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Robust stability analysis and control design for time-varying discrete-time polytopic systems with bounded parameter variation

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Abstract— **This paper investigates the problems of robust stability analysis and state feedback control design for discrete-time linear systems with time-varying parameters. It is assumed that the time-varying parameters lie inside a polytopic domain and have known bounds on their rate of variation. By exploiting geometric properties of the uncertainty domain, linear matrix inequality conditions that take into account the bounds on the rates of parameter variations are proposed. A feasible solution provides a parameter-dependent Lyapunov function assuring the robust stability of this class of systems. Extensions to deal with robust control design as well as gain-scheduling by state feedback are also provided in terms of linear matrix inequalities. Numerical examples illustrate the results.**

I. INTRODUCTION

The robust stability analysis and control design of uncertain linear systems have deserved special attention in the last years, mainly due to the development of numerical tools based on parameter-dependent Lyapunov functions. For instance, it is notorious the large number of methods devoted to the problem of robust stability analysis that have recently appeared in the literature [1–8].

Particularly in the context of uncertain linear systems with time-varying parameters, the use of parameter-dependent Lyapunov functions that consider finite bounds on the rates of the parameter variations provided less conservative results than the methods (e.g. quadratic stability approach) assuming arbitrary parameter variation. For continuous-time systems with bounds on the time-derivatives of the parameters, it is worth of mentioning the results in [9–13].

Concerning discrete-time systems, most of the existing results consider that the uncertain parameters can vary arbitrarily inside the uncertainty domain. In this special case, quadratic Lyapunov functions with a parameter-independent (constant) matrix can provide sufficient conditions for robust stability and state feedback control [14, 15]. In [16], a set of linear matrix inequality (LMI) conditions that provide an affine parameter-dependent Lyapunov function assuring the robust stability of a discrete-time system with arbitrary parameter variation inside a polytope has been proposed. The results are only sufficient, but contain the conditions based on quadratic stability as a particular case and can provide less conservative robust stability evaluations. Moreover, an extension to cope with control design has also been presented. More recently, it has been shown in [17, 18] that the asymptotic stability of discrete-time systems with arbitrary time-varying parameters can be characterized by an increasing union of LMI conditions, based on path-dependent Lyapunov functions, encompassing both the quadratic stability based conditions and the LMIs given in [16].

On the other hand, when the time-varying parameters have bounds on their rate of variation, the literature presents few results. As remarked in [19], this situation occurs quite frequently in real world applications [20]. In [21], the case of affine parameter-dependent systems with one single parameter is considered. Sufficient conditions are given by dividing the interval where the parameter lies accordingly to the variation rates. This technique has been extended to the case of multi-affine parameter-dependence in [19], where a gain-scheduling state feedback controller that assures a prescribed $\mathscr{L}_2$ performance for the closed-loop system is also provided. It is also worth of mentioning the robust filter design LMI conditions given in [22] for affine parameter-dependent systems with bounded rates of variation.

This paper investigates the robust stability of discrete-time linear systems with time-varying parameters lying inside a polytope and having bounded rates of variation. LMI conditions that take into account the bounds on the rates of parameter variations and exploit the geometry of the uncertainty domain are given, providing a parameter-dependent Lyapunov function to assess the robust stability of the time-varying system. By means of numerical experiments, it is shown that the use of recent results concerning polynomial matrices with parameters in the simplex [8, 23] to solve the proposed conditions can effectively improve the robust stability evaluations for this class of systems. Design conditions for robust and gain-scheduling state feedback control are also given and illustrated by examples.

II. PRELIMINARIES

Consider the time-varying discrete-time linear system described by

$$x(k+1) = A(\alpha(k))x(k) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $A(\alpha(k)) \in \mathbb{R}^{n\times n}$ is the dynamic matrix given in the form

$$A(\alpha(k)) = \sum_{i=1}^{N} \alpha_i(k)A_i$$

and $\alpha(k) \in [\alpha_1(k)\cdots\alpha_N(k)]'$ is the vector of time-varying parameters lying in the unit simplex Λ_N for all $0 \le k \in \mathbb{N}$, where

$$\Lambda_N = \Big\{\alpha \in \mathbb{R}^N : \sum_{i=1}^{N} \alpha_i = 1,\ \alpha_i \ge 0,\ i = 1,\ldots,N\Big\}. \tag{2}$$

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For stability analysis purposes, the time-varying vector of parameters is supposed to be uncertain.

The rate of variation of the parameters is given by

$$\Delta\alpha_i(k) = \alpha_i(k+1) - \alpha_i(k), \quad i = 1,\ldots,N \tag{3}$$

An immediate consequence from (2) and (3) is that

$$\sum_{i=1}^{N} \Delta\alpha_i(k) = 0 \tag{4}$$

Moreover, it is assumed that $\Delta\alpha_i(k)$ is bounded and satisfying the following condition

$$-b \leq \Delta\alpha_i(k) \leq b, \quad b \in \mathbb{R},\ b \in [0,1]. \tag{5}$$

The case $b = 0$ in (5) (i.e. frozen parameters) corresponds to the classical robust stability analysis problem of time-invariant uncertain discrete-time linear systems in polytopic domains (also known as *polytopic systems*), for which convergent LMI relaxations based on the existence of homogeneous polynomial solutions [23] are available in the literature [6, 8].

On the other hand, the case $b = 1$, where the parameters are allowed to vary arbitrarily inside Λ_N from the instant k to the instant $k+1$, has been handled by affine parameter-dependent Lyapunov functions in [16], yielding less conservative results than quadratic stability [14]. A more general result has been given in [18], encompassing both the quadratic stability and the results from [16] and also presenting convergent LMI relaxations to assess robust stability of time-varying discrete-time systems, a class which include switched systems. To the best of authors' knowledge, the situation $0 < b < 1$ was not investigated in the literature for polytopic systems and is the main concern of the present paper. As a matter of fact, many discrete-time systems with time-varying parameters have known bounds on their variation, as discussed for instance in [19, 22], which investigate robust stability and control/filtering design methods for affine or multi-affine linear systems with uncertain parameters with bounded rates of variation.

Next section discusses some geometric aspects concerning the constraints (5) and proposes a convex model to represent the vector $\Delta\alpha(k) = [\Delta\alpha_1(k)\ \cdots\ \Delta\alpha_N(k)]'$ taking into account that $\alpha \in \Lambda_N$ for all $k \in \mathbb{N}$.

A. Parameter variation modeling

Following some of the lines that have been used in the continuous-time case [11, 12], i.e. to model the space where the bounds on time-derivatives can lie, the vector $\Delta\alpha(k)$ can be assumed to belong, for all $k \geq 0$, to the compact set

$$\Gamma_b = \left\{ \delta \in \mathbb{R}^N : \delta = co\{h^1 \cdots h^M\}, \sum_{i=1}^{N} h_i^j = 0,\ j = 1,\ldots,M \right\} \tag{6}$$

defined as the convex combination of vectors $h^j,\ j = 1,\ldots,M$, given *a priori*. Since $\Delta\alpha(k) \in \Gamma_b$ (and $\alpha(k) \in \Lambda_N$) for all $k \geq 0$, from this point on the explicit dependence on k will be omitted whenever no confusion arises. Notice that this definition of Γ_b ensures (4) for all $k \geq 0$ and that $0 \in \mathbb{R}^N$ belongs to Γ_b. The vectors $h^j \in \Gamma_b,\ j = 1,\ldots,M$ can be constructed in a systematic way from a given b. Basically, the procedure searches recursively for all possible solutions of (4) using the extreme points of the constraints given in (5) for $i = 1,\ldots,N$.

At this point, the procedure used here would be similar to the one used in the continuous-time case [11, 12], except that the value of b is limited to 1 for the discrete-time case. However, this representation introduces conservativeness by considering that the bounds $\Delta\alpha_i$ are independent of α_i. Actually, the values of $\Delta\alpha_i$ are highly dependent of α_i, as illustrated in Figure 1. By considering (5), the whole gray region would be taken into account (thus producing conservative results), while in fact only the dashed region represents feasible values of $(\Delta\alpha_i, \alpha_i)$.

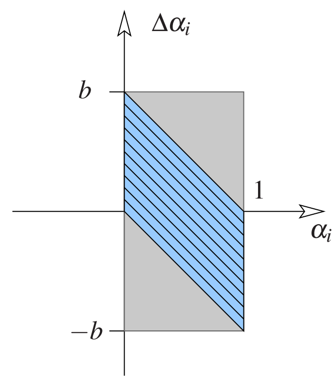

Fig. 1. Region on the plane $\Delta\alpha_i \times \alpha_i$ where $\Delta\alpha_i$ can assume values as a function of α_i (dashed region).The gray region is unreachable since $\alpha \in \Lambda_N$.

To consider only the valid region (dashed), $\Delta\alpha_i$ must be bounded by

$$-b\alpha_i \leq \Delta\alpha_i \leq b(1-\alpha_i), \quad i = 1,\ldots,N. \tag{7}$$

In this case, the algorithm used to solve (4) under (5) can be easily adapted to cope also with (7), producing the set $\Gamma_{b\alpha}$ with columns $h^j,\ j = 1,\ldots,M$ which now depend on both b and α.

Actually, the columns h_j of the set $\Gamma_{b\alpha}$ are given by

$$\begin{bmatrix} h^1 & h^2 & \cdots & h^M \end{bmatrix} = b \begin{bmatrix} 1-\alpha_1 & -\alpha_1 & -\alpha_1 & \cdots \\ -\alpha_2 & 1-\alpha_2 & -\alpha_2 & \cdots \\ \vdots & \vdots & \ddots & \cdots \\ -\alpha_N & -\alpha_N & -\alpha_N & 1-\alpha_N \end{bmatrix} \tag{8}$$

Taking the convex hull of the M columns of matrix $\Gamma_{b\alpha}$, one has for each $\Delta\alpha_i$ the following expression

$$\Delta\alpha_j = b\big(\beta_j - \alpha_j(\beta_1 + \cdots + \beta_M)\big) = b\big(\beta_j - \alpha_j\big) \tag{9}$$

since $(\beta_1 + \cdots + \beta_M) = 1$ (i.e. β belongs to Λ_M) and $M = N$.

Next section proposes LMI conditions based on parameter-dependent Lyapunov functions to investigate the *robust stability* of system (1), i.e. if $x(k) = 0$ is a (global) uniformly asymptotically stable equilibrium point of system (1) for all $\alpha \in \Lambda_N$ and $\Delta\alpha \in \Gamma_{b\alpha}$.

III. ROBUST STABILITY ANALYSIS

Next theorem presents a sufficient condition for the robust stability analysis of (1) based on the existence of a quadratic in the state Lyapunov function.

Lemma 1: System (1) is robustly stable if there exists a bounded positive definite matrix sequence $P(\alpha(k))$ verifying the inequality

$$\Phi \triangleq \begin{bmatrix} P(\alpha(k)) & A(\alpha(k))'P(\alpha(k+1)) \\ \star & P(\alpha(k+1)) \end{bmatrix} > 0 \qquad (10)$$

for all $\alpha(k) \in \Lambda_N$ and $\Delta\alpha(k) \in \Gamma_{b\alpha}$, $k \geq 0$ ($\star$ stands for symmetry).

Proof: Follows from the evaluation of the first difference of the Lyapunov function $v(x(k),\alpha(k)) = x(k)'P(\alpha(k))x(k)$ along the solutions of system (1) and the use of Schur complement. ∎

Since the dependence of matrix $P(\alpha(k))$ on the parameters $\alpha(k)$ is not known *a priori*, the conditions of Lemma 1 are not numerically verifiable. By considering the particular structure

$$P(\alpha(k)) = \alpha_1(k)P_1 + \cdots + \alpha_N(k)P_N, \quad \alpha(k) \in \Lambda_N \qquad (11)$$

i.e. that $P(\alpha(k))$ is affine in the parameters, the following finite dimensional sufficient condition can be obtained.

Theorem 2: If there exist symmetric matrices $P_i \in \mathbb{R}^{n\times n}$, $i = 1,\ldots,N$ such that the following LMIs hold

$$\Phi_{i\ell} = \begin{bmatrix} P_i & A_i'\big((1-b)P_i + bP_\ell\big) \\ \star & (1-b)P_i + bP_\ell \end{bmatrix} > 0,$$
$$i = 1,\ldots,N, \quad \ell = 1,\ldots,N \qquad (12)$$

$$\Phi_{ij\ell} = \begin{bmatrix} P_i + P_j & (1-b)\big(A_i'P_j + A_j'P_i\big) + b\big(A_i' + A_j'\big)P_\ell \\ \star & (1-b)P_i + (1-b)P_j + 2bP_\ell \end{bmatrix} > 0,$$
$$i = 1,\ldots,N, \quad \ell = 1,\ldots,N, \quad j = i+1,\ldots,N-1 \qquad (13)$$

then system (1) is robustly stable.

Proof: First note that if $P(\alpha(k))$ is given by (11), then $P(\alpha(k+1))$ can be written as

$$\begin{aligned} P(\alpha(k+1)) &= \alpha_1(k+1)P_1 + \cdots + \alpha_N(k+1)P_N \\ &= (\alpha_1(k) + \Delta\alpha_1(k))P_1 + \cdots + (\alpha_N(k) + \Delta\alpha_N(k))P_N \\ &= \sum_{i=1}^{N} \alpha_i P_i + \sum_{\ell=1}^{N} \Delta\alpha_\ell P_\ell \\ &= \sum_{i=1}^{N} \alpha_i P_i + \sum_{\ell=1}^{N} b\big(\beta_\ell - \alpha_\ell\big)P_\ell \\ &= \sum_{i=1}^{N} \alpha_i (1-b) P_i + \sum_{\ell=1}^{N} b\beta_\ell P_\ell \end{aligned}$$

where the two last equations were obtained using the model for $\Delta\alpha$ in Section II-A. In this case, Φ given in (10) can be written as

$$\Phi = \sum_{\ell=1}^{N}\sum_{i=1}^{N} \alpha_i^2 \beta_\ell \Phi_{i\ell} + \sum_{\ell=1}^{N}\sum_{i=1}^{N}\sum_{j=i+1}^{N-1} \alpha_i\alpha_j\beta_\ell\Phi_{ij\ell}.$$

The feasibility of LMIs (12–13) guarantees that Φ is positive definite, and, from Lemma 1, system (1) is robustly stable. ∎

Theorem 2 provides sufficient conditions in terms of LMIs to assess the robust stability of system (1) whose conservativeness is analyzed in the sequel for different values of b.

Remark 1: If $b = 0$, the conditions of Theorem 2 retrieve the LMIs from [8, Theorem 3] for $g = 1$ (degree of the homogeneous polynomial Lyapunov matrix) and $d = 0$ (number of Pólya's relaxations). In this case, there are two sources of conservativeness: *i)* a polynomial Lyapunov matrix of higher degree on α may be necessary to assess stability (see also [23] for homogeneous polynomial solutions to parameter-dependent LMIs with parameters in the simplex); *ii)* Pólya's relaxations may be necessary also [24, 25];

Remark 2: The case $b = 1$ (the parameters can vary arbitrarily inside the polytope) reveals some interesting points. In fact, condition (12) simplifies to

$$\begin{bmatrix} P_i & A_i'P_\ell \\ \star & P_\ell \end{bmatrix} > 0, \qquad (14)$$

Multiplying (14) by $\alpha_i\alpha_\ell$ and summing up for $i = 1,\ldots,N$, $\ell = 1,\ldots,N$ one has

$$\begin{bmatrix} P(\alpha) & A(\alpha)'P(\beta) \\ \star & P(\beta) \end{bmatrix} > 0, \qquad (15)$$

with $P(\alpha)$ given by (11) and $\beta(k) = \alpha(k+1)$. In other words, the conditions of Theorem 2 reduce to the necessary and sufficient conditions for the existence of an affine parameter-dependent Lyapunov function that guarantees robust stability for arbitrary time-varying parameters as given in [16]. Note that conditions (13) are, in this case, positive linear combinations of conditions (12), being always feasible whenever (12) hold.

The parameter-dependent inequality (15) allows to conclude about an interesting point concerning the use of more complex (higher degrees) homogeneous polynomial Lyapunov matrices, as it has been done in the case of time-invariant parameters ($b = 0$) [8, 23]. Due to the product $A(\alpha)'P(\beta)$ in the parameter-dependent LMI (15), if one tries to use homogeneous polynomial Lyapunov matrices of higher degrees and to impose that all the matrix coefficients must be positive definite, the resulting constraints will also contain the LMIs in (14). In this case, if a feasible solution to the constraints (14) does not exist, i.e. there does not exist an affine Lyapunov matrix assuring robust stability, then there is no higher degree homogeneous polynomial Lyapunov matrix assuring robust stability. To obtain better estimates of robust stability in the case $b = 1$, more sophisticated analysis methods should be used, as the path-dependent Lyapunov matrices proposed in [18].

Finally, in the case $0 < b < 1$, the use of homogeneous polynomial Lyapunov matrix of higher degrees can effectively improve the robust analysis results when compared to the existing methods, as it will be illustrated later.

IV. CONTROL DESIGN

A. Robust State Feedback

Consider the discrete-time linear system

$$x(k+1) = A(\alpha(k))x(k) + B(\alpha(k))u(k) \tag{16}$$

where $u(k) \in \mathbb{R}^m$ is the control input. As in the analysis case, $\alpha(k)$ is supposed to be uncertain and $\Delta\alpha_i(k)$ satisfy (3). The system matrices are given in the form

$$\big(A(\alpha(k)), B(\alpha(k))\big) = \sum_{i=1}^{N} \alpha_i(k)(A_i, B_i), \quad \alpha(k) \in \Lambda_N.$$

Suppose that $x(k)$ is available in real time for feedback by means of the state feedback control law

$$u(k) = Kx(k), \quad K \in \mathbb{R}^{m\times n}$$

The aim is to find a robust state feedback gain K such that the closed-loop system

$$x(k+1) = A_{cl}(\alpha(k))x(k),\; A_{cl}(\alpha(k)) = A(\alpha(k)) + B(\alpha(k))K$$

is robustly stable for all $\alpha(k) \in \Lambda_N$ and $\Delta\alpha(k) \in \Gamma_{b\alpha}$.

Next theorem presents a sufficient condition for the existence of such robust control gain using the slack variables approach [16, 26, 27] and the Lyapunov matrix given in (11).

Theorem 3: If there exist symmetric matrices $P_i \in \mathbb{R}^{n\times n}$, $i = 1,\ldots,N$, matrices $G \in \mathbb{R}^{n\times n}$ and $Z \in \mathbb{R}^{m\times n}$ such that the following LMIs hold

$$\begin{bmatrix} P_i & A_iG + B_iZ \\ \star & G + G' - \big((1-b)P_i + bP_j\big) \end{bmatrix} > 0, \quad i = 1,\ldots,N, \quad j = 1,\ldots,N \tag{17}$$

then the state feedback gain $K = ZG^{-1}$ assures that the closed-loop system (16) is robustly stable.

Proof: Multiply (17) by $\alpha_i\beta_j$ and sum for $i = 1,\ldots,N$ and $j = 1,\ldots,N$ to obtain

$$\begin{bmatrix} P(\alpha) & A(\alpha)G + B(\alpha)Z \\ \star & G + G' - P(\beta) \end{bmatrix} = \begin{bmatrix} P(\alpha) & A_{cl}(\alpha)G \\ \star & G + G' - P(\beta) \end{bmatrix} > 0.$$

and by $[\mathrm{I} \quad -A_{cl}(\alpha)]$ on the left and by its transpose on the right, yielding

$$P(\alpha) - A_{cl}(\alpha)P(\beta)A_{cl}(\alpha)' > 0$$

which, by Schur complement, implies

$$\begin{bmatrix} P(\alpha) & A_{cl}(\alpha)P(\beta) \\ \star & P(\beta) \end{bmatrix} > 0.$$

The robust stability is guaranteed by the negativity of the discrete-time Lyapunov difference along the trajectories of the transposed system $x(k+1) = A_{cl}(\alpha)'x(k)$. ■

The main source of conservativeness relying upon the conditions of Theorem 3 is the use of a constant slack variable G. As observed in [28], even within the range $0 \le b < 1$, a Lyapunov matrix with a more complex polynomial dependence could not improve the results. For $b = 1$, the resulting conditions are similar to the ones presented in [27]. For $b = 0$, the conditions of Theorem 3 reduce to the same conditions given in [26, Theorem 3]. The main contribution of Theorem 3 is the ability of computing robust state feedback gains when $0 < b < 1$.

B. Gain-Scheduled State Feedback

Finally, the case of state feedback stabilization by means of gain-scheduled controllers is investigated.

Suppose now that $\alpha(k)$ and $x(k)$ are available in real time for feedback by means of the state feedback control law

$$u(k) = K(\alpha(k))x(k), \quad K(\alpha(k)) \in \mathbb{R}^{m\times n} \text{ for all } k \ge 0.$$

The aim is to find a gain-scheduled state feedback gain $K(\alpha(k))$ such that the closed-loop system

$$x(k+1) = A_{cl}(\alpha(k))x(k), \quad A_{cl}(\alpha(k)) = A(\alpha(k)) + B(\alpha(k))K(\alpha(k))$$

is robustly stable for all $\alpha(k) \in \Lambda_N$ and $\Delta\alpha(k) \in \Gamma_{b\alpha}$.

Theorem 4: If there exist symmetric matrices $P_i \in \mathbb{R}^{n\times n}$, matrices $G_i \in \mathbb{R}^{n\times n}$ and $Z_i \in \mathbb{R}^{m\times n}$, $i = 1,\ldots,N$ such that the following LMIs hold

$$\Xi_{i\ell} \triangleq \begin{bmatrix} \big((1-b)P_i + bP_\ell\big) & A_iG_i + B_iZ_i \\ \star & G_i + G_i' - P_i \end{bmatrix} > 0, \quad i = 1,\ldots,N, \quad \ell = 1,\ldots,N \tag{18}$$

$$\Xi_{ij\ell} \triangleq \begin{bmatrix} \big((1-b)(P_i + P_j) + 2bP_\ell\big) & A_iG_j + A_jG_i + B_iZ_j + B_jZ_i \\ \star & G_i + G_i' + G_j + G_j' - P_i - P_j \end{bmatrix} > 0, \quad i = 1,\ldots,N, \quad \ell = 1,\ldots,N, \quad j = i+1,\ldots,N-1 \tag{19}$$

then the parameter-dependent state feedback gain $K(\alpha) = Z(\alpha)G(\alpha)^{-1}$ with

$$Z(\alpha) = \sum_{i=1}^{N} \alpha_i Z_i, \qquad G(\alpha) = \sum_{i=1}^{N} \alpha_i G_i, \tag{20}$$

i.e. $K(\alpha)$ is rational in the parameters, assures that the closed-loop system (16) is robustly stable.

Proof: Multiply (18) by $\alpha_i^2\beta_\ell$ and sum for $i = 1,\ldots,N$ and $\ell = 1,\ldots,N$. Multiply (19) by $\alpha_i\alpha_j\beta_\ell$ and sum for $i = 1,\ldots,N$, $\ell = 1,\ldots,N$ and $j = i+1,\ldots,N-1$. Summing the results yields

$$\Xi = \sum_{\ell=1}^{N}\sum_{i=1}^{N} \alpha_i^2\beta_\ell\Xi_{i\ell} + \sum_{\ell=1}^{N}\sum_{i=1}^{N}\sum_{j=i+1}^{N-1} \alpha_i\alpha_j\beta_\ell\Xi_{ij\ell}$$

with Ξ, from the definitions (11) and (20), given by

$$\Xi = \begin{bmatrix} P(\beta) & A(\alpha)G(\alpha) + B(\alpha)Z(\alpha) \\ \star & G(\alpha) + G(\alpha)' - P(\alpha) \end{bmatrix} = \begin{bmatrix} P(\beta) & A_{cl}(\alpha)G(\alpha) \\ \star & G(\alpha) + G(\alpha)' - P(\alpha) \end{bmatrix}. \tag{21}$$

Feasibility of (18–19) guarantees that $\Xi(\alpha)$ is positive definite. Now multiply (21) by $[\mathrm{I} \quad -A_{cl}(\alpha)]$ on the left and by its transpose on the right to obtain

$$P(\beta) - A_{cl}(\alpha)P(\alpha)A_{cl}(\alpha)' > 0 \Leftrightarrow \begin{bmatrix} P(\beta) & A_{cl}(\alpha)P(\alpha) \\ \star & P(\alpha) \end{bmatrix} > 0. \tag{22}$$

Now apply the following congruence transformation

$$\begin{bmatrix} 0 & P(\alpha)^{-1} \\ P(\beta)^{-1} & 0 \end{bmatrix} \begin{bmatrix} P(\beta) & A_{cl}(\alpha)P(\alpha) \\ \star & P(\alpha) \end{bmatrix} \begin{bmatrix} 0 & P(\beta)^{-1} \\ P(\alpha)^{-1} & 0 \end{bmatrix} > 0$$

to obtain

$$\begin{bmatrix} P(\alpha)^{-1} & A_{cl}(\alpha)'P(\beta)^{-1} \\ \star & P(\beta)^{-1} \end{bmatrix} > 0.$$

In this case, the Lyapunov function $v(x,\alpha) = x'P(\alpha)^{-1}x$ with $P(\alpha)^{-1}$ being a rational polynomial matrix on the parameters guarantees the robust stability of the closed-loop system. ■

The conditions of Theorem 4 present a similar scenario to the case of robust analysis conditions in terms of conservativeness for the range $0 \le b < 1$. Homogeneous polynomial Lyapunov matrices of higher degrees and Pólya's relaxations could further improve the results.

C. Different parameter bounds and perspectives

As a final remark, note that the approach proposed in this paper could also be used when the bounds on the parameter variations are considered different, i.e.

$$-b_i\alpha_i \le \Delta\alpha_i \le b_i(1-\alpha_i), \quad i = 1,\ldots,N.$$

What happens in this situation is that the columns of matrix $\Gamma_{b_i\alpha}$ would not present a systematic and simple law of formation as in (8). In fact, for a fixed N, the number of columns in matrix $\Gamma_{b_i\alpha}$ can vary according to the values of b_i. In this case, the resulting robust stability analysis and control design conditions would require more complex algorithms to construct the LMIs systematically. Such procedures combined with the used of homogeneous polynomial Lyapunov matrices of arbitrary degree are currently under investigation by the authors.

V. NUMERICAL EXPERIMENTS

The numerical complexity associated to an optimization problem based on LMIs can be estimated from the number V of scalar variables and the number L of LMI rows. Concerning the LMI conditions proposed in Theorems 2 (T2), 3 (T3) and 4 (T4) these numbers are: $V_{T2} = Nn(n+1)/2$, $L_{T2} = nN^2(N+1)$, $V_{T3} = V_{T2} + n(n+m)$, $L_{T3} = 2nN^2$, $V_{T4} = V_{T2} + Nn(n+m)$, $L_{T4} = L_{T3}$. In the worst case situation, the number V grows quadratically with n and the number L grows cubically with N for the proposed methods.

All the experiments have been performed in a Pentium IV 2.6 GHz, 512 MB RAM, using SeDuMi [29] and YALMIP [30] within the Matlab environment.

Example 1 (Stability Analysis): The time-varying discrete-time system

$$x(k+1) = \begin{bmatrix} 0.9979 & 0.008p(k) - 0.01 \\ 0.01 & 1 \end{bmatrix} x(k), \; |p(k)| \le 0.3,$$

borrowed from [21, Example 1], is analyzed by computing the maximum variation rate ρ of $p(k)$, such that robust stability is guaranteed for any $|\Delta p(k)| \le \rho$. In order to evaluate the precision and efficiency of the proposed approach, the conditions of Theorem 2 are compared with the robust stability conditions presented in [21, Theorem 1]. The maximum values of ρ obtained as well as the associated numerical complexity, given in terms of V, L and the computational times (in seconds), are shown in Table I.

TABLE I

MAXIMUM VALUES OF ρ AND NUMERICAL COMPLEXITY (V SCALAR VARIABLES; L LMI ROWS) OBTAINED IN THE ROBUST STABILITY ANALYSIS PROBLEM GIVEN IN EXAMPLE 1 USING THE CONDITIONS OF THEOREM 2 AND THE METHOD FROM [21, THEOREM 1].

Method	ρ_{max}	V	L	Time (s)
[21, Theorem 1]	0.0222	84	384	0.33
Theorem 2	0.0299	6	24	0.07

As it can be seen, the conditions of Theorem 2 are more efficient and less conservative than the ones presented in [21, Theorem 1], which demanded a bigger computational effort due to the division of the parameter domain into subintervals with size at most equal to ρ (in this case, 28 subintervals were needed). Note that the number of decision variables demanded by [21, Theorem 1] depends on the number of subintervals. The approach proposed here uses a different strategy, i.e. to assess the robust stability by approximating the solution of a parameter-dependent LMI by a homogeneous polynomial solution. In this case, a homogeneous polynomial of degree one (i.e. affine in the parameters) has been used. Higher degree polynomial solutions could further improve the results.

Example 2 (Control Design): Consider the system (16) for $n = 3$ and $N = 2$ with the following matrices

$$\left[\, A_1 \,\vdots\, A_2 \,\right] = \gamma \left[\begin{array}{ccc:ccc} 1 & 0 & -2 & 0 & 0 & -1 \\ 2 & -1 & 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & -2 & -1 \end{array}\right], \; B_1 = B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The aim is to determine the maximum value of γ such that the system is stabilized by a robust state feedback gain. For $b = 1$, i.e. the parameter α varies arbitrarily inside the polytope, both the conditions of Theorem 3 and the conditions from [27, Theorem 1] provide robust gains for $\gamma \le 0.5940$. For $\gamma > 0.5940$, the conditions from [27, Theorem 1] can no longer provide a feasible solution. On the other hand, the conditions from Theorem 3 can still provide robust gains for time-varying parameters with bounded rate of variation. Figure 2 shows the maximum variation rates b_{max} for $\gamma \in [0.58\ 0.72]$ such that robust gains can be synthesized by the conditions of Theorem 3.

For illustration purposes, the resulting robust gain for $\gamma = 0.64$ and $b = b_{max} = 0.496$ is given by

$$K = \begin{bmatrix} 0.6408 & -0.8600 & 1.0948 \end{bmatrix}$$

Example 3: Finally, consider system (16) for $n = 2$ and $N = 3$ with matrices $B_1 = B_2 = B_3 = \begin{bmatrix} 1 & 0 \end{bmatrix}'$ and

$$\left[\, A_1 \,\vdots\, A_2 \,\vdots\, A_3 \,\right] = \left[\begin{array}{cc:cc:cc} 0.5 & 0.7 & 1.0 & 0.7 & -0.1 & 1.1 \\ -1.6 & -0.1 & -0.6 & -1.2 & -1.0 & 0.9 \end{array}\right].$$

The aim now is to synthesize gain-scheduled state feedback stabilizing controllers. For $b = 1$, the conditions from

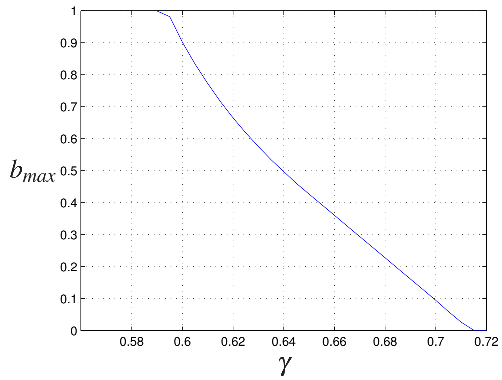

Fig. 2. Maximum parameter variation rate b_{max} such that the system given in Example 2 is stabilized by robust state feedback gain synthesized through the conditions of Theorem 3.

[16, Theorem 4] and the conditions of Theorem 4 cannot provide gain-scheduled stabilizing controllers. Note that the conditions from [16, Theorem 4] can only be applied when the input matrices are not affected by the time-varying parameters, which is the case in this example. On the other hand, Theorem 4 does not suffer from this limitation.

In this example, if limited rates of parameter variation are considered, the conditions of Theorem 4 can provide stabilizing gain-scheduled controllers for $b \leq 0.846$.

VI. CONCLUSION

The robust stability analysis and state feedback control design for time-varying discrete-time polytopic systems with bounded rates of parameter variation were investigated in this paper. LMI conditions for robust stability analysis, robust and gain-scheduled state feedback control have been proposed for this class of systems, yielding less conservative results than other available techniques, as illustrated by examples.

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