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Optimized Input-to-State Stabilization of Discrete-time Nonlinear Systems with Bounded Inputs

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Abstract— **In the problem of input-to-state stabilization of nonlinear systems, synthesis of input-to-state stabilizing feedback laws is usually carried out off-line. This results in a constant input-to-state stability (ISS) gain, which is guaranteed for the closed-loop system. As an alternative, we propose a finite dimensional optimization problem that allows for the simultaneous on-line computation of an ISS control action, and minimization of the ISS gain of the closed-loop system. The advantages of the developed controller are: ISS is guaranteed for any (feasible) solution of the optimization problem, constraints can be explicitly accounted for and feedback to disturbances is provided actively, on-line. The control scheme also has favorable computational properties for nonlinear systems affine in control. In this case the optimization problem can be formulated as a single quadratic or linear program.**

I. INTRODUCTION

A significant part of the literature on nonlinear control systems is dedicated to input-to-state stability (ISS), starting with the seminal works [1], [2]. Control Lyapunov functions (CLFs) [3], [4], [5] and ISS-CLFs, see [2], [6], [7], [8] and the references therein, represent a powerful tool for providing control laws achieving ISS. The usual approach is based on the design of an explicit feedback law off-line, which renders the derivative of the CLF (ISS-CLF) negative (to satisfy ISS conditions). An alternative to this approach is to construct an optimization problem such that any feasible solution renders the derivative of a candidate CLF negative. This method can be traced back to the early results presented in [9]. Synthesis of CLFs via on-line optimization has been the focus of the more recent articles [10], [11], where the usage of CLFs is elegantly combined with receding horizon control (RHC).

All of the above works mainly deal with continuous-time nonlinear systems and it is known that these results also hold for sampled-data systems, for small enough sampling intervals. In the discrete-time setting, it is worth to mention the result on sub-optimal receding horizon control presented in [12], where the finite horizon cost function of the RHC problem is employed as a discrete-time CLF for the closed-loop system.

In this paper we consider discrete-time constrained nonlinear systems subject to bounded inputs. Given a continuous and convex ISS-CLF we are interested to find among the corresponding ISS controllers one that has favorable properties in terms of a trade-off between performance and robustness. Based on the sufficient conditions for ISS presented in [13], [14] (and a natural extension to difference inclusions), we formulate a constrained optimization problem such that all its (feasible) solutions yield input-to-state stabilizing control actions. For bounded inputs, the proposed set-up leads to a finite dimensional optimization problem and more importantly, it leads to on-line optimization of the ISS gain of the resulting closed-loop system. This is achieved via additional optimization variables, which can be related to the closed-loop ISS gain via an explicit relation. By solving the developed optimization problem on-line, in a receding horizon fashion, feedback to disturbances is provided actively, as will be demonstrated by the example in Section IV. From a numerical point of view, for nonlinear systems affine in control and ISS-CLFs based on norms (e.g., $V(x) = \|Px\|_\infty$), the control algorithm can be implemented as a single linear or quadratic program. This is achievable by formulating the constraints enforced on the ISS-CLF as a finite number of linear inequalities.

A performance oriented finite horizon cost can be added to the optimization problem defined in this paper, leading to a combination of ISS-CLF synthesis and RHC. Furthermore, as continuity of the system dynamics is not necessary in the discrete-time setting [13], [14], the proposed control law can also be employed to achieve ISS of discrete-time discontinuous nonlinear and hybrid systems, including discontinuous piecewise affine systems [15].

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II. PRELIMINARIES

A. Basic notions and definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integers and the set of non-negative integers, respectively. We use the notation $\mathbb{Z}_{\geq c_1}$ and $\mathbb{Z}_{(c_1,c_2]}$ to denote the sets $\{k \in \mathbb{Z}_+ \mid k \geq c_1\}$ and $\{k \in \mathbb{Z}_+ \mid c_1 < k \leq c_2\}$, respectively, for some $c_1, c_2 \in \mathbb{Z}_+$. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, we denote by $\partial\mathcal{S}$ the boundary, by $\text{int}(\mathcal{S})$ the interior and by $\text{cl}(\mathcal{S})$ the closure of $\mathcal{S}$. For two arbitrary sets $\mathcal{S} \subseteq \mathbb{R}^n$ and $\mathcal{P} \subseteq \mathbb{R}^n$, let $\mathcal{S} \sim \mathcal{P} := \{x \in \mathbb{R}^n \mid x + \mathcal{P} \subseteq \mathcal{S}\}$ denote their Pontryagin difference. A polyhedron (or a polyhedral set) in $\mathbb{R}^n$ is a set obtained as the intersection of a finite number of open and/or closed half-spaces. Given $(n+1)$ *affinely independent* points $(\theta_0, \ldots, \theta_n)$ of $\mathbb{R}^n$, i.e. $(1\ \theta_0^\top)^\top, \ldots, (1\ \theta_n^\top)^\top$ are linearly independent in $\mathbb{R}^{n+1}$, we define a simplex S as

$$S := \text{Co}(\theta_0, \ldots, \theta_n) := \left\{ x \in \mathbb{R}^n \middle| x = \sum_{l=0}^{n} \mu_l \theta_l,\ \sum_{l=0}^{n} \mu_l = 1 \right.$$
$$\left. \mu_l \in \mathbb{R}_+ \ \text{ for } l = 0, 1, \ldots, n \right\},$$

where $\text{Co}(\cdot)$ denotes the convex hull.

The Hölder p-norm of a vector $x \in \mathbb{R}^n$ is defined as $\|x\|_p := (|[x]_1|^p + \ldots + |[x]_n|^p)^{\frac{1}{p}}$ for $p \in \mathbb{Z}_{[1,\infty)}$ and $\|x\|_\infty := \max_{i=1,\ldots,n} |[x]_i|$, where $[x]_i$, $i = 1, \ldots, n$, is the i-th component of x and $|\cdot|$ is the absolute value. For brevity, let $\|\cdot\|$ denote an arbitrary p-norm. For a matrix $Z \in \mathbb{R}^{m\times n}$ let $\|Z\| := \sup_{x\neq 0} \frac{\|Zx\|}{\|x\|}$ denote its corresponding induced matrix norm. It is well known that $\|Z\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |Z^{\{ij\}}|$, where $Z^{\{ij\}}$ is the ij-th entry of Z. Let $\mathbf{z} := \{z(l)\}_{l\in\mathbb{Z}_+}$ with $z(l) \in \mathbb{R}^o$ for all $l \in \mathbb{Z}_+$ denote an arbitrary sequence. Define $\|\mathbf{z}\| := \sup\{\|z(l)\| \mid l \in \mathbb{Z}_+\}$ and $\mathbf{z}_{[k]} := \{z(l)\}_{l\in\mathbb{Z}_{[0,k]}}$.

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is decreasing and $\lim_{k\to\infty} \beta(s, k) = 0$.

B. Input-to-state stability

Consider the discrete-time nonlinear system

$$x(k + 1) \in \Phi(x(k), w(k)), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state and $w(k) \in \mathbb{R}^l$ is an unknown disturbance input at the discrete-time instant k. The mapping $\Phi : \mathbb{R}^n \times \mathbb{R}^l \hookrightarrow \mathbb{R}^n$ is an arbitrary nonlinear, possibly discontinuous, set-valued function. For zero input in (1) we assume that $\Phi(0, 0) = \{0\}$. Suppose $w(k)$ takes a value in a bounded set $\mathbb{W} \subset \mathbb{R}^l$ for all $k \in \mathbb{Z}_+$.

Definition II.1 We call a set $\mathcal{P} \subseteq \mathbb{R}^n$ *robustly positively invariant (RPI)* for system (1) with respect to $\mathbb{W}$ if for all $x \in \mathcal{P}$ it holds that $\Phi(x, w) \subseteq \mathcal{P}$ for all $w \in \mathbb{W}$.

Definition II.2 Let $\mathbb{X}$ with $0 \in \text{int}(\mathbb{X})$ and $\mathbb{W}$ be subsets of $\mathbb{R}^n$ and $\mathbb{R}^l$, respectively. We call system (1) *ISS in* $\mathbb{X}$ *for inputs in* $\mathbb{W}$ if there exist a $\mathcal{KL}$-function $\beta(\cdot, \cdot)$ and a $\mathcal{K}$-function $\gamma(\cdot)$ such that, for each $x(0) \in \mathbb{X}$ and all $\mathbf{w} = \{w(l)\}_{l\in\mathbb{Z}_+}$ with $w(l) \in \mathbb{W}$ for all $l \in \mathbb{Z}_+$, it holds that all corresponding state trajectories of (1) satisfy $\|x(k)\| \leq \beta(\|x(0)\|, k) + \gamma(\|\mathbf{w}_{[k-1]}\|)$, $\forall k \in \mathbb{Z}_{\geq 1}$.

We call $\gamma(\cdot)$ an ISS gain of system (1).

Theorem II.3 *Let* $\mathbb{W}$ *be a subset of* $\mathbb{R}^l$ *and let* $\mathbb{X}$ *be a RPI set for* (1) *with respect to* $\mathbb{W}$*, with* $0 \in \text{int}(\mathbb{X})$*. Furthermore, let* $\alpha_1(s) := as^\delta$, $\alpha_2(s) := bs^\delta$, $\alpha_3(s) := cs^\delta$ *for some* $a, b, c, \delta \in \mathbb{R}_{>0}$, $\sigma \in \mathcal{K}$ *and let* $V : \mathbb{R}^n \to \mathbb{R}_+$ *be a function such that:*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{2a}$$
$$V(x^+) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|w\|) \tag{2b}$$

for all $x \in \mathbb{X}$, $w \in \mathbb{W}$ *and all* $x^+ \in \Phi(x, w)$*. Then the system* (1) *is ISS in* $\mathbb{X}$ *for inputs in* $\mathbb{W}$ *with*

$$\beta(s, k) := \alpha_1^{-1}(2\rho^k \alpha_2(s)), \quad \gamma(s) := \alpha_1^{-1}\left(\frac{2\sigma(s)}{1-\rho}\right),$$
$$\rho := 1 - \frac{c}{b} \in [0, 1). \tag{3}$$

If inequality (2b) *holds for* $w = 0$*, then the* 0*-input system* $x(k+1) \in \Phi(x(k), 0)$, $k \in \mathbb{Z}_+$*, is asymptotically stable in* $\mathbb{X}$.

The proof of Theorem II.3 is similar in nature to the proof given in [13], [14] by replacing the difference *equation* with the difference *inclusion* as in (1) and is omitted here for brevity. We call a function $V(\cdot)$ that satisfies the hypothesis of Theorem II.3 an *ISS Lyapunov function*.

III. OPTIMIZED INPUT-TO-STATE STABILIZATION

Consider the discrete-time constrained nonlinear system

$$\begin{aligned} x(k + 1) &= \phi(x(k), u(k), w(k)) \\ &:= f(x(k), u(k)) + g(x(k))w(k), \ k \in \mathbb{Z}_+, \end{aligned} \tag{4}$$

where $x(k) \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state, $u(k) \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control action and $w(k) \in \mathbb{W} \subset \mathbb{R}^l$ is an unknown disturbance input at the discrete-time instant k. $\phi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times l}$ are arbitrary nonlinear functions, possibly discontinuous, with $\phi(0, 0, 0) = 0$ and $f(0, 0) = 0$. Note that we allow that $g(0) \neq 0$. Naturally, we assume that $0 \in \text{int}(\mathbb{X})$ and $0 \in \text{int}(\mathbb{U})$. Next, let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and let $\sigma \in \mathcal{K}$.

Definition III.1 A function $V : \mathbb{R}^n \to \mathbb{R}_+$ that satisfies (2a) and for which there exists a control input $u \in \mathbb{U}$ such that

$$V(f(x, u)) - V(x) \leq -\alpha_3(\|x\|), \quad \forall x \in \mathbb{X},$$

is called a Control Lyapunov Function (CLF) for system $x(k + 1) = f(x(k), u(k))$, $k \in \mathbb{Z}_+$.

Definition III.2 A function $V : \mathbb{R}^n \to \mathbb{R}_+$ that satisfies (2a) and for which there exists a control input $u \in \mathbb{U}$ such that

$$\begin{aligned} &V(\phi(x, u, w)) - V(x) \leq \\ &\qquad - \alpha_3(\|x\|) + \sigma(\|w\|), \quad \forall w \in \mathbb{W}, \ \forall x \in \mathbb{X}, \end{aligned}$$

is called an Input-to-State Stability Control Lyapunov Function (ISS-CLF) for system (4).

Based on Definition III.1 we can formulate the following optimization problem.

Problem III.3 Let a CLF $V(\cdot)$ be given. At time $k \in \mathbb{Z}_+$ measure the state $x(k)$ and calculate a control action $u(k)$ that satisfies:

$$u(k) \in \mathbb{U}, \ \phi(x(k), u(k), 0) \in \mathbb{X}, \tag{5a}$$
$$V(\phi(x(k), u(k), 0)) - V(x(k)) + \alpha_3(\|x(k)\|) \leq 0. \tag{5b}$$

□

Let $\pi_0(x(k)) := \{u(k) \in \mathbb{R}^m \mid (5) \text{ holds}\}$. Let $x(k + 1) \in \phi_0(x(k), \pi_0(x(k))) := \{f(x(k), u) \mid u \in \pi_0(x(k))\}$ denote the difference inclusion corresponding to system (4) with 0-input in "closed-loop" with the set of feasible solutions obtained by solving Problem III.3 at each instant $k \in \mathbb{Z}_+$.

Theorem III.4 *Let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ of the form specified in Theorem II.3 and a CLF $V(\cdot)$ be given. Suppose that Problem III.3 is feasible for all states x in $\mathbb{X}$. Then the difference inclusion*

$$x(k+1) \in \phi_0(x(k), \pi_0(x(k))), \quad k \in \mathbb{Z}_+, \tag{6}$$

is asymptotically stable in $\mathbb{X}$.

Proof: Let $x(k) \in \mathbb{X}$ for some $k \in \mathbb{Z}_+$. Then, feasibility of Problem III.3 ensures that $x(k+1) \in \phi_0(x(k), \pi_0(x(k))) \subseteq \mathbb{X}$ due to constraint (5a). Hence, Problem III.3 remains feasible and thus, $\mathbb{X}$ is a positively invariant set for system (6). The result then follows directly from Theorem II.3. □

The above theorem establishes that feasible solutions of Problem III.3 are stabilizing feedback laws. In fact, if the set $\mathbb{X}$ is compact and the CLF $V(\cdot)$ is *continuous on $\mathbb{X}$ for all $u \in \mathbb{U}$*, it can be shown that by solving Problem III.3 one actually obtains an ISS feedback law. For simplicity, consider the case when $g(x)$ is bounded in norm, i.e. $\exists M \in \mathbb{R}_{>0}$ such that $\|g(x)\| \leq M$ for all $x \in \mathbb{X}$. Then, there exists $\sigma \in \mathcal{K}$ such that $|V(\phi(x, u, w)) - V(\phi(x, u, 0))| = |V(f(x, u) + g(x)w) - V(f(x, u))| \leq \sigma(M\|w\|)$ for all $x \in \mathbb{X}$ and all w. From this property, together with inequality (5b) we have that inequality (2b) holds. Thus, ISS of the perturbed difference inclusion $x(k+1) \in \phi_0(x(k), \pi_0(x(k))) + g(x(k))w(k)$, $k \in \mathbb{Z}_+$, follows from Theorem II.3 in this case.

However, this inherent ISS property of a feedback law calculated by solving Problem III.3 relies on a fixed, possibly large gain of the function σ, which depends on $V(\cdot)$. Note that this gain is related to the ISS gain of the closed-loop system via (3). To optimize the robustness of the closed-loop system it would be desirable to simultaneously find a control action $u(k) \in \mathbb{U}$ such that for all $w(k) \in \mathbb{W}$

$$V(\phi(x(k), u(k), w(k))) - V(x(k)) + \alpha_3(\|x\|) - \sigma(\|w(k)\|) \leq 0, \tag{7}$$

and minimize the gain of the function σ. Unfortunately, an optimization problem based directly on the constraint (7) is not finite dimensional in $w(k)$.

In what follows we demonstrate that by considering *continuous and convex*[1] CLFs and bounded polyhedral sets $\mathbb{X}, \mathbb{U}, \mathbb{W}$ (with non-empty interiors containing the origin) a solution to inequality (7) can be obtained via a finite set of inequalities that only depend on the vertices of $\mathbb{W}$.

Let w^e, $e = 1, ..., E$, be the vertices of $\mathbb{W}$ (notice that $E > l$, with l the dimension of the disturbance, as $\mathbb{W}$ is assumed to have a non-empty interior). Next, consider a finite set of simplices $S_1, \ldots, S_M$ with each simplex S_i equal to the convex hull of a subset of the vertices of $\mathbb{W}$ and the origin, and such that $\cup_{i=1}^M S_i = \mathbb{W}$. More precisely, $S_i = \text{Co}\{0, w^{e_{i,1}}, \ldots, w^{e_{i,l}}\}$ and $\{w^{e_{i,1}}, \ldots, w^{e_{i,l}}\} \subseteq \{w^1, \ldots, w^E\}$ (i.e. $\{e_{i,1}, \ldots, e_{i,l}\} \subseteq \{1, \ldots, E\}$) with $w^{e_{i,1}}, \ldots, w^{e_{i,l}}$ linearly independent. For an illustrative example see Figure 1: the polyhedron $\mathbb{W}$

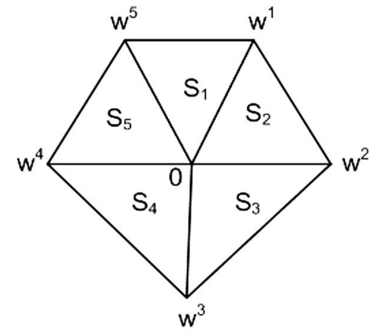

Fig. 1. An example of the set $\mathbb{W}$.

consists of $S_1, S_2, \ldots, S_5$, where, for instance, the simplex S_3 is generated by $0, w^{e_{3,1}}, w^{e_{3,2}}$, with $e_{3,1} = 2$ and $e_{3,2} = 3$. For each simplex S_i we define the matrix $W_i := [w^{e_{i,1}} \ \ldots \ w^{e_{i,l}}] \in \mathbb{R}^{l \times l}$, which is invertible.

Let $\lambda_e(k)$, $k \in \mathbb{Z}_+$, be optimization variables associated with each vertex w^e. Let $\alpha_3 \in \mathcal{K}_\infty$, suppose that $x(k)$ at time $k \in \mathbb{Z}_+$ is given and consider the following set of constraints depending on $u(k)$ and $\lambda_1(k), \ldots, \lambda_E(k)$:

$$V(\phi(x(k), u(k), 0)) - V(x(k)) + \alpha_3(\|x(k)\|) \leq 0, \tag{8a}$$

$$V(\phi(x(k), u(k), w^e)) - V(x(k)) + \alpha_3(\|x(k)\|) - \lambda_e(k) \leq 0 \tag{8b}$$

for all $e = 1, \ldots, E$.

Theorem III.5 *Let $V(\cdot)$ be a continuous and convex CLF. If for $\alpha_3 \in \mathcal{K}_\infty$ and $x(k)$ at time $k \in \mathbb{Z}_+$ there exist $u(k)$ and $\lambda_e(k)$, $e = 1, \ldots, E$, such that* (8a) *and* (8b) *hold, then* (7) *holds for the same $u(k)$, with $\sigma(s) := \eta(k)s$ and*

$$\eta(k) := \max_{i=1,\ldots,M} \|\bar{\lambda}_i(k) W_i^{-1}\|, \tag{9}$$

where $\bar{\lambda}_i(k) := [\lambda_{e_{i,1}}(k) \ \ldots \ \lambda_{e_{i,l}}(k)] \in \mathbb{R}^{1 \times l}$ and $\|\cdot\|$ is the corresponding induced matrix norm.

Proof: Let $\alpha_3 \in \mathcal{K}_\infty$ and $x(k)$ be given and suppose (8b) holds for some $\lambda_e(k)$, $e = 1, \ldots, E$. Let $w \in \mathbb{W} = \bigcup_{i=1}^M S_i$. Hence, there exists an i such that $w \in S_i = \text{Co}\{0, w^{e_{i,1}}, \ldots, w^{e_{i,l}}\}$, which means that there exist non-negative numbers $\mu_0, \mu_1, \ldots, \mu_l$ with $\sum_{j=0,1,\ldots,l} \mu_j = 1$ such that

$$w = \sum_{j=1,\ldots,l} \mu_j w^{e_{i,j}} + \mu_0 0 = \sum_{j=1,\ldots,l} \mu_j w^{e_{i,j}}.$$

In matrix notation we have that $w = W_i[\mu_1 \ \ldots \ \mu_l]^\top$ and thus $[\mu_1 \ \ldots \ \mu_l]^\top = W_i^{-1}w$. By multiplying each inequality in (8b) corresponding to the index $e_{i,j}$ and the inequality (8a) with $\mu_j \geq 0$, $j = 0, 1, \ldots, l$, summing up and using $\sum_{j=0,1,\ldots,l} \mu_j = 1$ yields:

$$\mu_0 V(\phi(x(k), u(k), 0)) + \sum_{j=1,\ldots,l} \mu_j V(\phi(x(k), u(k), w^{e_{i,j}})) - V(x(k)) + \alpha_3(\|x(k)\|) - \sum_{j=1,\ldots,l} \mu_j \lambda_{e_{i,j}}(k) \leq 0.$$

[1]This includes quadratic functions, $V(x) = x^\top Px$ with $P \succ 0$, and functions based on norms, $V(x) = \|Px\|$ with P a full-column rank matrix.

Furthermore, using $\phi(x(k), u(k), w^{e_{i,j}}) = f(x(k), u(k)) + g(x(k))w^{e_{i,j}}$, $\sum_{j=0,1,\ldots,l} \mu_j = 1$ and convexity of $V(\cdot)$ yields

$$V(\phi(x(k), u(k), \sum_{j=1,\ldots,l} \mu_j w^{e_{i,j}})) - V(x(k)) + \alpha_3(\|x(k)\|) - \sum_{j=1,\ldots,l} \mu_j \lambda_{e_{i,j}}(k) \leq 0,$$

or equivalently

$$V(\phi(x(k), u(k), w)) - V(x(k)) + \alpha_3(\|x(k)\|) - \bar{\lambda}_i(k)[\mu_1 \ \ldots \ \mu_l]^\top \leq 0.$$

Using that $[\mu_1 \ \ldots \ \mu_l]^\top = W_i^{-1}w$ we obtain (7) for $\sigma(s) = \eta(k)s$ and $\eta(k) \geq 0$ as in (9). □

Based on the result of Theorem III.5 we are now able to formulate a finite dimensional optimization problem that results in optimization of the closed-loop ISS gain, as follows.

For any $x \in \mathbb{X}$ let $\mathbb{W}_x := \{g(x)w \mid w \in \mathbb{W}\} \subset \mathbb{R}^n$ (note that $0 \in \mathbb{W}_x$) and assume that $\mathbb{X} \sim \mathbb{W}_x \neq \emptyset$. Let $\bar{\lambda} := [\lambda_1, \ldots, \lambda_E]^\top$ and let $J(\lambda_1, \ldots, \lambda_E) : \mathbb{R}^E \to \mathbb{R}_+$ be a function that satisfies $\alpha_4(\|\bar{\lambda}\|) \leq J(\lambda_1, \ldots, \lambda_E) \leq \alpha_5(\|\bar{\lambda}\|)$ for some $\alpha_4, \alpha_5 \in \mathcal{K}_\infty$. This implies $J(\lambda_1, \ldots, \lambda_E) \to 0 \Rightarrow \lambda_e \to 0$ for all $e = 1, \ldots, E$ and $J(0, \ldots, 0) = 0$.

Problem III.6 Let $\alpha_3 \in \mathcal{K}_\infty$, a cost $J(\cdot)$ and a CLF $V(\cdot)$ be given. At time $k \in \mathbb{Z}_+$ measure the state $x(k)$ and minimize the cost $J(\lambda_1(k), \ldots, \lambda_E(k))$ over $u(k), \lambda_1(k), \ldots, \lambda_E(k)$, subject to the constraints

$$u(k) \in \mathbb{U}, \ \lambda_e(k) \geq 0, \ f(x(k), u(k)) \in \mathbb{X} \sim \mathbb{W}_{x(k)}, \quad (10a)$$

$$V(\phi(x(k), u(k), 0)) - V(x(k)) + \alpha_3(\|x(k)\|) \leq 0, \quad (10b)$$

$$V(\phi(x(k), u(k), w^e)) - V(x(k)) + \alpha_3(\|x(k)\|) - \lambda_e(k) \leq 0 \quad (10c)$$

for all $e = 1, \ldots, E$. □

Let $\pi(x(k)) := \{u(k) \in \mathbb{R}^m \mid (10) \text{ holds}\}$ and let $x(k+1) \in \phi_{\text{cl}}(x(k), \pi(x(k)), w(k)) := \{\phi(x(k), u, w(k)) \mid u \in \pi(x(k))\}$ denote the difference inclusion corresponding to system (4) in "closed-loop" with the set of feasible solutions obtained by solving Problem III.6 at each $k \in \mathbb{Z}_+$.

Theorem III.7 *Let, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ of the form specified in Theorem II.3, a continuous and convex CLF $V(\cdot)$ and a cost $J(\cdot)$ be given. Suppose that Problem III.6 is feasible for all states x in $\mathbb{X}$. Then the difference inclusion*

$$x(k+1) \in \phi_{\text{cl}}(x(k), \pi(x(k)), w(k)), \quad k \in \mathbb{Z}_+ \quad (11)$$

is ISS in $\mathbb{X}$ for inputs in $\mathbb{W}$.

Proof: Let $x(k) \in \mathbb{X}$ for some $k \in \mathbb{Z}_+$. Then, feasibility of Problem III.6 ensures that

$$x(k+1) \in \phi_{\text{cl}}(x(k), \pi(x(k)), w(k)) \subseteq \mathbb{X}$$

for all $w(k) \in \mathbb{W}$, due to $g(x(k))w(k) \in \mathbb{W}_{x(k)}$ and constraint (10a). Hence, Problem III.6 remains feasible and thus, $\mathbb{X}$ is a RPI set with respect to $\mathbb{W}$ for system (11). From Theorem III.5 we also have that $V(\cdot)$ satisfies (2b) with $\sigma(s) := \eta(k)s$ and $\eta(k)$ as in (9). Let

$$\lambda^* := \max_{x \in \text{cl}(\mathbb{X}), u \in \text{cl}(\mathbb{U}), e=1,\ldots,E} \{V(\phi(x, u, w^e)) - V(x) + \alpha_3(\|x\|)\}.$$

Since $\mathbb{X}$ and $\mathbb{U}$ are assumed to be bounded sets, λ^* exists, and inequality (10c) is always satisfied for $\lambda_e(k) = \lambda^*$ for all $e = 1, \ldots, E$, $k \in \mathbb{Z}_+$, irrespective of x and u. This in turn, via (9) ensures the existence of a positive η^* such that $\eta(k) \leq \eta^*$ for all $k \in \mathbb{Z}_+$. Hence, we proved that inequality (7) holds, and thus, the continuous and convex CLF $V(\cdot)$ is an ISS-CLF. Then, due to RPI of $\mathbb{X}$, ISS in $\mathbb{X}$ for inputs in $\mathbb{W}$ follows directly from Theorem II.3. □

Note that in the above theorem we used a worst case evaluation of $\lambda_e(k)$ to prove ISS. However, in reality the gain $\eta(k)$ of the function σ can be much smaller for $k \geq k_0$, for some $k_0 \in \mathbb{Z}_+$. This is achieved via the minimization of the cost $J(\cdot)$, which produces small values of $\lambda_e(k)$, $e = 1, \ldots, E$. This in turn, via (9), will result in a small $\eta(k)$. Furthermore, this will ultimately yield a smaller ISS gain for the closed-loop system, due to the relation (3). *Hence, Problem III.6, although it inherently guarantees a constant ISS gain, it provides freedom to optimize the ISS gain of the closed-loop system, by minimizing the variables $\lambda_1(k), \ldots, \lambda_E(k)$ via the cost $J(\cdot)$.*

The result of Theorem III.7 holds for all inputs $u(k)$ for which Problem III.6 is feasible. In order to select online one particular control input from the set $\pi(x(k))$ and to improve closed-loop performance (in terms of settling time) it is useful to also penalize the state and the input. Let $F : \mathbb{R}^n \to \mathbb{R}_+$ and $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ with $F(0) = L(0,0) = 0$ be arbitrary, possibly discontinuous, nonlinear functions. For $N \in \mathbb{Z}_{\geq 1}$ let $\bar{\mathbf{u}}(k) := (\bar{u}(k), \bar{u}(k+1), \ldots, \bar{u}(k+N-1)) \in \mathbb{U}^N = \mathbb{U} \times \ldots \times \mathbb{U}$ and let $J_{\text{RHC}}(x(k), \bar{\mathbf{u}}(k), \lambda_1(k), \ldots, \lambda_E(k)) := F(\bar{x}(k+N)) + \sum_{i=0}^{N-1} L(\bar{x}(k+i), \bar{u}(k+i)) + J(\lambda_1(k), \ldots, \lambda_E(k))$, where $\bar{x}(k+i+1) := f(\bar{x}(k+i), \bar{u}(k+i))$ for $i = 0, \ldots, N-1$ and $\bar{x}(k) := x(k)$.

Algorithm III.8
Step 1: *Let $\alpha_3 \in \mathcal{K}_\infty$, $J(\cdot)$, $L(\cdot)$, $F(\cdot)$, $N \in \mathbb{Z}_{\geq 1}$ and a CLF $V(\cdot)$ be given. At time $k \in \mathbb{Z}_+$ measure the state $x(k)$ and minimize $J_{RHC}(\bar{\mathbf{u}}(k), \lambda_1(k), \ldots, \lambda_E(k))$ over $\bar{\mathbf{u}}(k), \lambda_1(k), \ldots, \lambda_E(k)$, subject to the constraints*

$$\bar{\mathbf{u}}(k) \in \mathbb{U}^N, \ \bar{x}(k+i) \in \mathbb{X}, \ i = 2, \ldots, N, \quad (12a)$$

$$\lambda_e(k) \geq 0, \ f(x(k), \bar{u}(k)) \in \mathbb{X} \sim \mathbb{W}_{x(k)}, \quad (12b)$$

$$V(\phi(x(k), \bar{u}(k), 0)) - V(x(k)) + \alpha_3(\|x(k)\|) \leq 0, \quad (12c)$$

$$V(\phi(x(k), \bar{u}(k), w^e)) - V(x(k)) + \alpha_3(\|x(k)\|) - \lambda_e(k) \leq 0 \quad (12d)$$

for all $e = 1, \ldots, E$. Let

$$\Pi(x(k)) := \{\bar{\mathbf{u}}(k) \in \{\mathbb{R}^m\}^N \mid \bar{\mathbf{u}}(k) \text{ satisfies } (12)\}$$

and let $\pi_{RHC}(x(k)) := \{\bar{u}(k) \in \mathbb{R}^m \mid \bar{\mathbf{u}}(k) \in \Pi(x(k))\}$.

Step 2: *Select a feasible sequence of inputs* $\bar{\mathbf{u}}(k) := (\bar{u}(k), \bar{u}(k+1), \ldots, \bar{u}(k+N-1)) \in \Pi(x(k))$ *and apply the input* $u(k) = \bar{u}(k) \in \pi_{RHC}(x(k))$ *to system (4), increment* k *by one and go to Step 1.* □

Corollary III.9 *Let* $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ *of the form specified in Theorem II.3, a continuous and convex CLF* $V(\cdot)$ *and* $J(\cdot)$, $L(\cdot)$, $F(\cdot)$, $N \in \mathbb{Z}_{\geq 1}$ *be given. Suppose that the optimization problem in Step 1 of Algorithm III.8 is feasible for all states* x *in* $\mathbb{X}$. *Then the closed-loop system*

$$x(k+1) = \phi_{cl}x(k), \pi_{RHC}(x(k)), w(k)), \quad k \in \mathbb{Z}_+ \quad (13)$$

is ISS in $\mathbb{X}$ *for inputs in* $\mathbb{W}$.

Remark III.10 In Algorithm III.8 we make a tradeoff between robustness (suppressing disturbances adequately) via a small $\eta(k)$ on one hand and performance on the other. Besides enhancing robustness, the constraints (12c)-(12d) also ensure that Algorithm III.8 recovers performance in terms of convergence when the state of the closed-loop system approaches the origin. Loosely speaking, when $x(k) \approx 0$, Algorithm III.8 will produce a control action $u(k) \approx 0$ (because of constraint (12c) and minimization of the cost $J_{\text{RHC}}(\cdot)$). This in turn yields $V(\phi(0,0,w^e)) - \lambda_e(k) \leq 0$, $e = 1, \ldots, E$, due to constraint (12d). Thus, Algorithm III.8 will not optimize each variable $\lambda_e(k)$ below the corresponding value $V(\phi(0,0,w^e))$, $e = 1, \ldots, E$, when the state reaches the equilibrium. This property is desirable, since it is known from min-max predictive control [16] that considering a worst case disturbance scenario leads to poor performance (convergence) when the real disturbance is small or vanishes.

A. Implementation issues

Next, we briefly discuss the ingredients which make it possible to implement Algorithm III.8 as a single linear or quadratic program. Firstly, we consider nonlinear systems of the form (4) that are affine in control. For this it is sufficient to assume that there exist functions $f_1 : \mathbb{R}^n \to \mathbb{R}^n$ with $f_1(0) = 0$ and $f_2 : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ such that:

$$\begin{aligned} x(k+1) &= \phi(x(k), u(k), w(k)) \\ &= f(x(k), u(k)) + g(x(k))w(k) \\ &= f_1(x(k)) + f_2(x(k))u(k) + g(x(k))w(k). \end{aligned} \quad (14)$$

Secondly, we restrict our attention to CLFs defined using the ∞-norm, i.e. $V(x) := \|Px\|_\infty$, where $P \in \mathbb{R}^{p \times n}$ is a matrix (to be determined) with full-column rank. We refer to [17], [14] for techniques to compute CLFs based on norms.

Then, the first step is to show that the ISS inequalities (12c)-(12d) can be specified, without introducing conservatism, via a finite number of linear inequalities. Recall that for any vector $x \in \mathbb{R}^n$, $[x]_i$ denotes the i-th element of x. Since by definition $\|x\|_\infty = \max_{i \in \mathbb{Z}_{[1,n]}} |[x]_i|$, for a constraint $\|x\|_\infty \leq c$ with $c > 0$ to be satisfied, it is *necessary and sufficient* to require that $\pm [x]_i \leq c$ for all $i \in \mathbb{Z}_{[1,n]}$ (in total, these are $2n$ linear inequalities in x). Therefore, as $x(k)$ in (12) is just the measured state, which is known at every $k \in \mathbb{Z}_+$, for (12c)-(12d) to be satisfied it is necessary and sufficient to require that:

$$\begin{aligned} &\pm [P(f_1(x(k)) + f_2(x(k))u(k))]_i - V(x(k)) + \alpha_3(\|x(k)\|) \leq 0 \\ &\pm [P(f_1(x(k)) + f_2(x(k))u(k) + g(x(k))w^e)]_i \\ &- V(x(k)) + \alpha_3(\|x(k)\|) - \lambda_e(k) \leq 0, \\ &\forall i \in \mathbb{Z}_{[1,p]}, \ e = 1, \ldots, E, \end{aligned}$$

which yields $2p(E+1)$ linear inequalities in the optimization variables $u(k), \lambda_1(k), \ldots, \lambda_E(k)$. If the sets $\mathbb{X}$, $\mathbb{U}$ and $\mathbb{W}_{x(k)}$ are polyhedra, which is a reasonable assumption, and for a unitary prediction horizon, then clearly the inequalities in (12a)-(12b) are also linear in the optimization variables $u(k), \lambda_1(k), \ldots, \lambda_E(k)$. Thus, a solution to the problem in Step 1 of Algorithm III.8 can be obtained by solving a nonlinear optimization problem subject to linear constraints. Furthermore, for $N = 1$ and quadratic or ∞-norm based costs, the optimization problem in Step 1 of Algorithm III.8 can be formulated as a single quadratic or linear program (see [14] for more details). Furthermore, notice that for a cost $J(\lambda_1, \ldots, \lambda_E)$ defined using quadratic forms or infinity norms, Problem III.6 can also be implemented as a single quadratic or linear program, respectively.

IV. ILLUSTRATIVE EXAMPLE

Consider the nonlinear system (14) where $x(k) \in \mathbb{X} = \{\xi \in \mathbb{R}^2 \mid \|\xi\|_\infty \leq 5\}$, $u(k) \in \mathbb{U} = \{\xi \in \mathbb{R} \mid |\xi| \leq 1\}$ and $w(k) \in \mathbb{W} = \{\xi \in \mathbb{R}^2 \mid \|\xi\|_1 \leq 0.2\}$, $k \in \mathbb{Z}_+$. The dynamics are given by:

$$f_1(x) = \begin{pmatrix} [x]_1 + 0.7[x]_2 + ([x]_2)^2 \\ [x]_2 \end{pmatrix},$$
$$f_2(x) = \begin{pmatrix} 0.245 + \sin([x]_2) \\ 0.7 \end{pmatrix}, \ g(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The technique of [17] was used to compute the weight $P \in \mathbb{R}^{2 \times 2}$ of the CLF $V(x) = \|Px\|_\infty$ for $\alpha_3(s) := 0.01s$ and the linearization of (14) around the origin, in closed-loop with $u(k) := Kx(k)$, $K \in \mathbb{R}^{2 \times 1}$, yielding

$$P = \begin{bmatrix} 2.7429 & 0.7121 \\ 0.1989 & 4.0173 \end{bmatrix}, \ K = \begin{bmatrix} -0.4379 & -1.5508 \end{bmatrix}.$$

To optimize robustness, 4 optimization variables $\lambda_1(k), \ldots, \lambda_4(k)$ were introduced, each one assigned to a vertex of the set $\mathbb{W}$. The RHC cost was chosen as $J_{\text{RHC}}(x(k), u(k), \lambda_i(k)) = \|Q_1(f_1(x(k)) + f_2(x(k))u(k))\|_\infty + \|Qx(k)\|_\infty + \|Ru(k)\|_\infty + \sum_{i=1}^4 \|\lambda_i(k)\|_\infty$, where $Q_1 = 4I_2$, $Q = 0.1I_2$ and $R = 0.4$. The resulting linear program has 11 optimization variables and 42 constraints. During the simulations, the worst case computational time required by the CPU over 4000 runs was 0.02 seconds, which shows the potential for controlling fast nonlinear systems.

In the simulation scenario we tested the closed-loop system response for $x(0) = [3, \ -1]^\top$ and for the following disturbance scenarios: $w(k) = [0, \ 0]^\top$ for $k \in \mathbb{Z}_{[0,40]}$ (nominal stabilization), $w(k)$ takes random values in $\mathbb{W}$ for $k \in \mathbb{Z}_{[41,80]}$ (robustness to random inputs), $w(k) = [0, \ 0.1]^\top$ for $k \in \mathbb{Z}_{[81,120]}$ (robustness to constant inputs) and $w(k) = [0, \ 0]^\top$ for $k \in \mathbb{Z}_{[121,160]}$ (to show that asymptotic stability is recovered for zero inputs).

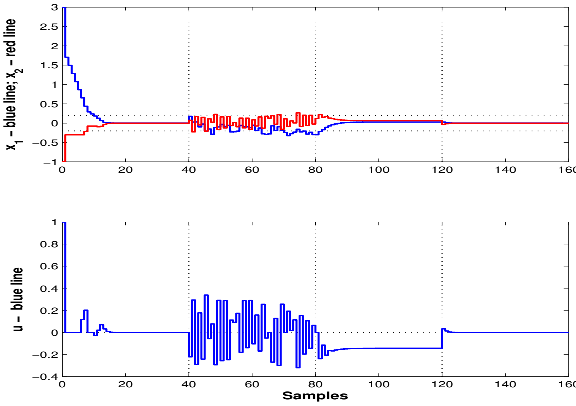

Fig. 2. Evolution of the closed-loop system state (top figure: red and blue lines) and of the control input (bottom figure: blue line).

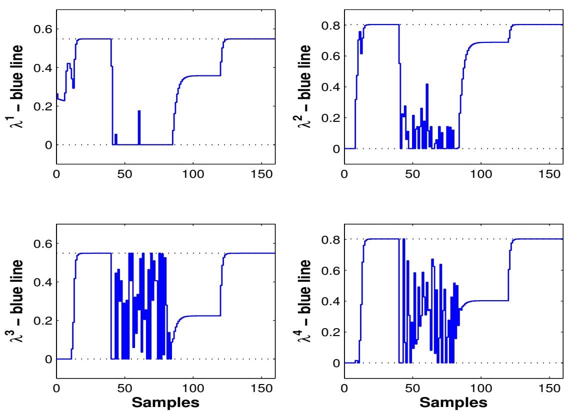

Fig. 3. Evolution of the optimization variables $\lambda_1(k), \ldots, \lambda_4(k)$.

In Figure 2 the time history of the states and control input is depicted. The dashed horizontal lines give an approximation of the bounded region in which the system's states remain despite disturbances, i.e. approximately within the interval $[-0.2,\ 0.2]$. The dashed vertical lines delimit the time intervals during which one of the four disturbance scenarios is active. One can observe that the feedback to disturbances is provided actively, resulting in good robust performance, while state and input constraints are satisfied at all times. In Figure 3 the time history of the optimization variables $\lambda_1(k), \ldots, \lambda_4(k)$ is presented. One can see that whenever the disturbance is acting on the system, or when the state is far from the origin (in the first disturbance scenario), these variables act so as to optimize the decrease of $V(\cdot)$. Whenever the equilibrium is reached, the optimization variables satisfy the constraint $V(\phi(0,0,w^e)) \leq \lambda_e(k),\ e = 1,\ldots,4$, as explained in Remark III.10. In Figure 3 the values of $V(\phi(0,0,w^e)$ for each vertex (0.5486 and 0.8432 for $w^1 = [0.2,\ 0]^\top$, $w^3 = [-0.2,\ 0]^\top$ and $w^2 = [0,\ -0.2]^\top$, $w^4 = [0,\ 0.2]^\top$, respectively) are depicted with dashed horizontal lines.

V. CONCLUSIONS

In this paper we studied the problem of input-to-state stabilization of discrete-time constrained nonlinear systems subject to bounded inputs. We presented a finite dimensional optimization problem that allows for the simultaneous on-line computation of an ISS control action, and minimization of the ISS gain of the resulting closed-loop system. ISS is guaranteed for the closed-loop system for any (feasible) solution of the optimization problem, while state and input constraints can be explicitly accounted for. A distinguishing, advantageous feature of the proposed controller is that it can provide feedback to disturbances actively, on-line, leading to improved robust performance. Furthermore, the controller can be implemented as a single linear or quadratic program for nonlinear systems affine in control, which brings the application to fast nonlinear systems within reach.

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