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On Observer Design for a Class of Impulsive Switched Systems

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Abstract— **In this work, the problem of state observation for a class of impulsive switched systems is addressed. Corresponding to each subsystem, an identity Luenberger observer is employed and a switching observer is constructed accordingly. The asymptotic stability property of the proposed switching observer is discussed and LMI-based algorithms are given which provide sufficient conditions for asymptotic stability of the switching observer for switching signals with an average dwell time greater than a specific value. Since switched systems without impulse are a special case of impulsive switched systems, the results of this work can be used to design observers for switched systems without impulse as well. Numerical examples are given to show the effectiveness of the proposed algorithms.**

I. Introduction

Switched systems are a class of hybrid systems and have increasingly been at the center of attention in recent years due to their wide range of applications in practice. In general, a switched system consists of several subsystems and a switching rule that orchestrates the switching between them. Unlike the stability problem of switched systems that has been extensively studied in the literature [6], [7], [14], [15], the observer design problem for switched systems has attracted less attention and not as many works are available in this area (see for example [3], [4]).

Switching can also be applied to control in order to cope with highly uncertain systems [1], [2], [17]. There are many examples of switched systems in power electronics [12], process control, biomedical and biochemical processes [9] and aerospace, to name only a few. Due to sudden changes in the state of the system at certain instants of switching, many practical switched systems exhibit impulsive dynamical behavior [8], [10], [13]. There are a number of works dealing with the impulsive behavior of switched systems and a few results have been developed in this area [18], [19]. In [18], impulsive phenomena are introduced into switched systems, and necessary and sufficient conditions for controllability and observability of impulsive switched control systems have been developed. On the other hand, necessary and sufficient conditions for stability of impulsive switched systems under arbitrary switching signals are obtained in [19]. Robust asymptotic stability of linear discrete impulsive systems and a class of uncertain nonlinear discrete impulsive systems are studied in [16]. Finally, input-to-state stability properties of impulsive systems are discussed in [11].

This work focuses on the problem of observer design for impulsive switched systems with either continuous or discrete linear subsystems. LMI-based algorithms are developed to guarantee the stability of the proposed switching observer for a class of impulsive switched systems with a constrained switching rule.

The remainder of the paper is organized as follows. Stability analysis for a class of linear impulsive switched systems is discussed in Section II. Section III focuses on the problem of switching observer design for impulsive switched systems as the main contribution of this work. To show the effectiveness of the proposed approach, two numerical examples are given in Section IV. Finally, some concluding remarks are provided in Section V.

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II. Stability of Impulsive Switched Systems Under Constrained Switching

In this section stability of impulsive switched systems with either continuous or discrete subsystems is investigated. In the first subsection sufficient conditions to guarantee the stability of impulsive switched systems with continuous subsystems is obtained while stability analysis for impulsive systems with discrete subsystems is discussed in the second subsection.

A. Stability of continuous impulsive switched systems

An impulsive switched system with N modes of operation obeys the following general formulation

$$\begin{cases} \dot{x}(t) = f_{\sigma(t)}(x(t), u(t)), & t \neq t_l,\ l \in \mathbb{Z}^+ \\ x(t) = g_{\sigma(t),\sigma(t^-)}(x(t^-)), & t = t_l,\ l \in \mathbb{Z}^+ \\ x(t_0) = x_0 \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and the input of the system, respectively. Furthermore, $f_{\sigma(t)}$ and $g_{\sigma(t),\sigma(t^-)}$ are a $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\mathbb{R}^n \to \mathbb{R}^n$ functions, respectively, and $\{t_i\}_{i\in\mathbb{Z}^+}$ is a sequence of increasing impulse times in $[t_0,\infty)$. The right continuous function $\sigma(t) : [t_0,\infty) \to \bar{\mathrm{N}}$ is the switching rule, where $\bar{\mathrm{N}} = \{1,2,\cdots,N\}$. By construction, the state of the system $x(t) : [t_0,\infty] \longrightarrow \mathbb{R}^n$ is right-continuous. Furthermore, $(.)^-$ denotes the left-limit operator, and $\sigma(t) = i$ shows that the the i-th mode is active ,$i \in \bar{\mathrm{N}}$. The relation $\sigma(t) = i$ while $\sigma(t^-) = j$ means that t is a switching instant, and that the system switches from the j-th mode to the i-th mode at time t. Without loss of generality, one can assume that the origin is the equilibrium point of this system when there is no input; i.e. $f_{\sigma(t)}(0,0) = 0, \quad \forall t > t_0$. Based on the general form of impulsive switched systems consider a class of linear impulsive switched control systems given by

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), & t \neq t_l,\ l \in \mathbb{Z}^+ \\ x(t) = G_{\sigma(t),\sigma(t^-)}x(t^-), & t = t_l,\ l \in \mathbb{Z}^+ \\ y(t) = C_{\sigma(t)}x(t) \\ x(t_0) = x_0 \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and $u \in \mathbb{R}^m$ are the state, the output and the input of the system respectively. $A_{\sigma(t)}$, $B_{\sigma(t)}$, $C_{\sigma(t)}$ and $G_{\sigma(t),\sigma(t^-)}$ are constant matrices with appropriate dimensions. Jumps in the state of the system at switching instants are represented by $N^2 - N$ matrices $G_{ij},\ \forall i,j \in \bar{\mathrm{N}}$, $i \neq j$.

The easiest way to represent constrained switching is to introduce a number $\tau_d > 0$, often called *dwell time* [14], and restrict the switching signal such that the time interval between every two consecutive switching instants is greater than τ_d. Since this can be a restrictive requirement in general, one can consider the *average dwell time* instead, which allows fast switchings in some instants, provided that their collective effect would be compensated by sufficiently slow switchings in some other instants [14].

Definition 1 [14]: Let the number of discontinuities of the switching signal $\sigma(t)$ on a given interval $[t_0, t)$ be denoted by $N(t,t_0)$. The signal $\sigma(t)$ is said to have an *average dwell time* τ_a if there exists two positive numbers N_0 and τ_a such that

$$N(t,t_0) \leq N_0 + \frac{t - t_0}{\tau_a} \qquad 0 \leq t_0 \leq\ t, \forall t \geq t_0 \tag{3}$$

In the sequel, sufficient conditions are derived for the stability of the impulsive switched system given in (1).

Lemma 1: Assume that there exist a C^1 Lyapunov function $V(x): \mathbb{R}^n \to \mathbb{R}$, and two class K_∞ functions α_1 and α_2 [15] satisfying

$$\alpha_1(\|x(t)\|) \leq V(x(t)) \leq \alpha_2(\|x(t)\|), \qquad \forall t \geq t_0 \tag{4}$$

where $\|.\|$ denotes any induced norm. Further, assume that there exist a number $\mu > 1$ and a strictly negative number λ_0 for which the derivative of $V(x)$ for the system (1) satisfies the inequalities

$$\dot{V}(x(t)) \leq 2\lambda_0 V(x(t)), \qquad \forall t \in (t_l, t_{l+1}),\ \forall l \in \mathbb{Z}^+ \tag{5}$$

and

$$V(x(t_l{}^+)) \leq \mu V(x(t_l^-)), \qquad \forall l \in\in \mathbb{Z}^+ \tag{6}$$

Then the equilibrium point $x = 0$ of (1) when $u(t) = 0$ is asymptotically stable for every switching signal $\sigma(t)$, with the average dwell time satisfying

$$\tau_a > \tau_{min} = \frac{\log \mu}{-2\lambda_0} \tag{7}$$

Proof: It can be deduced from (5) and (6) that

$$V(x(t)) \leq \mu^{N(t,t_0)} e^{2\lambda_0(t-t_0)} V(x(t_0))$$

(note that $N(t,t_0)$ is the number of switchings in the interval $[t_0,t)$). Using the definition of average dwell time (Definition 1) and replacing the minimum value of average dwell time given by (7), it follows that there must exist a positive number ε such that

$$\frac{1}{\tau_a} \leq \frac{-2\lambda_0}{\log \mu} - \varepsilon$$

and as a result

$$N(t,t_0) \leq (\frac{-2\lambda_0}{\log \mu} - \varepsilon)(t - t_0) + N_0$$

therefore

$$V(x(t)) \leq \mu^{N_0} \mu^{-\varepsilon(t-t_0)} V(x(t_0))$$

Now, it can be concluded from (4) that

$$\|x(t)\| \leq \alpha_1{}^{-1}(\mu^{N_0} \mu^{-\varepsilon(t-t_0)} \alpha_2(\|x(t_0)\|))$$

Let $\beta(\|x(t_0)\|, t) := \alpha_1{}^{-1}(\mu^{N_0}\mu^{-\varepsilon(t-t_0)}\alpha_2(\|x(t_0)\|))$. Since α_1 is a class K_∞ function so is $\alpha_1{}^{-1}$. On the other hand, α_2 is also a class K_∞ function and ε is a positive number. This implies that β is a class KL function and hence the proof is complete. ■

Remark 1: If the conditions of Lemma 1 hold for a quadratic Lyapunov function, then the equilibrium point in the switched system (1) will be exponentially stable.

Remark 2: If the inequality (6) in Lemma 1 holds for some $0 < \mu < 1$, then it can be shown that the switched system given by (1) is globally uniformly asymptotically stable for every arbitrary switching signal [14].

B. Stability of discrete impulsive switched systems

In this subsection, inspired by the concept of discrete impulsive systems described in [16] and previous works on continuous impulsive switched systems, a class of discrete impulsive switched systems consisting of M modes of operation (subsystems) are introduced as follows

$$\begin{cases} x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k), & k \neq k_l - 1,\ l \in \mathbb{Z}^+ \\ x(k+1) = G_{\sigma(k+1),\sigma(k)}x(k), & k = k_l - 1,\ l \in \mathbb{Z}^+ \\ y(k) = C_{\sigma(k)}x(k) \\ x(k_0) = x_0 \end{cases} \tag{8}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and $u \in \mathbb{R}^m$ are the state, the output and the input of the system, respectively. Moreover, $\{k_l\}_{l \in \mathbb{Z}^+}$ is a sequence of increasing impulse times, for which the following assumptions are satisfied

Assumption 1: The sequence $\{k_l\}$ has the property that $k_l \in \mathbb{Z}^+$, $k_0 = 0$ and $k_l < k_{l+1}, \forall l \in \mathbb{Z}^+$.

Assumption 2: $k_{l+1} - k_l > 1$, $l \in \mathbb{Z}^+$.

Define $\bar{\mathrm{M}} = \{1, 2, \cdots, M\}$. The function $\sigma(k): \mathbb{Z}+ \to \bar{\mathrm{M}}$ is the switching rule and determines which of the M modes is active at each time. For instance, $\sigma(k) = i$ and $\sigma(k - 1) = j$ indicates that k is a switching instant at which the system switches from the j-th mode to the i-th mode. Note that the state of this impulsive switched system $x(k)$: $\{0, 1, 2, \cdots\} \longrightarrow \mathbb{R}^n$ experiences impulses at the switching instants. Furthermore, $G_{\sigma(k_l),\sigma(k_l - 1)}$, $l \in \mathbb{Z}^+$ is a constant matrix which depends on the index of the active modes before and after that specific switching instant.

To represent constrained switching, a number $k_a > 0$ called average dwell time [14] is introduced here in a way similar to the continuous switched systems, which allows fast switchings in some instants, provided their that effects are compensated by sufficiently slow switchings at other instants [14].

Definition 2: Let the number of switching instants of the switching signal $\sigma(k)$ on a given interval $[k_0, k)$ be denoted by $N(k, k_0)$. The signal $\sigma(k)$ is said to have an *average dwell time* $k_a \in \mathbb{Z}+$ if there exists two positive numbers N_0 and k_a such that

$$N(k, k_0) \leq N_0 + \frac{k-k_0}{k_a} \qquad \forall k \geq k_0 \tag{9}$$

Inspired by the work [15], sufficient conditions for the stability of impulsive switched system (8) are provided.

Lemma 2: Consider the switched system (8) with the switching instants $\{k_1\}_{l\in\mathbb{Z}+}$. Suppose that there exists a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$, and two class K_∞ functions α_1 and α_2 for which the following inequality holds

$$\alpha_1(\|x(k)\|) \leq V(x(k)) \leq \alpha_2(\|x(k)\|), \qquad \forall k \geq k_0 \tag{10}$$

if there exists a number $0 < \beta < 1$ for which $V(x(k))$ along the solution of the system (8) satisfies the inequality

$$V(x(k+1)) \leq \beta V(x(k)) \qquad \forall k \in [k_l, k_{l+1}-1), \ \forall l \in \mathbb{Z}+ \tag{11}$$

and a number $\mu > 1$ such that

$$V(x(k+1)) \leq \mu V(x(k)), \qquad k = k_l - 1 \tag{12}$$

then the impulsive switched system (8) is asymptotically stable for every switching signal $\sigma(k)$ with the average dwell time

$$k_a > 1 - \frac{\log\mu}{\log\beta} \tag{13}$$

Proof: It can be deduced from (11) and (12) that

$$V(x(k)) \leq \mu^{N(k,k_0)}\beta^{k-k_0-N(k,k_0)}V(x(k_0)), \qquad \forall k \geq k_0$$

Using the definition of average dwell time for linear impulsive switched systems (see Definition 2) it follows that

$$\begin{aligned} V(x(k)) &\leq \mu^{N_0+\frac{k-k_0}{k_a}}\beta^{k-k_0-N_0-\frac{k-k_0}{k_a}}V(x(k_0)) \\ &= \mu^{N_0}\beta^{-N_0}\rho^{k-k_0}V(x(k_0)) \end{aligned}$$

where $\rho = \mu^{\frac{1}{k_a}}\beta^{\frac{k_a-1}{k_a}}$ (one can verify from (13) that $0 < \rho < 1$). Now it can be concluded from (10) that

$$\|x(k)\| \leq \alpha_1^{-1}(\mu^{N_0}\beta^{-N_0}\rho^{k-k_0}\alpha_2(\|x(k_0)\|))$$

Let $\beta(\|x(k_0)\|, k) := \alpha_1{}^{-1}(\mu^{N_0}\beta^{-N_0}\rho^{k-k_0}\alpha_2(\|x(k_0)\|))$. Since α_1 is a class K_∞ function, so is $\alpha_1{}^{-1}$. Moreover, α_2 is a class K_∞ function as well, and $0 < \rho < 1$. Hence it can be verified that β is a class KL function. This completes the proof. ∎

III. Observer Design for Linear Impulsive Switched Systems

The stability results obtained in the previous section will now be used to develop LMI-based algorithms for designing a switching observer for the impulsive switched systems given by (2) and (8) such that the stability of the observation error dynamics under constrained switching is guaranteed.

A. Observer design for continuous impulsive switched systems

Consider a switching observer O, for (2) as follows

$$\dot{\hat{x}}(t) = A_i\hat{x}(t) + B_iu(t) + L_i(y(t) - C_i\hat{x}(t)), \quad t_l < t < t_{l+1} \tag{14a}$$

$$\hat{x}(t^+) = H_{i,j}\hat{x}(t^-), \qquad t = t_l \tag{14b}$$

where $\sigma(t_l^+) = i$, $\sigma(t_l^-) = j$, $l \in \mathbb{Z}+$ and $\forall i, j \in \bar{\mathrm{N}}$, $i \neq j$. For each mode, an identity Luenberger observer namely O_i is designed and is employed when the corresponding mode is active. It is to be noted that $N^2 - N$ constant matrices H_{ij} suggest that the proposed observer O is an impulsive switched system by its construction. In the remainder of this subsection, an LMI-based algorithm is introduced to design the proposed observer i.e., to find the matrices L_i and H_{ij}, $\forall i, j \in \bar{\mathrm{N}}$, $i \neq j$ such that the following properties hold:

- The eigenvalues of $A_i - L_iC_i$, $\forall i \in \bar{\mathrm{N}}$ are placed to the left of the line $\mathrm{Re}\{s\} = \lambda_1$ and to the right of the line $\mathrm{Re}\{s\} = \lambda_2$, where λ_1 and λ_2 are given and $\lambda_2 < \lambda_1 < 0$.
- The stability of the observation error dynamics in the proposed switching observer O under the constrained switching is guaranteed.
- The required average dwell time in the switching observer is minimized. This means that the proposed observer can be used to observe the state of a large class of impulsive switched systems, in the sense that the switchings are allowed to occur relatively fast.

Remark 3: The first imposed property means that the error in the Luenberger observers O_i, $\forall i \in \bar{\mathrm{N}}$ converges exponentially to zero with rate of convergence greater than λ_1. However, if the poles are placed very far from the $j\omega$ axis, the resultant observer gains L_i, $\forall i \in \bar{\mathrm{N}}$ will be large. This can lead to a design highly sensitive to numerical errors and prone to implementation difficulties.

Algorithm 1: Consider the switched system described by (2). The following procedure is proposed to find the values of L_i and H_{ij}, $\forall i, j \in \bar{\mathrm{N}}$, $i \neq j$ such that all of the properties mentioned above hold.

Step 1: Given λ_1, λ_2 where $\lambda_2 < \lambda_1 < 0$, find the minimum value of μ for which there exist $P_i > 0$, X_i and $H_{i,j}$, satisfying the following matrix inequalities

$$A_i{}^TP_i + P_iA_i - C_i{}^TX_i{}^T - X_iC_i - 2\lambda_1P_i < 0, \quad \forall i \in \bar{\mathrm{N}} \tag{15}$$

$$2\lambda_2P_i - A_i{}^TP_i - P_iA_i + C_i{}^TX_i{}^T + X_iC_i < 0, \quad \forall i \in \bar{\mathrm{N}} \tag{16}$$

$$\begin{bmatrix} \mu P_j - G_{i,j}^TP_iG_{i,j} & -\mu P_j + G_{i,j}^TP_iH_{i,j} & 0 \\ -\mu P_j + H_{i,j}^TP_iG_{i,j} & \mu P_j & H_{i,j}^TP_i \\ 0 & P_iH_{i,j} & P_i \end{bmatrix} \geq 0, \quad \forall i, j \in \bar{\mathrm{N}}, \ i \neq j \tag{17}$$

It is to be noted that this minimization can be formulated as a BMI problem. PENBMI can solve this problem efficiently and can be used as a MATLAB function with PEN or YALMIP interface. Denote the optimum of the above non-convex optimization problem with μ^*.

Step 2: Using the matrices P_i and X_i $\left(\forall i \in \bar{\mathrm{N}}\right)$ obtained in Step 1, find L_i, the observer gains of O proposed in (14) as follows

$$L_i = P_i^{-1} X_i, \qquad \forall i \in \bar{\mathrm{N}} \tag{18}$$

Step 3: If $\mu^* > 1$, compute the minimum allowable dwell time as

$$\tau_{min} = \frac{\log \mu^*}{-2\lambda_1} \tag{19}$$

The above procedure arrives at the minimum value of μ, namely μ^*, and gives the matrices P_i, X_i, L_i $\left(\forall i \in \bar{\mathrm{N}}\right)$ and H_{ij}, $\forall i, j \in \bar{\mathrm{N}}$, $i \neq j$. The following result is obtained.

Theorem 1: If there exists N symmetric positive definite matrices P_i, N matrices X_i and $N^2 - N$ matrices $H_{i,j}$, $\forall i, j \in \bar{\mathrm{N}}$, $i \neq j$, which satisfy the LMIs (15), (16) and (17) then:

i) The eigenvalues of $A_i - L_iC_i$ satisfy the inequality $\lambda_2 < eig(A_i - L_iC_i) < \lambda_1, \quad \forall i \in \bar{\mathrm{N}}$

ii) If $\mu^* > 1$, then the error dynamic in the switching observer O is globally uniformly exponentially stable for the switching signal $\sigma(t)$ with any average dwell time τ_a greater than τ_{min} given by (19). Otherwise $(0 < \mu^* \leq 1)$, the error dynamic in the switching observer O is globally uniformly exponentially stable for any arbitrary switching signal.

Proof: Since $L_i = P_i^{-1} X_i$ or equivalently $X_i = P_iL_i$, the inequalities (15) and (16) can be rewritten as

$$(A_i - L_iC_i)^T P_i + P_i(A_i - L_iC_i) - 2\lambda_1 P_i < 0, \qquad \forall i \in \bar{\mathrm{N}}$$

$$2\lambda_2 P_i - (A_i - L_iC_i)^T P_i - P_i(A_i - L_iC_i) < 0, \qquad \forall i \in \bar{\mathrm{N}}$$

Now it can be easily concluded from the Lyapunov theory that these two LMIs are equivalent to this inequality $\lambda_2 < eig(A_i - L_iC_i) < \lambda_1$.

Assume that the active observer in the time intervals $[t_{l-1}, t_l)$ and $[t_l, t_{l+1})$ are j and i, respectively. The error dynamic in the proposed switching observer denoted by $\tilde{x}(t)$ can be described by

$$\dot{\tilde{x}}(t) = (A_i - L_iC_i)\tilde{x}(t), \qquad t_{l-1} < t < t_l, \ l \in \mathbb{Z}+ \tag{20a}$$

$$\tilde{x}(t^+) = G_{i,j}x(t^-) - H_{i,j}\widehat{x}(t^-), \qquad t = t_l, \ l \in \mathbb{Z}+ \tag{20b}$$

Define a switched Lyapunov function as

$$V(\tilde{x}(t)) = \tilde{x}(t)^T P_i \tilde{x}(t), \ t_{l-1} < t < t_l, \ l \in \mathbb{Z}+, \ \forall i \in \bar{\mathrm{N}} \tag{21}$$

where i is the index of active modes at each time and P_i, $\forall i \in \bar{\mathrm{N}}$ are obtained in Step 1. Since

$$\min_i \{\lambda_{min}(P_i)\} \|\tilde{x}\|^2 < V(\tilde{x}(t)) < \max_i \{\lambda_{max}(P_i)\} \|\tilde{x}\|^2, \forall i \in \bar{\mathrm{N}}$$

then the Lyapunov function V satisfies (4), where $\alpha_1(\|\tilde{x}\|)$ and $\alpha_2(\|\tilde{x}\|)$ are defined as

$$\alpha_1(\|\tilde{x}\|) = \min_i \{\lambda_{min}(P_i)\} \|\tilde{x}\|^2, \qquad \forall i \in \bar{\mathrm{N}}$$

$$\alpha_2(\|\tilde{x}\|) = \max_i \{\lambda_{max}(P_i)\} \|\tilde{x}\|^2, \qquad \forall i \in \bar{\mathrm{N}}$$

according to (18), since $X_i = P_iL_i$, (15) can be written as

$$(A_i - L_iC_i)^T P_i + P_i(A_i - L_iC_i) < 2\lambda_1 P_i$$

Considering the definition of switched Lyapunov function given by (21) and the equality (20a), the above inequality can be rewritten as

$$\dot{V}(\tilde{x}(t)) < 2\lambda_1 V(\tilde{x}(t)), \quad t_{l-1} < t < t_l, \ l \in \mathbb{Z}+$$

In other words (5) is satisfied by this choice of V. According to (20b), the last condition of Lemma 1 by this choice of V can be rewritten as

$$(x(t_l^-)^T G_{i,j}^T - \widehat{x}(t_l^-)^T H_{i,j}^T) P_i (G_{i,j}x(t_l^-) - H_{i,j}\widehat{(t_l^-)}) \leq \mu(x(t_l^-)^T - \widehat{x}(t_l^-)^T) P_j (x(t_l^-) - \widehat{x}(t_l^-))$$

or equivalently

$$X^T \begin{bmatrix} \mu P_j - G_{i,j}^T P_i G_{i,j} & -\mu P_j + G_{i,j}^T P_i H_{i,j} \\ -\mu P_j + H_{i,j}^T P_i G_{i,j} & \mu P_j - H_{i,j}^T P_i H_{i,j} \end{bmatrix} X \geq 0$$

where $X = \begin{bmatrix} x(t_l^-) \\ \widehat{x}(t_l^-) \end{bmatrix}$. Again this inequality can be rewritten as

$$\begin{bmatrix} \mu P_j - G_{i,j}^T P_i G_{i,j} & -\mu P_j + G_{i,j}^T P_i H_{i,j} \\ -\mu P_j + H_{i,j}^T P_i G_{i,j} & \mu P_j \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & H_{i,j}^T P_i H_{i,j} \end{bmatrix} \geq 0 \tag{22}$$

replacing the second term in the above inequality by $\begin{bmatrix} 0 \\ P_iH_{i,j} \end{bmatrix}^T P_i^{-1} \begin{bmatrix} 0 & P_iH_{i,j} \end{bmatrix}$ and using the Schur-complement formula one can verify that the above inequality becomes the same as (17). Thus all the conditions of Lemma 1 are satisfied by this choice of V. This completes the proof. ■

The optimization problem introduced above is non-convex, in general. In the following algorithm, some additional assumptions are made on the structure of the proposed observer, to turn the above non-convex optimization problem to a convex LMI problem.

Algorithm 2: Consider the switched system described by (2). The following procedure is followed to find the values of L_i and H_{ij}, $\forall i, j \in \bar{\mathrm{N}}$, $i \neq j$ in the structure of the proposed observer O.

Step 1: Find the minimum value of μ for which there exist N matrices X_i, $\forall i \in \bar{\mathrm{N}}$, and N positive definite symmetric matrices P_i, $\forall i \in \bar{\mathrm{N}}$ satisfying the following LMIs

$$A_i{}^T P_i + P_iA_i - C_i{}^T X_i{}^T - X_iC_i - 2\lambda_1 P_i < 0, \quad \forall i \in \bar{\mathrm{N}} \tag{23}$$

$$2\lambda_2 P_i - A_i{}^T P_i + P_iA_i - C_i{}^T X_i{}^T - X_iC_i < 0, \quad \forall i \in \bar{\mathrm{N}} \tag{24}$$

$$\mu P_j \geq G_{ij}^T P_i G_{ij}, \ \forall i, j \in \bar{\mathrm{N}}, \ i \neq j \tag{25}$$

It is to be noted that this minimization can be formulated as a GEVP problem. (MATLAB can solve this problem efficiently). Moreover, denote the optimum of the above convex optimization problem with μ^*.

Step 2: Using the matrices P_i and X_i, $\forall i \in \bar{\mathrm{N}}$ obtained in Step 1, find L_i, the observer gains of O proposed in (14a) as follows

$$L_i = P_i^{-1} X_i, \qquad \forall i \in \bar{\mathrm{N}} \tag{26}$$

Step 3: If $\mu^* > 1$, compute the minimum allowable dwell time

$$\tau_{min} = \frac{\log \mu^*}{-2\lambda_1} \tag{27}$$

Theorem 2: Using this algorithm to obtain the minimum value of μ namely μ^*, matrices P_i, X_i, L_i $\left(\forall i \in \bar{\mathrm{N}}\right)$ and H_{ij}, $\forall i, j \in \bar{\mathrm{N}}$, $i \neq j$, we have the following result. If there exists N symmetric positive definite matrices P_i and N matrices X_i, $\forall i \in \bar{\mathrm{N}}$ which satisfy the LMIs (23), (24) and (25), then:

i) The eigenvalues of $A_i - L_iC_i$ satisfy the inequality $\lambda_2 < eig(A_i - L_iC_i) < \lambda_1, \quad \forall i \in \bar{\mathrm{N}}$

ii) If $\mu^* > 1$, the error dynamic in the corresponding switching observer O is globally uniformly exponentially stable for the switching signal $\sigma(t)$ with any average dwell time τ_a greater than τ_{min} given by (27). Otherwise $(0 < \mu^* \leq 1)$, the error dynamic in the switching observer O is globally uniformly exponentially stable for arbitrary switching signals.

Proof: Proof is similar to the proof of the previous theorem, in fact LMIs (23) and (24) guarantees that convergence rate of each Luenberger observer is in the desired region, also by substituting $H_{ij} = G_{ij}$, one can verify that LMI's (25) and (17) are the same. Thus if the same Lyapunov function V as in the previous theorem is considered all the conditions of Lemma 1 are satisfied by this choice of V and error in the observer is asymptotically stable for any impulsive switched system with the average dwell time greater than (27). ■

Remark 4: Using this algorithm the proposed switching observer will have the same jumps at switching instants as in the impulsive switched system, in other words $H_{ij} = G_{ij}$ will be imposed on the structure of the observer.

Remark 5: The proposed two algorithms in this section can be applied to switched systems without impulse, in fact the systems without impulse are a special case of impulsive systems when $G_{ij} = I$.

B. Observer design for discrete impulsive switched systems

Consider a switching observer for (8) as follows

$$\hat{x}(k+1) = A_i\hat{x}(k) + B_iu(k) + L_i(y(k) - C_i\hat{x}(k)), \quad k \neq k_l - 1 \tag{28a}$$

$$\hat{x}(k+1) = H_{i,j}\hat{x}(k), \quad k = k_l - 1 \tag{28b}$$

where $\sigma(k_l) = i$, $\sigma(k_{l-1}) = j$ and $l \in \mathbb{Z}+$. For each mode, an identity Luenberger observer, namely O_i, is designed and is employed when the corresponding mode is active. It is to be noted that $M^2 - M$ constant matrices H_{ij} show that the proposed observer O is a discrete impulsive switched system by its construction.

In the remainder of this subsection, an LMI-based algorithm is introduced to design the proposed observer (i.e., to obtain the matrices L_i and H_{ij}) such that the following properties in the observer are held by the observer:

- The eigenvalues of $A_i - L_iC_i$ are placed inside the circle centered at the origin with the radius $r = \beta$ where $\beta \in (0,1)$ is given.
- The stability of the observation error dynamics in the proposed switching observer O under constrained switching is guaranteed while the required average dwell time is minimized.

Remark 6: Unlike the case of continuous dynamics where placing the eigenvalues of Luenberger observer far from the imaginary axis in LHP results in high gains, in the case of discrete dynamics placing the eigenvalues of Luenberger observer near the origin does not result in high gains. In fact, an observer with all of its eigenvalues located at the origin is desirable, and is referred to as a dead-beat observer. Unlike the continuous impulsive switching observer design, here the desired region for the eigenvalues of a discrete Luenberger observer is given only by one parameter β (which directly reflects the speed of convergence).

Algorithm 3: Consider the switched system described by (8). The following steps should be followed to obtain the values of L_i and H_{ij}, $\forall i, j \in \bar{\mathrm{M}}$, $i \neq j$ such that all the properties mentioned above are satisfied by the proposed switched observer O.

Step 1: For the given $0 < \beta < 1$, find the minimum value of μ for which there exist M matrices X_i, $\forall i \in \bar{\mathrm{M}}$, and M positive definite symmetric matrices P_i, $\forall i \in \bar{\mathrm{M}}$, and $M^2 - M$ matrices H_{ij} satisfying the following LMIs

$$\begin{bmatrix} \beta^2 P_j & A_i^T P_i - C_i^T X_i^T \\ P_iA_i - X_iC_i & P_i \end{bmatrix} > 0 \quad \forall i, j \in \bar{\mathrm{M}} \tag{29}$$

$$\begin{bmatrix} \mu P_i - G_{i,j}^T P_i G_{i,j} & -\mu P_i + G_{i,j}^T P_i H_{i,j} & 0 \\ -\mu P_i + H_{i,j}^T P_i G_{i,j} & \mu P_i & H_{i,j}^T P_i \\ 0 & P_i H_{i,j} & P_i \end{bmatrix} \geq 0, \quad \forall i, j \in \bar{\mathrm{M}}, \ i \neq j \tag{30}$$

Denote the optimum value of the above non-convex optimization problem with μ^*.

Step 2: Using the matrices P_i and X_i^1, $\forall i \in \bar{\mathrm{M}}$ obtained in Step 1, find L_i, the observer gain of O_i (given in (28) for each mode) as follows

$$L_i = P_i^{-1} X_i, \qquad \forall i \in \bar{\mathrm{M}} \tag{31}$$

Step 3: If $\mu^* > 1$, compute the minimum allowable dwell time

$$\tau_{min} = 1 - \frac{\log \mu^*}{\log \beta^2} \tag{32}$$

Using this algorithm to obtain the minimum value of μ namely μ^*, matrices P_i, X_i, L_i $\left(\forall i \in \bar{\mathrm{M}}\right)$ and H_{ij}, $\forall i, j \in \bar{\mathrm{M}}$, $i \neq j$, we have the following result.

Theorem 3: If there exists M symmetric positive definite matrices $P_i > 0$, and M matrices X_i, $\forall i \in \bar{\mathrm{M}}$ and $M^2 - M$ matrices H_{ij}, $\forall i, j \in \bar{\mathrm{M}}$, $i \neq j$ which satisfy the LMIs (29), (30), then:

i) The eigenvalues of $A_i - L_iC_i$ satisfy the inequality $|eig(A_i - L_iC_i)| < \beta, \quad \forall i \in \bar{\mathrm{M}}$.

ii) If the $\mu^* > 1$, the error dynamic in the switching observer O is globally uniformly exponentially stable for the switching signal $\sigma(k)$ with any average dwell time k_a greater than k_{min} given by (32). Otherwise $(0 < \mu^* \leq 1)$, the error dynamic in the switching observer O is globally uniformly exponentially stable for arbitrary switching signals.

Proof: The proof is omitted here due to space restriction. ■

In general the above problem is non-convex. To relax the above non-convex problem to an LMI problem in the next algorithm the same restriction on the structure of the proposed observer is made and H_{ij} in the structure of the observer are assumed to be the same as G_{ij} in the system.

Algorithm 4: Consider the switched system described by

(8). Similar to the previous algorithm the following steps are followed to obtain the values of gains L_i and H_{ij}, $\forall i, j \in \bar{\mathrm{M}}$, $i \neq j$.

Step 1: For the given $0 < \beta < 1$, find the minimum value of μ for which there exist M matrices X_i, $\forall i \in M$, and M positive definite symmetric matrices P_i, $\forall i \in M$, satisfying the following LMIs

$$\begin{bmatrix} \beta^2 P_j & A_i^T P_i - C_i^T X_i^T \\ P_i A_i - X_i C_i & P_i \end{bmatrix} > 0 \quad \forall i, j \in \bar{\mathrm{M}} \tag{33}$$

$$\mu P_i \geq G_{ij}^T P_i G_{ij} \quad \forall i, j \in \bar{\mathrm{M}},\ i \neq j \tag{34}$$

Denote the optimum of the above convex optimization problem with μ^*.

Step 2: Using the matrices P_i and X_i^1, $\forall i \in M$ obtained in Step 1, find L_i, the observer gain of O_i proposed in (28) as follows

$$L_i = P_i^{-1} X_i, \qquad \forall i \in \bar{\mathrm{M}} \tag{35}$$

Step 3: If $\mu^* > 1$, compute the minimum allowable dwell time

$$\tau_{min} = 1 - \frac{\log \mu^*}{\log \beta^2} \tag{36}$$

Theorem 4: If there exists M symmetric positive definite matrices $P_i > 0$, and M matrices X_i, $\forall i \in \bar{\mathrm{M}}$ which satisfy the LMIs (33), (34), then if the $\mu^* > 1$, the error dynamic in the switching observer O is globally uniformly exponentially stable and for the switching signal $\sigma(k)$ with any average dwell time k_a greater than k_{min} given by (36). Otherwise ($0 < \mu^* \leq 1$), the error dynamic in the switching observer O is globally uniformly exponentially stable for arbitrary switching signals. Moreover, $|eig(A_i - L_i C_i)| < \beta$, $\forall i \in \bar{\mathrm{M}}$.

Proof: The proof is omitted here due to space restriction. ■

In the following, inspired by the work given in [6], alternate sufficient conditions for the LMI (29) are proposed.

Proposition 1: Assume C_i, $\forall i \in \bar{\mathrm{M}}$, in (29) are full-row rank. Given $\beta \in (0,1)$, if there exists M positive definite matrices $S_1, \cdots, S_M$, M matrices $U_1, \cdots, U_M$, M matrices $G_1, \cdots, G_M$, and M matrices $V_1, \cdots, V_M$ satisfying

$$\begin{bmatrix} \beta^2 G_i + \beta^2 G_i^T - \beta^2 S_j & G_i^T A_i^T - C_i^T U_i^T \\ A_i G_i - U_i C_i & S_i \end{bmatrix} > 0 \qquad \forall i, j \in \bar{\mathrm{M}} \tag{37}$$

and

$$V_i C_i = C_i G_i, \quad \forall i \in \bar{\mathrm{M}} \tag{38}$$

then there exist M positive definite matrices $P_1, \cdots, P_M$ and M matrices $L_1, \cdots, L_M$ satisfying

$$\begin{bmatrix} \beta^2 P_j & (A_i - L_i C_i)^T P_i \\ P_i (A_i - L_i C_i) & P_i \end{bmatrix} > 0 \quad \forall i, j \in \bar{\mathrm{M}} \tag{39}$$

Proof: From (37), it can be concluded that for all $\forall i, j \in \bar{\mathrm{M}}$

$$\beta^2 G_i + \beta^2 G_i^T > \beta^2 S_j \tag{40}$$

and thus the matrix G_i, $\forall i \in \bar{\mathrm{M}}$, is full-rank. Furthermore, since C_i, $\forall i \in \bar{\mathrm{M}}$, is assumed to be full-row rank, the matrix V_i satisfying (38) is nonsingular, $\forall i \in \bar{\mathrm{M}}$ matrices. Define $L_i = U_i V_i^{-1}$ and hence, rewrite (37) as

$$\begin{bmatrix} \beta^2 G_i + \beta^2 G_i^T - \beta^2 S_j & G_i^T (A_i^T - C_i^T L_i^T) \\ (A_i - L_i C_i) G_i & S_i \end{bmatrix} > 0 \tag{41}$$

It follows from positive definiteness of S_i, $\forall i \in \bar{\mathrm{M}}$ that

$$\beta^2 (S_j - G_i)^T S_j^{-1} (S_j - G_i) \geq 0 \quad \forall i, j \in \bar{\mathrm{M}}$$

or, equivalently

$$\beta^2 G_i^T S_j^{-1} G_i \geq \beta^2 G_i^T + \beta^2 G_i - \beta^2 S_j \tag{42}$$

It follows from (41) and (42) that

$$\begin{bmatrix} \beta^2 G_i^T S_j^{-1} G_i & G_i^T (A_i^T - C_i^T L_i^T) \\ (A_i - L_i C_i) G_i & S_i \end{bmatrix} > 0$$

Therefore,

$$\begin{bmatrix} G_i^T & 0 \\ 0 & S_i \end{bmatrix} \begin{bmatrix} \beta^2 S_j^{-1} & (A_i^T - C_i^T L_i^T) S_i^- 1 \\ S_i^{-1} (A_i - L_i C_i) & S_i^{-1} \end{bmatrix} \begin{bmatrix} G_i & 0 \\ 0 & S_i \end{bmatrix} > 0$$

which is equivalent to

$$\begin{bmatrix} \beta^2 S_j^{-1} & (A_i^T - C_i^T L_i^T) S_i^- 1 \\ S_i^{-1} (A_i - L_i C_i) & S_i^{-1} \end{bmatrix} > 0$$

Let S_i^{-1} be denoted by P_i, $\forall i \in \bar{\mathrm{M}}$. Then, one can obtain the following matrix inequality

$$\begin{bmatrix} \beta^2 P_j & (A_i - L_i C_i)^T P_i \\ P_i (A_i - L_i C_i) & P_i \end{bmatrix} > 0 \quad \forall i, j \in \bar{\mathrm{M}}$$

This completes the proof. ■

It is to be noted that in the inequality given by (29), X_i is equal to $P_i L_i$. As a result, the LMI conditions in (29) and (39) are the same.

Remark 7: According to Proposition 1, if the LMI (39) does not hold, the proposed alternate conditions (37) and (38) are also infeasible. To specify the advantage of using the proposed alternate LMI conditions, consider an impulsive parameter varying switched system. In this case, it is aimed to search for a parameter dependent Lyapunov function corresponding to each uncertain modes in the switched system. In this context, introducing slack variables G_i in (37) and (38) is of great importance which leads to a less degree of conservatism [5].

Similarly, alternate sufficient conditions for LMI (33) can be obtained.

Proposition 2: Assume C_i, $\forall i \in \bar{\mathrm{M}}$, in (33) are full-row rank. For $0 < \beta < 1$, if there exist M symmetric matrices $S_1, \cdots, S_M$ and M matrices $U_1, \cdots, U_M$, M matrices $G_1, \cdots, G_M$, and M matrices $V_1, \cdots, V_M$ satisfying

$$\begin{bmatrix} \beta^2 G_i + \beta^2 G_i^T - \beta^2 S_i & G_i^T A_i^T - C_i^T U_i^T \\ A_i G_i - U_i C_i & S_i \end{bmatrix} > 0 \quad \forall i \in \bar{\mathrm{M}} \tag{43}$$

and

$$V_i C_i = C_i G_i, \quad \forall i \in \bar{\mathrm{M}} \tag{44}$$

Then there exist M symmetric matrices $P_1, \cdots, P_M$ and M matrices $L_1, \cdots, L_M$ satisfying

$$\begin{bmatrix} \beta^2 P_i & (A_i - L_i C_i)^T P_i \\ P_i (A_i - L_i C_i) & P_i \end{bmatrix} > 0 \quad \forall i \in \bar{\mathrm{M}} \tag{45}$$

Proof: Proposition 2 can be regarded as a special case of Proposition 1 when all the indexes are the same. ■

IV. Numerical Examples

In this section, two numerical examples are given to show the effectiveness of the proposed algorithms.

Example 1: Consider a continuous impulsive switched system given by (2) where the switching signal $\sigma(t)$ is a piecewise constant function with the set of images equal to $\{1,2\}$ and the system in different modes is represented by

$$A_1 = \begin{bmatrix} -2 & 0 \\ 2 & -3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -1 & -4 \\ 3 & -8 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 2 & -1 \end{bmatrix}$$

and

$$G_{12} = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}, \quad G_{21} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The desired region for both Luenberger observers' rate of convergence is assumed to be $-10 < \lambda_0 < -4$. Using Algorithm 2, the observer gains are obtained as follows

$$L_1 = \begin{bmatrix} -16.6293 \\ -22.6511 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 2.6276 \\ 1.6847 \end{bmatrix}$$

The minimum value of μ using GEVP (generalized eigenvalue problem) is obtained as $\mu^* = 36.0086$, which implies that the proposed observer is stable for the given impulsive switched system for any switching signal with the average dwell time greater than $\tau_{min} = \frac{\log(36.0086)}{(-2)(-4)} = 0.4480$.

Example 2: Consider a discrete impulsive switched system given by (8) consisting of two modes represented by the state-space matrices

$$A_1 = \begin{bmatrix} -0.2 & 0 & 0.1 \\ 0.2 & -0.3 & 0.8 \\ 0.5 & -0.3 & 0.6 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1 \\ 0.5 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.1 & -0.4 & 0 \\ 0.3 & -0.8 & 0.5 \\ 0.1 & 0 & 0.7 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & -1 & -2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}$$

and

$$G_{12} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad G_{21} = \begin{bmatrix} -2 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix}$$

Assume that it is desired to have the eigenvalues of each Luenberger observer O_i designed for each mode inside the circle in the s-plane centered at the origin and with the radius equal to 0.3; that is, $\beta = 0.3$. The minimum value of μ using GEVP is obtained as $\mu^* = 206.7415$. Since $\tau_{min} = 1 - \frac{\log(206.7415)}{\log(0.09)} = 3.2141$, the proposed observer is stable for the given impulsive switched system for any switching signal with the average dwell time greater than or equal to 4. The two Luenberger observer gains are

$$L_1 = \begin{bmatrix} -0.0292 \\ -0.2206 \\ -0.1575 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -0.6840 \\ -0.5613 \\ 0.6031 \end{bmatrix}$$

Using the alternative form of Algorithm 4 which has fewer LMI conditions will result in $\mu^* = 9.1325$, and the corresponding minimum required average dwell time will be greater than or equal to 3.

V. Conclusions

In this work, sufficient conditions for the stability of continuous and discrete impulsive switched systems are presented. LMI-based algorithms are developed subsequently to design observers for impulsive switched systems. Using these algorithms, asymptotic stability of the error dynamics in the switching observers can be determined for a special class of impulsive switched systems under constrained switching. Numerical examples are provided to demonstrate the efficacy of the proposed results.

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