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Neighboring Extremal Solution for Discrete-Time Optimal Control Problems with State Inequality Constraints

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***Abstract*— A neighboring extremal control method is proposed for discrete-time optimal control problems subject to a general class of inequality constraints. The approach generalizes the method proposed in [7] to the case when the results of [7] become inapplicable: one with constraints which depend only on states but not inputs and another with over-determined input-state constraints. The application of the proposed method leads to a computationally efficient model predictive control algorithm, which is described in conjunction with a numerical example, to illustrate the utility of the proposed approach.**

I. Introduction

Efficient numerical methods which solve finite horizon optimal control problems can broaden the range of applications of optimization-based control, including Model Predictive Control. One approach to reduce the computational time and effort is to use an approximate solution derived using the neighboring extremal method.

The neighboring extremal solution associated with a perturbed initial state, in the absence of state or input constraints, is presented in [1], [2], for continuous time systems, while its counterpart for discrete time systems can be found in [3], [4]. Moreover, the neighboring extremal solution for continuous time systems with inequality constraints and discontinuities can be derived using multi-point boundary value techniques, as presented in [5].

For discrete time systems, the dynamic optimal control problem can be transformed (or transcripted) into a nonlinear programming problem. Consequently, exploiting sensitivity analysis for the nonlinear programming problem, the neighboring extremal solution can be calculated as shown in [6]. The drawback of this method is that solving the resulting high-dimensional quadratic programming problem can be computationally expensive.

In an attempt to alleviate the computational burden associated with the nonlinear programming, a neighboring extremal method was proposed in [7] for discrete-time systems with input and state inequality constraints. This method is based on linearization of the necessary optimality conditions of the original problem along a nominal optimal trajectory, thereby leading to a set of Riccati-like backward recursive equations that can be used to calculate the neighboring extremal solution. This method has the advantage that the numerical effort for calculating the neighboring extremal solutions grows only linearly with respect to the length of the horizon, as compared to the cubical growth rate for the nonlinear programming method. However, the method proposed in [7] is not applicable if there are pure state inequality constraints, or when the state inequality constraints are over-determined at some time instants.

In this paper we propose a modified neighboring extremal method for discrete-time systems which can handle more general classes of constraints. These classes of constraints include cases when there are constraints dependent only on states but not inputs, as well as the cases when the active inequality constraints outnumber the control inputs at some time instant over the horizon. We will also illustrate the application of the proposed approach to Model Predictive Control, using a 5th order nonlinear ship maneuvering model as an example.

II. Problem Formulation

In this section we review the neighboring extremal control approach for discrete-time systems with mixed input and state inequality constraints, proposed in [7]. Consider the problem of minimizing a cost function,

$$J[u] = \sum_{k=0}^{N-1} L(x(k), u(k)) + \Phi(x(N)), \tag{1}$$

over all feasible control sequences $u : [0, N] \to \mathbb{R}^m$ and all state vectors $x : [0, N] \to \mathbb{R}^n$ subject to the following constraints:

$$x(k+1) = f(x(k), u(k)); \quad f : \mathbb{R}^{n+m} \to \mathbb{R}^n \tag{2}$$

$$x(0) = x_0; \; x_0 \in \mathbb{R}^n \tag{3}$$

$$C(x(k), u(k)) \leq 0, \; C : \mathbb{R}^{n+m} \to \mathbb{R}^l. \tag{4}$$

Here, it is assumed that the number of active inequality constraints at each time instant is not greater than the number of control inputs.

Let $x^o(k)$, $u^o(k)$, $k \in [0, N]$ be the state and control trajectories corresponding to the optimal solution in the problem of minimizing (1) subject to the constraints (2)-(4) with the initial condition $x(0)$. The solution $x^o(k)$, $u^o(k)$ is referred to as the nominal solution. Let $C^a(k)$ be a vector consisting of the constraints that are active at the time instant k and $\mu(k)$ be the corresponding Lagrange variable[1]. Moreover, letting $\lambda(\cdot)$ be the sequence of co-states associated

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[1]Note that the dimension of $C^a(k)$ could vary for different k. It is an empty vector if no constraint is active at the time instant k.

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with the dynamics of the system, the Hamiltonian function can be defined as follows:

$$\begin{aligned} H(k) &= L(x(k),u(k)) + \lambda(k+1)^T f(x(k),u(k)) \\ &+ \mu(k)^T C^a(x(k),u(k)). \end{aligned} \tag{5}$$

As shown in [11], if a perturbation $\delta x(0)$ in the initial state $x(0)$ does not change the activity status of the constraints, then the corresponding optimal solution to the problem defined by the cost function (1) and constraints (2), (4) and initial state $x(0) = x_0 + \delta x(0)$ can be approximated, in a neighboring extremal sense, as $x^o(k) + \delta x(k)$ and $u^o(k) + \delta u(k),\ k \in [0, N]$, provided

$$Z_{uu}(k) \succ 0 \text{ for } k \in [0, N] \tag{6}$$

for the nominal solution, where[2]

$$\begin{aligned} Z_{uu}(k) &= H_{uu}(k) + f_u^T(k)S(k+1)f_u(k), \\ Z_{ux}(k) &= Z_{xu}(k)^T = H_{ux}(k) + f_u^T(k)S(k+1)f_x(k), \\ Z_{xx}(k) &= H_{xx}(k) + f_x^T(k)S(k+1)f_x(k) \end{aligned} \tag{7}$$

and $S(k)$ in equation (7) is given by:

$$\begin{aligned} S(i) &= Z_{xx}(i) - [Z_{xu}(i)\ C_x^T(i)]K_0(i) \left[\begin{array}{c} Z_{ux}(i) \\ C_x(i) \end{array} \right], \\ S(N) &= \Phi_{xx}(N). \end{aligned} \tag{8}$$

Moreover, the following explicit relation between state and input variations can be derived to calculate the perturbed solution:

$$\delta u(k) = K^*(k)\delta x(k), \tag{9}$$

$$K^*(k) = -[I\ 0]K_0(k) \left[\begin{array}{c} Z_{ux}(k) \\ C_x^a(k) \end{array} \right], \tag{10}$$

and

$$K_0(k) = \left[\begin{array}{cc} Z_{uu}(k) & C_u^{aT}(k) \\ C_u^a(k) & 0 \end{array} \right]^{-1}. \tag{11}$$

Since it is assumed that the matrix $Z_{uu}(\cdot)$ is positive definite over the entire horizon, the matrix $K_0(k)$ is well defined as long as $C_u^a(k)$ is full row rank for $k = 0, \cdots, N-1$. If $C_u^a(k)$ is not full row rank at some time instant k, the matrix $K_0(k)$ is not well defined and the proposed algorithm fails. Two special cases of such situation can be easily identified: one is when the constraint is a function of state $x(\cdot)$ and not input $u(\cdot)$ (in this case, $C_u = 0$), another is when the number of active inequality constraints at the time instant k is greater than the number of control inputs, m. The goal of the subsequent sections is to circumvent the technical difficulties in the these cases, and propose a general approach that can deal with a broader class of problems.

[2]Following the notation used in [8], H_{uu}, H_{ux}, H_{xx}, Φ_{xx}, etc., denote the partial derivatives with respect to x and/or u, with the exception for Z_{uu}, Z_{ux}, Z_{xx}, which are defined by (7).

III. Perturbation analysis for discrete time optimal control problem subject to constraints

In this section we consider the case where there are general point-wise-in-time input-state constraints. Let us consider the optimization problem of minimizing the cost (1) subject to the following constraints:

$$x(k+1) = f(x(k),u(k)); \quad f: \mathbb{R}^{n+m} \to \mathbb{R}^n, \tag{12}$$

$$x(0) = x_0;\ x_0 \in \mathbb{R}^n, \tag{13}$$

$$C(x(k),u(k)) \le 0,\ C: \mathbb{R}^{n+m} \to \mathbb{R}^l, \tag{14}$$

$$\bar{C}(x(k)) \le 0\ \ \bar{C}: \mathbb{R}^n \to \mathbb{R}^{\bar{l}}, \tag{15}$$

where C and $\bar{C}$ denote the mixed state-input constraints and state-only constraints, respectively.

In order to describe the neighboring extremal solution to this problem, we first introduce the following notations. Let $x(k), u(k),\ k \in [0, N]$ be referred to as the nominal solution for the state and control corresponding to the optimal solution in the problem of minimizing (1) subject to the constraints (12)-(15). With $\lambda(\cdot)$ defined as in Section II, the Hamiltonian function can be defined as follows:

$$\begin{aligned} H(k) &= L(x(k),u(k)) + \lambda(k+1)^T f(x(k),u(k)) \\ &+ \mu(k)^T C^a(x(k),u(k)) + \bar{\mu}(k)^T \bar{C}^a(x(k)), \end{aligned} \tag{16}$$

where $\mu(\cdot)$ and $\bar{\mu}(\cdot)$ are vectors of Lagrange multipliers associated with the active parts of constraints (14) and (15), respectively. Now we define matrix sequences $\tilde{C}_u(\cdot)$, $\tilde{C}_x(\cdot)$ and $S(\cdot)$ using the following backward recursive equations. Let

$$\begin{aligned} \hat{C}_x(N) &:= \bar{C}_x^a(x(N)), \\ S(N) &:= \Phi_{xx}(N) + \frac{\partial}{\partial x}(\bar{C}_x^T(x(N))\bar{\mu}(N)), \end{aligned} \tag{17}$$

and, at the time instant k, we define

$$\begin{aligned} C_{aug}(k) &:= \left[\begin{array}{c} C_u^a(k) \\ \hat{C}_x(k+1)f_u(k) \end{array} \right], \\ \tilde{r}_k &:= rank(C_{aug}(k)). \end{aligned} \tag{18}$$

At each time instant k, there is a matrix $P(k)$ that transforms matrix $C_{aug}(k)$ into an upper triangular form, namely

$$P(k) \left[\begin{array}{c} C_u(k) \\ \hat{C}_x(k+1)f_u(k) \end{array} \right] = \left[\begin{array}{c} \tilde{C}_u(k) \\ 0 \end{array} \right], \tag{19}$$

with $\tilde{C}_u(k) \in \mathbb{R}^{\tilde{r}_k \times m}$ having independent rows.

By defining

$$\Gamma(k) := \left[\begin{array}{c} P(k) \left[\begin{array}{c} C_x^a(k) \\ \hat{C}_x(k+1)f_x(k) \end{array} \right] \\ \bar{C}_x^a(k) \end{array} \right],$$

and assuming that γ_k is the number of rows of matrix $\Gamma(k)$, we can define

$$\begin{aligned} \tilde{C}_x(k) &:= [I_{\tilde{r}_k \times \tilde{r}_k}\ 0_{\tilde{r}_k \times (\gamma_k - \tilde{r}_k)}]\Gamma(k) \in \mathbb{R}^{\tilde{r}_k \times m}, \\ \hat{C}_x(k) &:= [0_{(\gamma_k - \tilde{r}_k) \times \tilde{r}_k}\ I_{(\gamma_k - \tilde{r}_k) \times (\gamma_k - \tilde{r}_k)}]\Gamma(k) \in \mathbb{R}^{(\gamma_k - \tilde{r}_k) \times m}. \end{aligned} \tag{20}$$

Having $Z_{uu}(\cdot)$, $Z_{ux}(\cdot)$ and $Z_{xx}(\cdot)$ defined in (7), the matrix $S(k)$ can be defined as follows

$$S(k) = Z_{xx}(k) - [Z_{xu}(k)\ \tilde{C}_x^T(k)]K_0(k)\begin{bmatrix} Z_{ux}(k) \\ \tilde{C}_x(k) \end{bmatrix}, \quad (21)$$

where

$$K_0(k) = \begin{bmatrix} Z_{uu}(k) & \tilde{C}_u(k)^T \\ \tilde{C}_u(k) & 0 \end{bmatrix}^{-1}. \quad (22)$$

Using equation (17) as an initial condition for backward iterating, we can apply equations (19), (20) and (21) to calculate matrix sequences $Z_{uu}(\cdot)$, $Z_{ux}(\cdot)$, $Z_{xx}(\cdot)$, $\tilde{C}_u(\cdot)$, $\tilde{C}_x(\cdot)$ and $P(\cdot)$. Having the above matrix sequences calculated, we introduce the following theorem which gives a sufficient condition for existence of the neighboring extremal solution to the optimal control problem with the perturbed initial state. The theorem is followed by a corollary which gives the neighboring extremal solution.

Theorem 3.1: If $rank(\hat{C}_x(0)) = 0$, then a sufficient condition for the existence of the neighboring extremal control subject to the inequality constraints and initial state perturbation $\delta x(0)$ is

$$Z_{uu}(k) \succ 0 \text{ for } k \in [0, N-1]. \quad (23)$$

Remark 3.1: Condition (23) guarantees the convexity of the quadratic programming problem resulting from the second order variational analysis, which is performed to calculate the neighboring extremal solution.

Remark 3.2: The condition, $rank(\hat{C}_x(0)) = 0$, can be interpreted as follows: For the cases where the constraints involve only states and not inputs, or the cases where the constraint variations caused by control variations are dependent, the state equation $x(k) = f(x(k-1), u(k-1))$ will be used to back-propagate the constraints to the time $k-1$. The condition $rank(\hat{C}_x(0)) = 0$ implies that such back-propagation, when applied consecutively as needed, will not result in a constraint on the initial state variation $\delta x(0)$.

Corollary 3.1: If a perturbation $\delta x(0)$ in the initial state $x(0)$ does not change the activity status of the constraints and

$$Z_{uu}(k) \succ 0 \text{ for } k \in [0, N-1], \quad (24)$$

then the corresponding optimal solution to the problem defined by the cost function (1) and constraints (12)-(15) and initial state $x(0) = x_0 + \delta x(0)$ can be approximated as $x(k) + \delta x(k)$ and $u(k) + \delta u(k),\ k \in [0, N]$ where

$$\delta u(k) = K^*(k)\delta x(k), \quad (25)$$

$$K^*(k) = -[I\ 0]K_0(k)\begin{bmatrix} Z_{ux}(k) \\ \tilde{C}_x(k) \end{bmatrix}. \quad (26)$$

We describe the derivation of the neighboring extremal solution given in equation (25) next.

Derivation of the neighboring extremal solution

Using the second order variational analysis, we minimize the variation of the Hamiltonian function in the optimization problem of minimizing the cost (1) subject to constraints (12)-(15). Namely, we consider the problem of minimizing the following functional

$$\delta^2 \bar{J} = 1/2\delta x(N)^T \left(\Phi_{xx}(N) + \frac{\partial}{\partial x}(\bar{C}_x^T(x(N))\bar{\mu}(N))\right)\delta x(N) + 1/2\sum_{k=0}^{N-1}\begin{bmatrix} \delta x(k) \\ \delta u(k) \end{bmatrix}^T \begin{bmatrix} H_{xx}(k) & H_{xu}(k) \\ H_{ux}(k) & H_{uu}(k) \end{bmatrix}\begin{bmatrix} \delta x(k) \\ \delta u(k) \end{bmatrix}, \quad (27)$$

subject to the constraints:

$$\delta x(k+1) = f_x(k)\delta x(k) + f_u(k)\delta u(k), \quad (28)$$
$$\delta x(0) = \delta x_0, \quad (29)$$
$$C_x^a(k)\delta x(k) + C_u^a(k)\delta u(k) = 0, \quad (30)$$
$$\bar{C}_x^a(k)\delta x(k) = 0, \quad (31)$$

which are obtained by linearizing (12)-(15) at the nominal solution. Let us assume that $\delta\lambda(\cdot)$, $\delta\mu(\cdot)$ and $\delta\bar{\mu}(\cdot)$ are the Lagrange multipliers associated with constraints (28), (30) and (31), respectively. Hereafter, the superscript a is dropped for notational simplicity, assuming that the constraints appearing in the equations are active.

By applying the Karush-Kuhn-Tucker (KKT) conditions to the problem (27) for the time instant $k = N$, we have

$$\begin{aligned} \delta\lambda(N) &= \left(\Phi_{xx}(N) + \frac{\partial}{\partial x}(\bar{C}_x^T(x(N))\bar{\mu}(N))\right)\delta x(N) \\ &+ \bar{C}_x^T(x(N))\delta\bar{\mu}(N), \\ &\bar{C}_x(x(N))\delta x(N) = 0. \end{aligned} \quad (32)$$

Defining $\delta\hat{\mu}(N) := \delta\bar{\mu}(N)$, $T(N) := 0$, $\hat{C}_x(N) := \bar{C}_x(x(N))$ and

$$S(N) := \Phi_{xx}(N) + \frac{\partial}{\partial x}(\bar{C}_x^T(x(N))\bar{\mu}(N)),$$

the first equality in (32) can be expressed as

$$\delta\lambda(N) = S(N)\delta x(N) + T(N) + \hat{C}_x^T(N)\delta\hat{\mu}(N). \quad (33)$$

Now assume that for the time instant $k+1$,

$$\delta\lambda(k+1) = S(k+1)\delta x(k+1) + T(k+1) + \hat{C}_x^T(k+1)\delta\hat{\mu}(k+1) \quad (34)$$

and

$$\hat{C}_x(k+1)\delta x(k+1) = 0. \quad (35)$$

From equations (35), (28) and (30), we have:

$$C_u(k)\delta u(k) + C_x(k)\delta x(k) = 0, \quad (36)$$
$$\hat{C}_x(k+1)f_u(k)\delta u(k) + \hat{C}_x(k+1)f_x(k)\delta x(k) = 0 \quad (37)$$
$$\bar{C}_x(k)\delta x(k) = 0. \quad (38)$$

Applying the Karush-Kuhn-Tucker (KKT) conditions to the problem (27) at time k, we have

$$\begin{aligned} \delta\lambda(k) &= H_{xx}\delta x(k) + H_{xu}\delta u(k) + f_x^T(k)\delta\lambda(k+1) \\ &+ C_x^T(k)\delta\mu(k) + \bar{C}_x^T(k)\delta\bar{\mu}(k). \end{aligned} \quad (39)$$

Substituting expression of $\delta\lambda(k+1)$ given by (34), equation (39) becomes:

$$\delta\lambda(k) = Z_{xx}(k)\delta x(k) + Z_{xu}(k)\delta u(k) + f_x^T(k)T(k+1) + \begin{bmatrix} C_x(k) \\ \hat{C}_x(k+1)f_x(k) \\ \bar{C}_x(k) \end{bmatrix}^T \begin{bmatrix} \delta\mu(k) \\ \delta\hat{\mu}(k+1) \\ \delta\bar{\mu}(k) \end{bmatrix}. \tag{40}$$

By defining

$$\delta\tilde{\mu}(k) := [I_{\tilde{r}_k\times\tilde{r}_k}\ 0_{\tilde{r}_k\times(\gamma_k-\tilde{r}_k)}]\begin{bmatrix} P(k)^{-T}\begin{bmatrix} \delta\mu(k) \\ \delta\hat{\mu}(k+1) \end{bmatrix} \\ \delta\bar{\mu}(k) \end{bmatrix},$$

$$\delta\hat{\mu}(k) := [0_{(\gamma_k-\tilde{r}_k)\times\tilde{r}_k}\ I_{(\gamma_k-\tilde{r}_k)\times(\gamma_k-\tilde{r}_k)}]\begin{bmatrix} P(k)^{-T}\begin{bmatrix} \delta\mu(k) \\ \delta\hat{\mu}(k+1) \end{bmatrix} \\ \delta\bar{\mu}(k) \end{bmatrix} \tag{41}$$

where $\delta\tilde{\mu}(k) \in \mathbb{R}^{\tilde{r}_k}$, $\delta\hat{\mu}(k) \in \mathbb{R}^{(\gamma_k-\tilde{r}_k)}$, and referring to (19) for the definition of $\tilde{C}_x(\cdot)$ and $\hat{C}_x(\cdot)$, we have

$$\delta\lambda(k) = Z_{xx}(k)\delta x(k) + Z_{xu}(k)\delta u(k) + f_x^T(k)T(k+1) + \tilde{C}_x(k)^T\delta\tilde{\mu}(k) + \hat{C}_x(k)\delta\hat{\mu}(k). \tag{42}$$

On the other hand, from equations (36), (37), (19) and (20) we have

$$\tilde{C}_u(k)\delta u(k) + \tilde{C}_x(k)\delta x(k) = 0. \tag{43}$$

In addition, by applying the Karush-Kuhn-Tucker (KKT) conditions to the problem (27)-(31), $\delta x(k)$, $\delta u(k)$, $\delta\lambda(k)$ and $\delta\mu(k)$ should satisfy the following equation,

$$H_{ux}(k)\delta x(k) + H_{uu}(k)\delta u(k) + f_u(k)^T\delta\lambda(k+1) + C_u(k)^T\delta\mu(k) = 0. \tag{44}$$

Using equations (44), (34), and (19) we have

$$Z_{uu}(k)\delta u(k) + \tilde{C}_u^T(k)\delta\tilde{\mu}(k) = -Z_{ux}(k)\delta x(k) - f_u^T(k)T(k+1). \tag{45}$$

Since $\tilde{C}_u(k)$ is full row rank and $Z_{uu}(k)$ is positive definite, the matrix

$$K_0(k) = \begin{bmatrix} Z_{uu}(k) & \tilde{C}_u(k)^T \\ \tilde{C}_u(k) & 0 \end{bmatrix}^{-1} \tag{46}$$

is well defined and from equations (43) and (45) we have

$$\begin{bmatrix} \delta u(k) \\ \delta\tilde{\mu}(k) \end{bmatrix} = -K_0(k)\begin{bmatrix} Z_{ux}(k) \\ \tilde{C}_x(k) \end{bmatrix}\delta x(k) - K_0(k)\begin{bmatrix} f_u^T(k)T(k+1) \\ 0 \end{bmatrix}. \tag{47}$$

Applying equation (47) to (42), we have

$$\delta\lambda(k) = S(k)\delta x(k) + T(k) + \hat{C}_x^T(k)\delta\hat{\mu}(k) \tag{48}$$

where $\hat{C}_x^T(k)$ and $S(k)$ are calculated from equation (20) and (21), respectively, and

$$T(k) \equiv 0.$$

If $rank(\hat{C}_x(0)) \neq 0$, then either $\hat{C}_x(0)\delta x(0) \equiv 0$, which means that the linear equations resulted from linearizing active constraints are redundant, or the variation $\delta x(0)$ for the initial state $x(0)$ is infeasible. In both cases, further modification will be needed in order to apply the proposed algorithm. This modification will be addressed in our future work.

The procedure for determining the neighboring extremal solution can be summarized as follows.

- Initialize matrices $S(N)$ and $\hat{C}_x(N)$ using equation (17).
- Calculate, in a backward run, matrix sequences $P(\cdot)$ (using equation (19)), $\tilde{C}_u(\cdot)$ and $\tilde{C}_x(\cdot)$ (using equations (19 and (20))), $Z_{uu}(\cdot)$, $Z_{ux}(\cdot)$, $Z_{xx}(\cdot)$ (using equation (7)) and $S(\cdot)$ (using equation (21)).
- Given initial state variation $\delta x(0)$, in a forward run, calculate $\delta x(\cdot)$ and $\delta u(\cdot)$ using equation (25) and (28).

Figure 1 illustrates the computation sequence and equations involved in calculating $\delta u(\cdot)$ and $\delta x(\cdot)$, which involves a step to calculate $\mu(\cdot)$, $\hat{\mu}(\cdot)$ and $\lambda(\cdot)$.

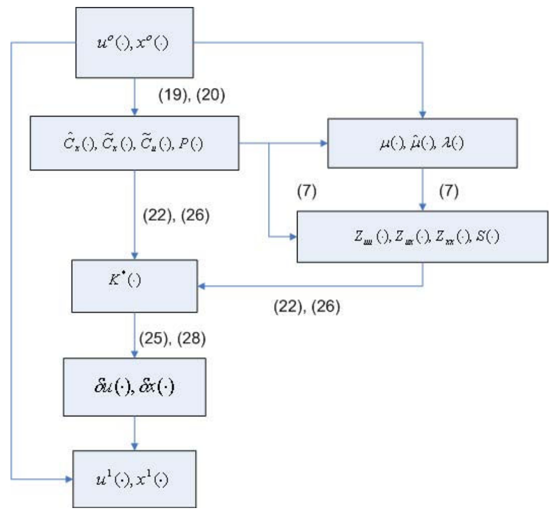

Fig. 1. Computation sequence for calculating the neighboring extremal solution.

IV. Applications in Model Predictive Control

The neighboring extremal method developed in the previous sections can be useful in developing fast model predictive control algorithms. In this section, we illustrate two approaches based on the neighboring extremal method proposed in Section III in the context of MPC. One approach is using the neighboring extremal method to develop a local compensation mechanism for the Forecasting MPC (FMPC) proposed in [9], [10]. Another one is the combination of the neighboring extremal method with the Sequential Quadratic Programming, as proposed in [11], where it is referred to as IPA-SQP.

Consider a class of discrete-time systems represented by the following difference equation

$$x(k+1) = f(x(k), u(k)) \tag{49}$$

where the state and control must satisfy the constraints

$$C(x(k), u(k)) \leq 0,\ \ C: \mathbb{R}^{n+m} \to \mathbb{R}^l. \tag{50}$$

$$\bar{C}(x(k)) \leq 0,\ \ \bar{C}: \mathbb{R}^n \to \mathbb{R}^{\bar{l}}. \tag{51}$$

Based on the FMPC strategy [12], [13], the following optimal control problem is solved within the time interval $[k-1, k]$:

$$\mathscr{P}_N(\hat{x}) : V_N^*(\hat{x}) = \min_{\mathbf{u}} \{V_N(x, \mathbf{u})\},$$

$$V_N(\hat{x}, \mathbf{u}) = \sum_{i=k}^{k+N-1} L(x(i), u(i)) + \Phi(x(k+N)), x(k) = \hat{x}(k) \tag{52}$$

where

$$\hat{x}(k) = f(x(k-1), u(k-1)), \tag{53}$$

$$\mathbf{u} = \{u(k), u(k+1), ..., u(k+N-1)\} \tag{54}$$

and $x(k-1)$ is the measured state at the time instant $k-1$, while $\hat{x}(k)$ is the predicted state at the time instant k. Denoting the optimal control sequence as $\mathbf{u}^*(\hat{x}(k))$, the FMPC control law at the time instant k is

$$u(k) = \mathbf{u}_1^*(\hat{x}(k)), \tag{55}$$

where $\mathbf{u}_1^*$ denotes the first element in the sequence $\mathbf{u}^*$. In the presence of disturbances or model uncertainties, the predicted state $\hat{x}(k)$ and actual state $x(k)$ may not match. Such difference may be accounted for as the perturbation in the initial condition of the optimal control problem $\mathscr{P}_N(\hat{x})$. By utilizing the neighboring extremal method proposed in the previous sections, the optimal solution corresponding to $x(k)$, the measured state at time k, can be derived by approximating it with the nominal solution, computed in advance, and the perturbation solution which is computed once the state measurement is available at the time instant k. Namely,

$$u(x(k)) \approx \mathbf{u}_1^*(\hat{x}(k)) + \delta u(\delta x_0) \tag{56}$$

where $\delta x_0 = x(k) - \hat{x}(k)$. This strategy can be expanded to states predicted more than a single step in advance, and its advantage is that it permits more time to perform computations.

The second example is the IPA-SQP approach [11], described in the following, that is based on approximating the optimal solution at each time instant using the optimal solution computed at the previous time instant.

Let us assume that at the time instant $k+1$, the state $x(k+1)$ is observed and the optimal control problem $\mathscr{P}_N(x(k+1))$ must be solved. Suppose that by the time instant $k+1$, the solution to the problem $\mathscr{P}_N(x(k))$ is available. Defining

$$dx(k) := x(k+1) - x(k), \tag{57}$$

the solution of the problem $\mathscr{P}_N(x(k+1))$ can be approximated using the solution of $\mathscr{P}_N(x(k))$ and the neighboring extremal method when the perturbation on the initial state $dx(k)$ is assumed. Clearly, the time and effect involved in computing the approximation are smaller than computing the optimal solution from scratch. However, since the neighboring extremal method is based on an approximation, large error in the optimality condition ($H_u(k) = 0$) may develop if such a strategy is employed repeatedly. Such considerations provided motivation for introducing IPA-SQP method in [11] which unifies the Perturbation Analysis and SQP with the active set method to achieve faster convergence in calculating the perturbed optimal solution, given the nominal optimal solution.

V. A NUMERICAL EXAMPLE

In this section, we consider a ship maneuvering problem. Our objective is to steer a ship to a desired location while avoiding an obstacle. For instance, such an obstacle may represent an oil rig or another ship. Given the constraints involved in this control problem, we choose the MPC as the control approach. The following ship model, taken out from [14], is used for numerical simulation:

$$\begin{aligned} \dot{x}_1 &= x_5 cos(x_3) - (r_1 x_4 + r_3 x_4^3) sin(x_3), \\ \dot{x}_2 &= x_5 sin(x_3) + (r_1 x_4 + r_3 x_4^3) cos(x_3), \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -a x_4 - b x_4^3 + c u_r, \\ \dot{x}_5 &= -f x_5 - W x_4^2 + u_t, \end{aligned} \tag{58}$$

where x_1 and x_2 are the ship's coordinates (in nautical miles (nm)) in the $x_1 - x_2$ plane, x_3 is the heading angle (in radians (rad)), x_4 the yaw rate (rad/min), and x_5 the forward velocity (nm/min). The two control inputs are: u_r, the rudder angle (rad), and u_t, the propeller's thrust (nm/min^2).

The constant parameters in the ship model are summarized in Table 1. With these parameters, the ship has a maximum speed of .25 nm/min = 15 knots for a maximum thrust of 0.235 nm/min^2. For maximal rudder angle of $35^o (0.61 rad)$, the stationary rate of turn is $1^o/$sec.

Table I: Constant parameters of ship model.

Parameter	value	unit
a	1.084	1/min
b	0.62	min/rad^2
c	3.553	1/min^2
r_1	-0.0375	nm/rad
r_3	0	Nm.min^2/rad^3
f	0.86	1/min
W	0.067	nm/rad^2

The control objective is to steer the ship from any initial condition to a neighborhood around the origin described by a circle with a radius 0.1 (nm) with a minimum energy consumption and control effort. Moreover, there is an obstacle described by a circle centered at $(x_1, x_2) = (1.5, 0)$ with radius 2.5 (nm). The obstacle is represented by the following inequality constraint, which depends only on the state,

$$(x_1 - 1.5)^2 + x_2^2 \geq (0.25)^2.$$

To implement MPC to solve this problem, since the consumed energy is proportional to $u_t(\cdot)^{3/2}$, we define the cost (1) with

$$\begin{aligned} L(x(k), u(k)) &= 0.1 u_r(k)^2 + 10 u_t(k)^{3/2}, \\ \Phi(x) &= 2000(x_1^2 + x_2^2). \end{aligned} \tag{59}$$

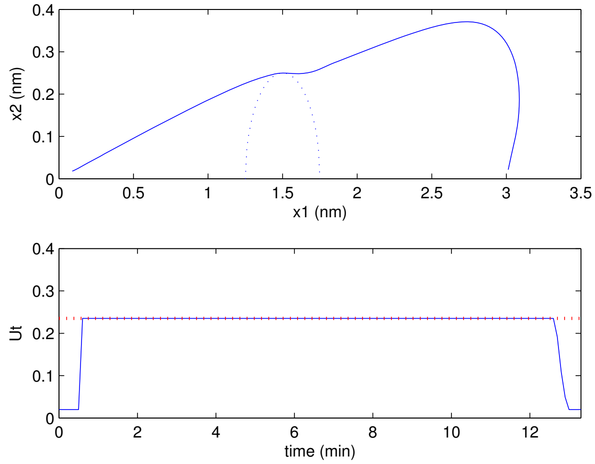

Fig. 2. Ship position on x1-x2 plane (top) and time history of the propeller thrust (bottom). Dashed lines indicate constraints.

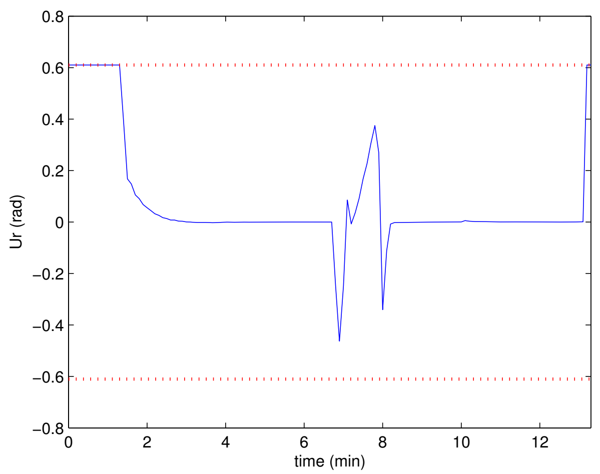

Fig. 3. Time history of the rudder angle. Dashed lines indicate constraints.

Figure 2 shows the ship trajectory in $x_1 - x_2$ plane and the propeller's thrust using IPA-SQP approach for initial condition of $x(0) = [3, 0, \pi/3, 0, 0.25]$. It is shown that the obstacle is avoided. In addition, Figure 3 shows the trajectory of the rudder angle.

VI. Conclusion

In this paper we have extended the neighboring extremal control method for constrained discrete-time systems to handle broader classes of constraints, including pure state constraints and more general mixed input-state constraints. Our approach is based on the use of Riccati-like backward recursive equations and a constraint back-propagation approach. Thanks to our use of Riccati-like backward recursive equations the computing effort grows only linearly in the length of the optimization horizon. The proposed method can be incorporated into Model Predictive Control, and a simulation example, based on a 5th order nonlinear ship maneuvering model, has been reported.

References

[1] J. V. Breakwell, J. L. Speyer, and A. E. Bryson, "Optimization and control of nonlinear systems using the second variation," *SIAM Journal on Control*, Vol. 1, pp. 193-223, 1963.

[2] J. V. Breakwell and Y. C. Ho,"On the conjugate point condition for the control problem," *International Journal of Engineering and Science*, Vol. 2, pp. 565-579, 1965.

[3] S. R. McReynolds and A. E. Bryson,"A successive sweep method for optimal programming problems," Auto control conference, 1965.

[4] F. L. Lewis, *Optimal Control,* Wiley, New York, 1986.

[5] B. Kugelmann, H. J. Pesch, "A new general guidance method in constrained optimal control, Part 1: Numerical method," *Journal of Optimization Theory and Applications*, Vol. 67, pp. 421-435, 1990.

[6] C. Büskens, H. Maurer, "Sensitivity analysis and real-time control of parametric optimal control problems using nonlinear programming method, In M. Grötschel, S. O. Krumke, J. Rambau (editors):, *Online Optimization of Large Scale Systems,* Springer-Verlag, Berlin, 2001.

[7] R. Ghaemi, J. Sun, I. V. Kolmanovsky, "Model Predictive Control for constrained ciscrete time systems: An Optimal Perturbation Analysis Approach," *Proceedings of the American Control Conference*, New York City, pp. 3757-3762, July 2007.

[8] A. E. Bryson and Y. Ho, *Applied Optimal Control,* Hemisphere Publishing Corp, Washington D. C., 1975.

[9] W. H. Chen, D. J. Ballance and J. O'Reilly, "Model predictive control of nonlinear systems: computational burden and stability" *IEE Proc.-Control Theory Appl.*, Vol. 147, N0. 4, July 2000.

[10] M.B. Milam, R. Franz, J. E. Hauser, R. Murray,"Receding horizon control of a vectored thrust flight experiment," *IEE Proc.-Control Theory Appl.*, Vol. 152, No. 3, May 2005.

[11] R. Ghaemi, J. Sun, I. Kolmanovsky, "Model Predictive Control for constrained discrete time systems: An integrated perturbation analysis and sequential quadratic programming approach," *Proceedings of the 46th IEEE Conference on Decision and Control*, New Orleans, to appear.

[12] B. Gholami, B. W. Gordon, C. A. Rabbath, "Uncertain nonlinear receding horizon control systems subject to non-zero computation time," *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference 2005*, Sevelle, Spain, December 12-15, 2005.

[13] R. Ghaemi, J. Sun, I. Kolmanovsky, "Computationally efficient model predictive control with explicit disturbance mitigation and constraint enforcement," *Proceedings of the 45th IEEE Conference on Decision and Control*, San Diego, pp. 4842-4847, December 2006.

[14] M. H. Casado, A. Fernandez, and J. Iglesis,"Optimization of the course in the ship's movement by input-output linearization", *Proceedings of IFAC Conference on Control Application and Marine Systems*, pp. 419-424, 2001.