antsaklis and J. Baillieul, "Special issue on networked control systems," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, 2004.

[2] D. H. Chyung, "Feedforward Digital Tracking Controller," IEEE Transactions on Automatic Control, 36(10): 1199-1204, 1991.

[3] A. Cohen, U. Shaked, *"Linear discrete-time Hinf-optimal tracking with preview,"* IEEE Transactions on Automatic Control, Volume 42, Issue 2, Feb 1997 Page(s):270 - 276

[4] E. J. Davidson, *"The robust control of a servomechanism problem for linear time-invariant multivariable plants,"* IEEE Transactions on Automatic Control, vol 21, no 1, Feb 1976, pages 25-34

[5] B. Francis, *"The optimal linear-quadratic time-invariant regulator with cheap control,"* IEEE Transactions on Automatic Control, vol AC-24, no 4, Aug. 1979, pages 616-621

[6] B. Francis and W. Wonham, *"The internal model principle of control theory,"* Automatica, vol 12, no 5, 1976, pages 457-465

[7] V. Gupta, T. Chung, B. Hassibi and R. M. Murray, "On a Stochastic Sensor Selection Algorithm with Applications in Sensor Scheduling and Dynamic Sensor Coverage," *Automatica*, Vol. 42, No. 2, February 2006, Pages: 251-260.

[8] V. Gupta, A. F. Dana, J. P. Hespanha, R. M. Murray and B. Hassibi, "Data Transmission over Networks for Estimation and Control," *IEEE Transactions on Automatic Control*, 2006, Submitted.

[9] V. Gupta, B. Hassibi, and R. M. Murray, "Optimal LQG control across a packet-dropping link," *Systems and Controls Letters*, Vol. 56, No. 6, June 2007, Pages: 439-446.

[10] V. Gupta, N. C. Martins and J. Baras, "Optimal Output Feedback Control Using Two Remote Sensors over Packet Erasure Channels," *IEEE Transactions on Automatic Control*, 2007, Submitted. Available online at http://hdl.handle.net/1903/6622

[11] M. E. Halpern, *"Preview tracking for discrete-time SISO systems,"* IEEE Transactions on Automatic Control, Volume 39, Issue 3, Mar 1994 Page(s):589 - 592

[12] B. Hassibi, A. H. Sayed, and T. Kailath, "Indefinite-Quadratic Estimation and Control", *Studies in Applied and Numerical Mathematics*, 1999.

[13] H. Katoh, *"H_∞-optimal preview controller and its performance limit,"* IEEE Transactions on Automatic Control, vol 48, Issue 11, Nov 2004, Pages 2011-2017

[14] A. Kojima, S. Ishijima, *"LQ preview synthesis: optimal control and worst case analysis,"* IEEE Transactions on Automatic Control, vol 44, Issue 2, Feb 1999, Pages 352-357

[15] A. Kojima, *"H_2 performance on preview feedforward action,"* Proceedings of the 16th MTNS, 2004.

[16] G. Marro, E. Zattoni, *"H_2 optimal rejection with preview in continuous-time domain,"* Automatica, 41(5), May 2005, Pages 815-821.

[17] L. Mirkin, *"On the H_∞ fixed-lag smoothing: How to exploit the information preview,"* Automatica, Volume 39, no 8, 2003, Page(s): 1495-1504.

[18] L. Mirkin, G. Tadmor, *"Yet another H_∞ discretization,"* Technical report ETR-2002-3, Technion—IIT (it can be downloaded from http://meeng.technion.ac.il/Research/TReports/2002/ETR-2002-03.html)

[19] A. A. Moelja, G. Meinsma, *"H_2 control of preview systems,"* Automatica, 42(6), June 2006, 945-952.

[20] E. Mosca, G. Zappa, *"Matrix fraction solution to the discrete-time LQ stochastic tracking and servo problems,"* IEEE Transactions on Automatic Control, Volume 34, Issue 2, Feb 1989 Page(s):240 - 242

[21] J. P. Hespanha, P. Naghshtabrizi and Y. Xu, "Networked Control Systems: Analysis and Design," *Proc. of the IEEE*, 95(1), Jan. 2007, pp. 138-162.

[22] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla and S. S. Sastry, "Foundations of Control and Estimation over Lossy Networks", *Proc. of the IEEE*, 95(1), Jan. 2007, pp. 163-187.

[23] Sebek, M.; Kucera, V., *"Polynomial approach to quadratic tracking in discrete linear systems,"* IEEE Transactions on Automatic Control, Volume 27, Issue 6, Dec 1982 Page(s): 1248 - 1250

[24] M. Sternad and T. Soderstrom, *"LQG-Optimal feedforward regulators,"* Automatica, vol. 24, no. 4, pages 557-561, 1988.

[25] G. Tadmor, L. Mirkin, *"H_∞ Control and Estimation With Preview - Part I: Matrix ARE Solutions in Continuous Time, "*, Automatic Control, IEEE Transactions on, Volume 50, No 1, Jan 2005, pp. 19 - 28

[26] G. Tadmor, L. Mirkin, *"H_∞ Control and Estimation With Preview - Part II: Fixed-Size ARE Solutions in Discrete Time, "*, Automatic Control, IEEE Transactions on, Volume 50, No 1, Jan 2005, pp. 29 - 40

[27] M. Tomizuka, *"Optimal continuous finite preview problem,"* Automatic Control, IEEE Transactions on, Volume 20, Issue 3, March 1975, Page(s): 362-265