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On the Stability of the Recursive Kalman Filter for Linear Time-Invariant Systems

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***Abstract*— Stability of the Kalman filter for linear time-invariant systems is studied from the perspective of divergence of the estimation error under incorrect noise measurement. We provide testable, necessary and sufficient conditions for the filter to be stable, stable with respect to perturbations in the initial error covariance, or semi-stable, respectively meaning that the estimate error covariance is bounded, bounded for perturbations in the initial error covariance or that it does not diverge exponentially. Previous conditions present some degrees of conservativeness or address only stability.**

I. Introduction

Consider the linear, time-invariant system defined in a fundamental probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\Phi : \begin{cases} x(k+1) = Ax(k) + Bw(k), \quad x(0) = x_0, \\ y(k) = Cx(k) + Dv(k), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^r$ is the observed variable, $w \in \mathbb{R}^p$ and $v \in \mathbb{R}^q$ form stationary zero-mean independent white noise processes satisfying $\mathcal{E}\{w(k)w(k)'\} = I$ and $\mathcal{E}\{v(k)v(k)'\} = I$, and the (independent) random variable x_0 is such that $\mathcal{E}\{x_0\} = \bar{x_0}$ and $\mathcal{E}\{x_0 x_0'\} = \Psi$. Assume that matrices A, C and D of appropriate dimensions and with $DD' > 0$, are known. Regarding B and Ψ, the available data are E and Σ, respectively.

It is a well-known fact that, for the Kalman filter (KF) in the above scenario, the *calculated* state estimates (taking into account E and Σ) may diverge from the *actual* ones (taking into account the actual data B and Ψ) with a fast rate, a phenomena usually referred to as divergence under incorrect noise measurement. It is particularly problematic for applications that there can be no indication of divergence when implementing the filter, in the sense that the calculated error covariance P_k is bounded, whereas the actual error covariance $\tilde{X}_k$ may diverge exponentially as $k \to \infty$.

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Divergence for KFs has been studied in the context of time-invariant systems and other setups, see e.g. [2], [8], [10], [13], [14], [15]. However, there are two important gaps. Firstly, available conditions present some degrees of conservativeness, as they rely on detectability assumptions or on the existence of a stationary solution to the filtering problem; a more complete assessment of available results is presented in Section II. However, there may exist neither a stationary solution nor a limiting gain, see e.g. [5, Example 3]. Secondly, available results concern only stability, meaning from the point of view of divergence analysis that the the actual error covariance is *bounded* for perturbations on *both* E and Σ. In many applications (particularly in finite-time contexts) it may be acceptable that the actual error does not diverge exponentially, or diverges with a “specified” rate; on the other hand, imprecise Σ can be handled without requiring stability.

In this paper, we consider *semi-stability*, requiring that X_k does not diverge exponentially for both incorrect E and Σ, *stability with respect to (w.r.t.)* Ψ requiring that X_k is bounded for incorrect Σ, and *stability*, requiring that X_k is bounded, for both incorrect E and Σ. We are concerned with divergence of the actual error under bounded calculated error, hence we assume that P_k is bounded. Based on the recent results in [5], [6], we obtain a testable necessary and sufficient condition for semi-stability, relying only on A, E and Σ, the relevant data for this purpose. This result is combined with existing conditions to obtain necessary and sufficient conditions for stability and stability w.r.t. Ψ.

The derived condition for semi-stability of the KF can be interpreted as requiring that Σ completely excites unstable modes of A that are not already excited by E, formally,

$$\ker\{J_H \Sigma J_H'\} \cap \mathcal{J}_H = \{0\}.$$

where J_H is the similarity matrix such that $J_H A H J_H^{-1}$ is in Jordan form, H stands for the orthogonal projection into the non-controllable subspace of (A,E) and $\mathcal{J}_H$ is the unstable subspace of $J_H A H J_H^{-1}$. This condition appears in the literature of ARE, as a necessary and sufficient condition for Σ to belong to the basin of attraction of the strong solution of the ARE for *detectable systems*, see Proposition 3. From this standpoint, the present paper is relevant as it clarifies the meaning of the above condition in the scenario of non-detectable systems.

The paper is organised as follows. In Section II we formalise the stability notions and present preliminary results concerning KFs and AREs. In Section III we derive the testable condition for semi-stability of the KF, and in Section IV we extend the result to stability. Finally, Section V provides concluding remarks.

II. Definitions and Preliminary Results

Let $\mathbb{D}$ (respectively, $\bar{\mathbb{D}}$) be the open (closed) unit disk in the complex plane. Let $\mathcal{R}^{r,s}$ (respectively, $\mathcal{R}^{r}$) represent the normed linear space formed by all $r \times s$ real matrices (respectively, $r \times r$) and $\mathcal{R}^{r*}$ ($\mathcal{R}^{r0}$) the cone $\{U \in \mathcal{R}^r : U = U'\}$ (the closed convex cone $\{U \in \mathcal{R}^r : U = U' \geq 0\}$) where U' denotes the transpose of U; $U \geq V$ signifies that $U - V \in \mathcal{R}^{r0}$. For $U \in \mathcal{R}^n$, $\lambda_i(U)$, $i = 1, \ldots, n$, stands for an eigenvalue of U. $\lambda_i(U)$ lying in $\bar{\mathbb{D}}$ (respectively, $\mathbb{D}$) is referred to as a semi-stable[1] (stable) eigenvalue of U; an associated eigenvector $v \in \mathbb{R}^n$ is semi-stable (stable), otherwise it is unstable. The space spanned by stable eigenvectors of U is referred to as the stable subspace of U, and similarly for the semi-stable and unstable spaces.

The Kalman filter estimates is given by $\hat{x}(0) = \bar{x}_0$ and

$$\hat{x}(k+1) = A\hat{x}(k) + L_k[y(k) - C\hat{x}(k)], \quad k \geq 0, \tag{2}$$

where $L_k = AP_kC'[CP_kC' + DD']^{-1}$ is referred to as the Kalman gain, calculated for the available noise covariances E and Σ via the following Riccati difference equations (RDEs),

$$P_{k+1} = A[P_k - P_kC'(CP_kC' + DD')^{-1}CP_k]A' + EE', \quad k \geq 0, \tag{3}$$

with initial condition $P_0 = \Sigma$. P_k describes the covariance of the estimation error at time k when $E = B$ and $\Sigma = \Psi$, and it is referred to as the *calculated* error covariance. Since we are concerned with divergence of actual error *under bounded calculated error*, it is natural to consider the next assumption.

Assumption 1. *For each $\Sigma \in \mathcal{R}^{n0}$ there is $\bar{X} \in \mathcal{R}^{n0}$ such that $P_k \leq \bar{X}$, $k \geq 0$.*

The *actual* error covariance $\tilde{X}_k(B,\Psi) = \mathcal{E}\{\tilde{x}(k)\tilde{x}(k)'\}$, where $\tilde{x}(k) = \hat{x}(k) - x(k)$, is calculated from (1) and (2), see e.g. [14], $\tilde{X}_0(B,\Psi) = \Psi$ and, for $k \geq 0$,

$$\tilde{X}_{k+1}(B,\Psi) = (A - L_kC)\tilde{X}_k(B,\Psi)(A - L_kC)' + L_kDD'L_k' + BB'. \tag{4}$$

We omit variables B and Ψ when $B = E$ and $\Psi = \Sigma$, respectively, e.g. we write $\tilde{X}_k(E,\Psi) = \tilde{X}_k(\Psi)$.

This paper is devoted to study the behaviour of $\tilde{X}_k$, which can be a difficult task in view of the following facts: L_k is a time-varying gain; it is calculated via the RDE; it is not the "optimal" gain in the sense that it is calculated for the available data E and Σ whereas the initial condition and forcing terms of (4) are related to the actual data B and Ψ. The task is much simpler when $L_k = L$ is a fixed gain (stationary filter), e.g. it is a well-known fact that if $A - LC$ is not stable and the forcing terms excites all modes of $A - LC$, then $\tilde{X}$ diverges at least linearly; we present the next related result, for later reference.

Proposition 1. *Assume $L_k = L$ is such that $A - L_kC$ is not a stable matrix, and $BB' > 0$. Then there exists $M \geq 0$ such that $\|\tilde{X}_M\| \geq \|\tilde{X}_0\| + 1$.*

The stability notions for the KF are as follows.

Definition 1. *We say that the KF is:*
(i) stable *if, for each $B \in \mathcal{H}^{n,p}$ and $\Psi \in \mathcal{R}^{n0}$, there exists $\bar{X}$ such that $\tilde{X}_k(B,\Psi) \leq \bar{X}$, $k \geq 0$.*
(ii) stable w.r.t. Ψ *if, for each $\Psi \in \mathcal{R}^{n0}$ there exists $\bar{X}$ such that $\tilde{X}_k(\Psi) \leq \bar{X}$, $k \geq 0$.*
(iii) semi-stable *if, for each $B \in \mathcal{H}^{n,p}$, $\Psi \in \mathcal{R}^{n0}$ and $0 \leq \xi < 1$, there exists $\bar{X}$ such that $\tilde{X}_k(B,\Psi) \leq \xi^{-2k}\bar{X}$, $k \geq 0$.*

Classical sufficient conditions for stability of the KF rely on detectability and stabilizability and can be traced back to the sixties, see [13]. More recent results appear in [14]. Consider the algebraic Riccati equation (ARE) in the variable $P \in \mathcal{R}^{n0}$, associated with the RDE (3),

$$P = A[P - PC'(CPC' + DD')^{-1}CP]A' + BB'. \tag{5}$$

P and L are said to be stabilising when the stationary gain $L = APC'[CPC' + DD']^{-1}$ is such that $A - LC$ is stable, and similarly for semi-stabilizing P and L. P and L are said to be strictly semi-stabilizing when they are semi-stabilizing but not stabilizing. Consider the following hypothesis:

H1. (A,C) is detectable.
H2. Unreachable modes of (A,E) do not lie on the unit circle.

Remark 1. H1 implies Assumption 1, see e.g. [1].

Proposition 2 ([14])**.** *Assume H1–H2 hold. If P_k converges to a stabilising P or, in particular, if either (A,E) is stabilizable or $\Sigma > 0$, then the KF is stable.*

A useful condition for $P_0 = \Sigma$ to converge to a semi-stabilizing solution of the ARE, related to the condition of Proposition 2, is presented in [3, Theorem 1]. We now convert this condition into a form convenient for comparisons purposes, and which involves the use of the orthogonal projection H into the non-controllable subspace. Consider the controllability matrix $\mathcal{C} = \begin{bmatrix} E & AE & \cdots & A^{n-1}E \end{bmatrix}$, let $H = I - (\mathcal{C}\mathcal{C}')^*\mathcal{C}\mathcal{C}'$ where $(\mathcal{C}\mathcal{C}')^*$ stands for the pseudo-inverse of $\mathcal{C}\mathcal{C}'$, J_H be the similarity transformation such that $J_HAHJ_H^{-1}$ is in Jordan form and $\mathcal{J}_H$ be the unstable space of $J_HAHJ_H^{-1}$.

H3. $\ker\{J_H\Sigma J_H'\} \cap \mathcal{J}_H = \{0\}$.

Proposition 3 ([3])**.** *Assume H1 holds. H3 is a necessary and sufficient condition for $\lim_{k\to\infty} P_k = P_S$, given $P_0 = \Sigma$, where P_S is the semi-stabilizing solution of the ARE* (5).

We shall also need the following result characterizing P_S.

[1] Terminology borrowed from [1].

Proposition 4 ([7], [9], [12]). *Assume H1 holds. The semi-stabilizing solution of the ARE* (5) *is stabilising if and only if H2 holds.*

It is quite standard and natural to consider H1 in the quest for stability of the KF, but it is constrictive when dealing with semi-stability. Actually, even the weaker assumption of convergence of P_k is conservative, see e.g. a semi-stable filter with periodic P_k in [5, Example 3]. The main consequence for the study of semi-stability is that it can not benefit from tools like orderings and comparisons involving P, P_k and $\tilde{X}$, which are common in the literature of KF and RDE. In fact, rather than employing those tools, our approach for semi-stability explores links between a condition for semi-stability of the KF and a condition for partial stability of a certain non-linear system, which have been recently developed in [5], [6]. The results that are essential for our approach are presented hereafter. For $U,V \in \mathcal{R}^n$ and $W \in \mathcal{R}^{n0}$, consider the non-linear system

$$\Theta(U,V,W):\begin{cases} Z_{k+1} = H_k U Z_k U' H_k', \\ X_{k+1} = U X_k U' + V V', \qquad k \geq 0, \\ (Z_0, X_0) = (H_0 H_0', W) \end{cases} \tag{6}$$

where (Z_k, X_k) is the state, $Z_k, X_k \in \mathcal{R}^{n0}$; H_k, $k \geq 0$, stands for the orthogonal projection onto $\ker\{X_k\}$. We employ the notation $X_k(U,V,W)$ and $Z_k(U,V,W)$ to emphasise the dependence on the variables U,V and W. Note that

$$\begin{aligned} X_k(U,V,W) &= U^k W U^{k\prime} + \sum_{\ell=0}^{k-1} U^\ell V V' U^{\ell\prime} \\ Z_k(U,V,W) &= \Big(\Pi_{\ell=1}^k (H_\ell U)\Big)(H_0 H_0')\Big(\Pi_{\ell=1}^k (H_\ell U)\Big)' \end{aligned} \tag{7}$$

where $H_k = H_k(U,V,W)$ stands for the projection onto the null space of $X_k(U,V,W)$.

Proposition 5 ([5]). *Consider Assumption 1. Consider system $\Theta(A,E,\Sigma)$ and the corresponding state component $Z_k(A,E,\Sigma)$. The KF is semi-stable if and only if, for each $0 \leq \xi_Z < 1$ there exists $\bar{Z} \in \mathcal{R}^{n0}$ such that $\xi_Z^{2k} Z_k \leq \bar{Z}$, $k \geq 0$.*

Proposition 6 ([6]). *Consider system $\Theta(A,0,\Sigma)$ and the corresponding state component $Z_k(A,0,\Sigma)$. Let J represent the similarity matrix for which JAJ^{-1} is in Jordan form and let $\mathcal{J}$ stand for the unstable space of JAJ^{-1}. For each $0 \leq \xi_Z < 1$ there exists $\bar{Z} \in \mathcal{R}^{n0}$ such that $\xi_Z^{2k} Z_k \leq \bar{Z}$, $k \geq 0$, if and only if*

$$\ker\{J\Sigma J'\} \cap \mathcal{J} = \{0\}. \tag{8}$$

Moreover, if $V \in \mathcal{R}^{n0}$ satisfies $\ker\{V\} \cap \mathcal{J} = \{0\}$ then $\ker\{A^k V A'^k\} \cap \mathcal{J} = \{0\}$, $k \geq 0$.

III. A necessary and sufficient condition for semi-stability

In this section we start adapting the condition in Proposition 6 to a general system $\Theta(A,E,\Sigma)$. The adapted condition is directly extended to semi-stability of the KF via Proposition 5. The basic idea is to employ an intermediate form $\Theta(HA,0,\Sigma)$ matching the setup of Proposition 6 and simultaneously related to the setup of Proposition 5. Let $X_{\mathrm{nf},k}$ and $X_{\mathrm{f},k}$ stand for the free and forced solutions of system $\Theta(A,\mathcal{C},\Sigma)$, given by

$$\begin{aligned} X_{\mathrm{nf},k} &= A^k \Sigma A^{k\prime}; \\ X_{\mathrm{f},k} &= \sum_{\ell=0}^{k-1} A^\ell \mathcal{C}\mathcal{C}' A^{\ell\prime}, k \geq 1; \;\; X_{\mathrm{f},0} = 0, \end{aligned} \tag{9}$$

in such a manner that $X_k(A,\mathcal{C},\Sigma) = X_{\mathrm{nf},k} + X_{\mathrm{f},k}$. We introduce the orthogonal projections $H, H_{\mathrm{f},k} \in \mathcal{R}^n$ as

$$\begin{aligned} H &= I - (\mathcal{C}\mathcal{C}')^*(\mathcal{C}\mathcal{C}'), \\ H_{\mathrm{f},k} &= I - (X_{\mathrm{f},k})^*(X_{\mathrm{f},k}). \end{aligned} \tag{10}$$

The next result follows from basic matrix properties [11].

Lemma 1. $H_k = (I - (H_{\mathrm{f},k} X_{\mathrm{nf},k} H_{\mathrm{f},k}')^*(H_{\mathrm{f},k} X_{\mathrm{nf},k} H_{\mathrm{f},k}'))H_{\mathrm{f},k}$.

Lemma 2. *The following statements hold.*
(i) $\ker\{X_{f,k}\} = \ker\{\mathcal{C}\mathcal{C}'\}$, $k \geq 0$.
(ii) $H_{\mathrm{f},k} = H$, $k \geq 1$, *and* $H_{\mathrm{f},0} = I$.

Proof. (i). From (9) it follows that $X_{\mathrm{f},1} = \sum_{\ell=0}^{0} A^\ell \mathcal{C}\mathcal{C}' A^{\ell\prime} = \mathcal{C}\mathcal{C}'$, so it is enough to show that $\ker\{X_{\mathrm{f},k}\} = \ker\{X_{\mathrm{f},1}\}$, $k \geq 0$. For each $v \in \mathbb{R}^n$ such that $v' X_{f,1} v = 0$, we have that

$$\begin{aligned} 0 &= v' X_{\mathrm{f},1} v = v' \mathcal{C}\mathcal{C}' v \\ &= v'(EE' + AEE'A' + \cdots + A^{n-1}EE'A'^{n-1})v, \end{aligned}$$

leading to $v' A^\ell EE' A'^\ell v = 0$, $0 \leq \ell \leq n-1$. Employing the Cayley-Hamilton theorem one can check that $v' A^\ell EE' A'^\ell v = 0$, $\ell \geq 0$, yielding

$$\begin{aligned} v' X_{\mathrm{f},k} v &= v' \Big[\sum_{\ell=0}^{k-1} A^\ell \mathcal{C}\mathcal{C}' A^{\ell\prime}\Big] v \\ &= v' \Big[\sum_{\ell=0}^{k-1} A^\ell (EE' + AEE'A' + \cdots + A^{n-1}EE'A'^{n-1}) A^{\ell\prime}\Big] v = 0 \end{aligned}$$

and we conclude that $\ker\{X_{\mathrm{f},1}\} \subset \ker\{X_{\mathrm{f},k}\}$. For the converse relation, note that

$$X_{\mathrm{f},k} = \sum_{\ell=0}^{k-1} A^\ell \mathcal{C}\mathcal{C}' A^{\ell\prime} = A\Big(\sum_{\ell=0}^{k-2} A^\ell \mathcal{C}\mathcal{C}' A^{\ell\prime}\Big)A' + \mathcal{C}\mathcal{C}',$$

hence for each $v \in \mathbb{R}^n$ for which $v' X_{\mathrm{f},k} v = 0$ we can write $v' X_{\mathrm{f},1} v = v' \mathcal{C}\mathcal{C}' v = 0$, and thus $\ker\{X_{\mathrm{f},1}\} \supset \ker\{X_{\mathrm{f},k}\}$. (ii) the result is trivial for $k = 0$, as $H_{\mathrm{f},0}$ is the projection onto the null space of $X_{\mathrm{f},0} = 0$; for $k \geq 1$ it follows immediately from assertion (i). □

Lemma 3. $(HA)^k \sigma = HA^k \sigma$, $k \geq 1$.

Proof. For $k = 1$ the result is trivial. From (10), H is the orthogonal projection onto $\ker\{\mathcal{C}\mathcal{C}'\} = \ker\{\mathcal{C}'\}$. Then, for each $v \in \mathbb{R}^n$, $(I-H)v \perp \ker\{\mathcal{C}'\}$. Since the space orthogonal to $\ker\{\mathcal{C}\}$ is A-invariant [4], we have that $A(I-H)v \perp \ker\{\mathcal{C}'\}$ and $HA(I-H)v = 0$. Therefore setting $v = A\sigma$, yields

$$\begin{aligned} (HA)^k \sigma &= (HA)^{k-1}[A\sigma] + (HA)^{k-1}[(H-I)A\sigma] \\ &= (HA)^{k-1} A\sigma, \; k \geq 2, \end{aligned}$$

and the result follows in a straightforward manner. $\square$

Lemma 4. *The following statements hold:*
(i) $H_0 = I - \Sigma^* \Sigma$;
(ii) $H_k = (I - ((HA)^k \Sigma (HA)^{k\prime})^* (HA)^k \Sigma (HA)^{k\prime}) H$, $k \geq 1$.

Proof. (i) is immediate from the definition of H_0. (ii) We employ Lemma 3 and assertion (ii) of Lemma 2, respectively, to write

$$(HA)^k \Sigma (HA)^{k\prime} = HA^k \Sigma A^{k\prime} H = H_{\mathrm{f},k} A^k \Sigma A^{k\prime} H'_{f,k} = H_{\mathrm{f},k} X_{\mathrm{nf},k} H'_{f,k}$$

and Lemma 1 completes the proof. $\square$

Lemma 5. *Consider the systems* $\Theta(A, \mathcal{C}, \Sigma)$ *and* $\Theta(HA, 0, \Sigma)$ *and the associated state trajectories.* $Z(A, \mathcal{C}, \Sigma)$ *is bounded if and only if* $Z(HA, 0, \Sigma)$ *is bounded.*

Proof. For ease of notation, let $Z_k(A, \mathcal{C}, \Sigma)$, $X_k(A, \mathcal{C}, \Sigma)$, $Z_k(HA, 0, \Sigma)$ and $X_k(HA, 0, \Sigma)$ be denoted respectively by Z_k, X_k, $\tilde{Z}_k$ and $\tilde{X}_k$. Let H_k and $\tilde{H}_k$ stand respectively for the projection onto $\ker\{X_k\}$ and $\ker\{\tilde{X}_k\}$. It is simple to check by inspection that $\tilde{X}_k = (HA)^k \Sigma (HA)^{k\prime}$ and

$$\tilde{H}_k = I - \tilde{X}_k^* \tilde{X}_k = I - ((HA)^k \Sigma (HA)^{k\prime})^* (HA)^k \Sigma (HA)^{k\prime})$$

in such a manner that from Lemma 4 we have that $H_k = \tilde{H}_k H$ and that

$$Z_{k+1} = H_k A Z_k A' H'_k = \tilde{H}_k (HA) Z_k (HA)' \tilde{H}'_k, \quad k \geq 0. \quad (11)$$

Since $Z_0 = H_0 H'_0 = (I - \Sigma^* \Sigma)(I - \Sigma^* \Sigma)' = \tilde{H}_0 \tilde{H}'_0 = \tilde{Z}_0$, employing (11) in a recursive fashion for $k = 0, 1, \ldots$, yields

$$\tilde{Z}_k = Z_k, \quad k \geq 0. \quad \square$$

Corollary 1. *Consider the system* $\Theta(A, \mathcal{C}, \Sigma)$ *and the associated state component* $Z_k(A, \mathcal{C}, \Sigma)$. *There exists* $\bar{Z} \in \mathcal{R}^{n0}$ *such that* $\xi_Z^{2k} Z_k \leq \bar{Z}$, $k \geq 0$, $0 \leq \xi_Z < 1$, *if and only if H3 holds.*

Proof. Proposition 6 with A replaced by HA provides that H3 holds if and only if $Z_k(HA, 0, \Sigma)$ is bounded, and this is equivalent to require that $Z_k(A, \mathcal{C}, \Sigma)$ is bounded, as stated in Lemma 5. $\square$

Lemma 6. *Consider the systems* $\Theta(A, E', \Sigma)$, $\Theta(A, \mathcal{C}, \Sigma)$ *and* $\Theta(A, \mathcal{C}, A^n \Sigma A'^n)$ *and the associated state trajectories. The following statements are equivalent:*
(i) $Z(A, E, \Sigma)$ *is bounded;*
(ii) $Z(A, \mathcal{C}, \Sigma)$ *is bounded;*
(iii) $Z(A, \mathcal{C}, A^n \Sigma A'^n)$ *is bounded.*

Proof. (i) $\Rightarrow$ (ii). Since $\mathcal{C}\mathcal{C}' = EE' + \cdots + A^{n-1} EE' A'^{n-1}$, one has that $EE' \leq \mathcal{C}\mathcal{C}'$, which yields $A^k EE' A^{k\prime} \leq A^k \mathcal{C}\mathcal{C}' A^{k\prime}$, $k \geq 0$, and

$$\begin{aligned} X_k(A, E, \Sigma) &= A^k \Sigma A^{k\prime} + \sum_{\ell=0}^{k-1} A^\ell EE' A^{\ell\prime} \\ &\leq A^k \Sigma A^{k\prime} + \sum_{\ell=0}^{k-1} A^\ell \mathcal{C}\mathcal{C}' A^{\ell\prime} = X_k(A, \mathcal{C}, \Sigma). \end{aligned} \quad (12)$$

Then, for each v such that $X_k(A, \mathcal{C}, \Sigma) v = 0$ we have that $v' X_k(A, E, \Sigma) v \leq v' X_k(A, \mathcal{C}, \Sigma) v = 0$, allowing to conclude that $X_k(A, E, \Sigma) v = 0$. This means that

$$\ker\{X_k(A, E, \Sigma)\} \supset \ker\{X_k(A, \mathcal{C}, \Sigma)\}$$

and recalling that $H(\cdot)$ is the projection onto the null space of the associated $X(\cdot)$, it is simple to check that for each $v \in \mathbb{R}^n$, $H_k(A, E, \Sigma) v = H_k(A, \mathcal{C}, \Sigma) v + r$ for some $r \perp H_k(A, \mathcal{C}, \Sigma) v$. This allows to evaluate, for each $V \in \mathcal{R}^{n0}$,

$$H_k(A, E, \Sigma) V H_k(A, E, \Sigma) \geq H_k(A, \mathcal{C}, \Sigma) V H_k(A, \mathcal{C}, \Sigma)$$

and to obtain the ordering condition

$$Z_k(A, E, \Sigma) \geq Z_k(A, \mathcal{C}, \Sigma).$$

(ii) $\Rightarrow$ (iii). The basic tool for this proof is provided by Proposition 6, with A replaced by HA and the associated similarity matrix J_H for which $\bar{A} = J_H (AH) J_H^{-1}$ is in Jordan form. Since $Z(A, \mathcal{C}, \Sigma)$ is bounded, from Corollary 1 we have that $\ker\{J_H \Sigma J_H^{-1}\} \cap \mathcal{J}_H = \{0\}$ and Proposition 6 yields

$$\ker\{\bar{A}^n (J \Sigma J^{-1}) \bar{A}'^n\} \cap \mathcal{J}_H = \{0\}. \quad (13)$$

Note that

$$\begin{aligned} \bar{A}^n (J \Sigma J^{-1}) \bar{A}'^n &= J_H \,(HA)^n J_H^{-1} \,(J_H \Sigma J'_H)(J_H \,(HA)^n J_H^{-1})' \\ &= J_H \,(HA)^n \Sigma (HA)^{n\prime} J'_H \end{aligned}$$

and from (13) we get that $\ker\{J_H \,(HA)^n \Sigma (HA)^{n\prime} J'_H\} \cap \mathcal{J}_H = \{0\}$. Employing the fact that $(HA)^n = HA^n$, see Lemma 3, we have that

$$\ker\{J_H \tilde{\Sigma} J'_H\} \cap \mathcal{J}_H = \{0\}$$

where $\tilde{\Sigma} = HA^n \Sigma A^{n\prime} H'$. Proposition 6 implies that $Z(HA, 0, \tilde{\Sigma})$ is bounded, and from Lemma 5 we get that $Z(A, \mathcal{C}, \tilde{\Sigma})$ is bounded. Now, recalling H is an orthogonal projection, one can check that

$$\tilde{\Sigma} = (W_0^{-1} W_n) HA^n \Sigma A'^n H' (W_0^{-1} W_n)' \leq A^n \Sigma A'^n,$$

hence

$$\begin{aligned} X_k(A, \mathcal{C}, \tilde{\Sigma}) &= A^k (\tilde{\Sigma}) A^{k\prime} + \sum_{\ell=0}^{k-1} A^\ell \mathcal{C}\mathcal{C}' A^{\ell\prime} \\ &\leq A^k (A^n \Sigma A'^n) A^{k\prime} + \sum_{\ell=0}^{k-1} A^\ell \mathcal{C}\mathcal{C}' A^{\ell\prime} \leq X_k(A, \mathcal{C}, A^n \Sigma A'^n). \end{aligned}$$

Note that this is in analogy with (12), and arguments similar to the ones of part (i) lead to

$$\begin{aligned} \ker\{X_k(A, \mathcal{C}, \tilde{\Sigma})\} &\supset \ker\{\bar{\kappa} X_k(A, \mathcal{C}, A^n \Sigma A'^n)\} \\ &= \ker\{X_k(A, \mathcal{C}, A^n \Sigma A'^n)\} \end{aligned}$$

and to the ordering condition

$$Z_k(A, \mathcal{C}, \tilde{\Sigma}) \geq Z_k(A, \mathcal{C}, A^n \Sigma A^n). \quad (14)$$

(iii) $\Rightarrow$ (i). In this part of the proof we shall need as an auxiliary system the version of system $\Theta(U, V, W)$ in (6) with the initial condition on the Z-component, $H_0 H'_0$, replaced by $H_0 \Delta H'_0$ with $\Delta \in \mathbb{R}^{n0}$; we refer to this system by $\Gamma(U, V, W, \Delta)$. Consider the system $\Gamma(A, \mathcal{C}, A^n \Sigma A'^n +$

$\mathcal{C}\mathcal{C}',\Delta)$, and let the associated state components be denoted simply by X_k and $Z_k(\Delta)$ and the associated projection onto the null space of the X-component be denoted by H_k. Consider the system $\Theta(A,E,\Sigma)$ and let the associated state components and projection be denoted by $\tilde{X}_k$, $\tilde{Z}_k$ and $\tilde{H}_k$. We set

$$\Delta = A\,(\tilde{H}_{n-1}A)\cdots(\tilde{H}_1A)\tilde{H}_0\tilde{H}_0'(\tilde{H}_1A)'\cdots(\tilde{H}_{n-1}A)'A'.$$

With this notation,

$$\begin{aligned}Z_0(\Delta) = H_0\big(A\,(\tilde{H}_{n-1}A)\cdots(\tilde{H}_1A)\tilde{H}_0\cdot \\ \cdot\tilde{H}_0'(\tilde{H}_1A)'\cdots(\tilde{H}_{n-1}A)'A'\big)H_0',\end{aligned}$$

Moreover, from (7) we have that

$$\tilde{X}_n = A^n\Sigma A^{n\prime} + \sum_{\ell=0}^{n-1} A^\ell EE'A^{\ell\prime} = A^n\Sigma A^{n\prime} + \mathcal{C}\mathcal{C}' = X_0, \qquad (15)$$

which yields $\tilde{H}_n = H_0$ and allows to write

$$\begin{aligned}\tilde{Z}_n &= (\tilde{H}_nA)\cdots(\tilde{H}_1A)(\tilde{H}_0\tilde{H}_0')(\tilde{H}_1A)'\cdots(\tilde{H}_nA)' \\ &= H_0\big(A(\tilde{H}_{n-1}A)\cdots(\tilde{H}_1A)(\tilde{H}_0\cdot \qquad (16)\\ &\quad \cdot\tilde{H}_0')(\tilde{H}_1A)'\cdots(\tilde{H}_{n-1}A)'A\big)H_0' = Z_0(\Delta).\end{aligned}$$

Equations (15) and (16) yield, for $k \geq 0$,

$$X_k = \tilde{X}_{k+n}, \quad Z_k(\Delta) = \tilde{Z}_{k+n}. \qquad (17)$$

Now we are ready to present the main arguments: since $Z_k(A,\mathcal{C},A^n\Sigma A'^n)$ is bounded by hypothesis, (omitting details) we have that there exists $\bar{Z}$ such that $Z_k \leq \bar{Z}$, $k \geq 0$, and (17) immediately leads to $\tilde{Z}_{k+n} \leq \bar{Z}$. This allows to write that $\tilde{Z}_k \leq \bar{z}I$, $k \geq 0$, where we set $\bar{z} = \max(\|\bar{Z}\|, \|\tilde{Z}_0\|, \ldots, \|\tilde{Z}_{n-1}\|)$. □

Lemma 6 allows for a direct extension of the necessary and sufficient condition for semi-stability of Corollary 1.

Theorem 1. *Consider Assumption 1. The KF is semi-stable if and only if H3 holds.*

Remark 2. H3 is not equivalent to $\ker\{\Sigma\}\cap\mathcal{N} = \{0\}$, where $\mathcal{N}$ stands for the unstable subspace of HA. This means that the condition cannot be expressed in terms of the original bases. For example, setting

$$A = \begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix},\ \Sigma = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},\ E = 0,$$
we have $H = I$, $J_HAHJ_H^{-1} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$, $J_H\Sigma J_H' = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$,

yielding $\ker\{\Sigma\}\cap\mathcal{N} \neq \{0\}$ but $\ker\{J_H\Sigma J_H'\}\cap\mathcal{J}_H = \{0\}$.

Remark 3. Assume H1. In this context, combining Proposition 3 and Theorem 1 we conclude that the KF is semi-stable if and only if P_k converges. This implies that a periodic behaviour for P_k is not allowed.

IV. Stability of the Kalman filter and comparisons with previous conditions

There is no available proof of necessity for the conditions in Proposition 2 for stability of the KF. The necessity of H1 (detectability of (A,C)) is a simple matter. However, if we disregard the results for semi-stability of the KF obtained in Section III, necessity of convergence of the RDE solution is far from obvious because, in principle, the solution could present a complex behaviour (e.g. periodicity) that is difficult to analyse. In a sense, this analysis is carried out in Section III where we indirectly show that such complex solutions are related to KFs that are not even semi-stable, see Remark 3.

Theorem 2. *The KF is stable if and only if H1, H2 and H3 hold.*

Proof. (Sufficiency). Propositions 4 and 3 yield that P_k converges to the stabilising solution of the ARE, thus satisfying the conditions of Proposition 2, which leads to the result.
(Necessity). Theorem 1 trivially makes clear that H3 is necessary for a KF to be stable, otherwise it is not even semi-stable. Regarding H1, if we assume that (A,C) is not detectable then there is at least one unobservable mode of A that is not stable, hence $A-L_kC$ preserves the same mode no matter how L_k is chosen and thus the KF is not stable. It only remains to show that H2 is required for stability of the KF, *under* H1 and H3. Let us deny this assertion and assume that H2 is not satisfyied and the KF is stable, hence from Definition 1 we have that for $\tilde{X}_k$ defined by

$$\tilde{X}_{k+1} = (A-L_kC)\tilde{X}_k(A-L_kC)' + L_kDD'L_k' + I, \qquad (18)$$

there is $\bar{X} \in \mathcal{R}^{n0}$ such that

$$\tilde{X}_k \leq \bar{X},\ k \geq 0. \qquad (19)$$

Proposition 3 states that P_k converges to the semi-stabilizing solution P_S, in such a manner that, for each $\varepsilon > 0$ exists $k_\varepsilon \geq 0$ such that

$$\|L_k - L\| \leq \varepsilon, \quad k \geq k_\varepsilon, \qquad (20)$$

where, according to Proposition 4, L is the (strictly) semi-stabilising gain $L = AP_SC'[CP_SC' + DD']^{-1}$. Let us define $\bar{X}_k \in \mathcal{R}^{n0}$, $k \geq k_\varepsilon$, by $\bar{X}_{k_\varepsilon} = \tilde{X}_{k_\varepsilon}$ and

$$\bar{X}_{k+1} = (A-LC)\bar{X}_k(A-LC)' + LDD'L' + I, \quad k \geq k_\varepsilon. \qquad (21)$$

One can check from (18), (20) and (21) that the deviation between $\bar{X}$ and $\tilde{X}$ is proportional to time and ε or, more precisely, that

$$\|\tilde{X}_k - \bar{X}_k\| \leq o(k-k_\varepsilon,\varepsilon)\|\tilde{X}(k_\varepsilon)\| \leq o(k,\varepsilon)\|\bar{X}\|, \quad k \geq k_\varepsilon, \qquad (22)$$

where $o(k,\varepsilon)$ is increasing with respect to k and ε, with $o(\cdot,0) = o(0,\cdot) = 0$. For the semi-stable matrix $A-LC$ and $\bar{X}$ as in (19), set M as in Proposition 1 and ε in such a manner that $o(M,\varepsilon) < 1/2\|\bar{X}\|^{-1}$. Then, we employ respectively the triangular inequality, Proposition 1 and (22) to evaluate

$$\|\tilde{X}_{k_\varepsilon+M}\| \geq -\|\tilde{X}_{k_\varepsilon+M} - \bar{X}_{k_\varepsilon+M}\| + \|\bar{X}_{k_\varepsilon+M}\| \\ \geq -o(M,\varepsilon)\|\bar{X}\| + \|\tilde{X}_{k_\varepsilon}\| + 1 > \|\tilde{X}_{k_\varepsilon}\| + 1/2.$$

For each integer $m \geq 1$ we can proceed similarly as above (replacing $\bar{X}_{k_\varepsilon}$ and $\tilde{X}_{k_\varepsilon}$ with $\bar{X}_{k_\varepsilon+mM}$ and $\tilde{X}_{k_\varepsilon+mM}$, respectively) to conclude that $\|\tilde{X}_{k_\varepsilon+mM}\| \geq \|\tilde{X}_{k_\varepsilon+(m-1)M}\| + 1/2$, $m \geq 1$, yielding

$$\|\tilde{X}_{k_\varepsilon+mM}\| \geq \|\tilde{X}_{k_\varepsilon}\| + m/2 \geq m/2, \quad m \geq 1,$$

which is an absurd in view of (19). □

Remark 4. H3 requires that Σ completely excites unstable spaces of A that are not already excited by E. H2 further requires that the strictly semi-stable (i.e., semi-stable but not stable) spaces of A are excited by E. Indeed, for granting stability it is not enough to excite semi-stable modes via Σ, since this noise may vanish as time evolves and the KF may "deteriorate" to a semi-stable one.

The situation when all unstable modes of A are excited by Σ or E, but some of the semi-stable modes of A are excited only by Σ, yields stability w.r.t. Ψ. The proof for this fact involves several adaptations of results in [5], [6] and therefore is not presented.

Proposition 7. *Let H be as in* (10)*, J_H be the similarity transformation such that $J_H A H J_H^{-1}$ is in Jordan form and $\mathcal{J}_S$ be the subspace spanned by the eigenvectors associated with the eigenvalues of $J_H A H J_H^{-1}$ lying inside $\mathbb{D}$. If $\ker\{J_H \Sigma J_H'\} \cap \mathcal{J}_S^\perp = \{0\}$ then the KF is stable w.r.t. Ψ.*

We finish the section with some links with classical conditions for stability of KF.

Proposition 8. (A,E) *semi-stabilizable or* $\Sigma > 0$ *implies H3.*

Proof. Provided (A,E) is semi-stabilizable, it is a straightforward matter to check that HA is a semi-stable matrix, recalling that the projection H "cancels" controllable dynamics of A, and semi-stabilizability of (A,B) yields that the remaining dynamics are semi-stable. A semi-stable HA leads to $\mathcal{J}_H = \{0\}$ and H3 holds. In particular, for controllable (A,B) one can easily check that $H = 0$. For $\Sigma > 0$, it is immediate that $\ker\{J_H \Sigma J_H'\} = \{0\}$. □

Example 1. *Consider system Φ with*

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},\ C = D = I,\ \Sigma = \sigma'\sigma,\ E = v'v,$$

where $\sigma, v \in \mathbb{R}^{1,n}$. Consider the following setups. (i) $\sigma = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $v = 0$. It is simple to check that: Assumption 1 holds (since H1 holds), $\mathcal{C} = 0$, $H = I$, AH is in Jordan form and $J_H = I$. Although (A,Σ) is stabilizable, we have that $\ker\{\Sigma\} \cap \mathbb{R}^n = [\mu]$ where $\mu = [1 \quad 0]'$, thus H3 does not hold and the KF is not semi-stable. (ii) $\sigma = 0$ and $v = \begin{bmatrix} 0 & 1 \end{bmatrix}$. A stabilizable (A,E) is enough to provide stability; indeed, now $\mathcal{C} = I$, $H = 0$ and $AH = 0$, providing $J_H = \{0\}$ and H3 holds trivially. Note that Σ in (i) equals E in (ii); this illustrates in what sense Σ has to "completely" excite the unstable spaces of A (whereas E has only to excite them).

V. Concluding remarks

We have shown that KFs presenting bounded error covariance P_k are semi-stable if and only if H3 holds, i.e., the non-persistent noise model employed in calculations (characterised by Σ) has to completely excite unstable modes of A that are not already excited by E. The result is valid independently of convergence of P_k or any conditions on C, hence clarifying the meaning of the condition H3 for non-detectable systems (or non-stabilizable systems in the dual control scenario), complementing available results for detectable systems [3]. We combine the condition H3 with conditions H1 and H2 to obtain a necessary and sufficient condition for stability of the KF. A condition for stability w.r.t. Ψ is also derived, which holds trivially when $\Sigma > 0$, see Proposition 7.

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