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Global Path Following for the Unicycle and Other Results

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Abstract— **We address the maneuver regulation of the kinematic unicycle to a circle. Our control approach is passivity-based, and we frame the control design objective as a set stabilization problem. We present two main results. First, we provide a smooth, time-invariant, static feedback that globally asymptotically stabilizes the motion on the circle in a desired direction and constant velocity. Second, we provide a smooth time-varying feedback that almost globally asymptotically stabilizes the set of configurations corresponding to the unicycle centre of mass on the circle with desired heading on the circle.**

I. INTRODUCTION

Equilibrium stabilization for nonholonomic systems has been studied extensively over the past years. Brockett's necessary condition for smooth stabilization, in [1], shows that no time-invariant static-state feedback stabilizer exists for these systems. The main approaches for equilibrium stabilization include discontinuous control [2], [3], [4], [5], time-varying control [6], [7], [8] and hybrid feedback techniques [9]. There are also many results for tracking control [10], [11].

Path following for nonholonomic systems has also acquired considerable attention. Results include techniques previously used in equilibrium stabilization [4], [10], and more recent results using techniques such as virtual target tracking [12], transverse feedback linearization [13], and backstepping [14]. Path following results can also be found in conjunction with those on formation control of multi-agent systems. For example, in [15] linearization techniques and gain scheduling are used for path following control and applied to approximate stabilization of the circular motion. In [16] the authors present a Lie group setting for control of formations and curvature control design. The specialization of the results to a single vehicle and a beacon is the circular path following problem where the Lyapunov-based control provides almost-global stabilization. Most notably, in [17] Ceccarelli *et al.* present a switching control that globally asymptotically stabilizes the circular motion of the unicycle with desired orientation.

In this paper, we consider the kinematic unicycle and investigate the stabilization of *sets* in the state space that correspond to specific maneuvers. Specifically, we present a smooth, time-invariant, static feedback that globally asymptotically stabilizes the circular motion of the unicycle with desired orientation. Remarkably, our controller is extremely simple and, to the best of our knowledge, there is no other result reported in the literature showing that the circular path following problem can be solved globally by a smooth time-invariant static feedback. The closest result is that in [17] where the controller is discontinuous. We also consider the problem of maneuvering the unicycle to the unit circle centred at the origin, with desired heading on the circle. We present a time-varying, static feedback that almost globally solves this problem, in the sense that the unicycle will accomplish the desired maneuver for any initial condition except when its centre of mass is initialized at the origin. Our two controllers are based on a new result which builds upon our previous work in [18]. See also relevant work by Shiriaev and co-workers in [19]. The advantage of posing the two control problems in the set stabilization framework, rather than that of tracking, is that the resulting feedback guarantees invariance of the set in question. So, for instance, our circular path following controller guarantees that when the unicycle is initialized on the circle with heading tangent to the circle and appropriate orientation, the unicycle will follow the circle without leaving it. This essential invariance property, distinguishing path following from tracking, is often neglected in the literature.

II. PASSIVITY-BASED SET STABILIZATION

In [18] we provided results for the use of passivity-based control in stabilizing non-compact sets. In this section we present a new result, extending part of our work in [18], which is used in Sections III and IV to design controllers for the unicycle.

Given a closed nonempty set $\Gamma \subset \mathcal{X}$, where $\mathcal{X}$ is a vector space, a point $\xi \in \mathcal{X}$, and a vector norm $\|\cdot\| : \mathcal{X} \to \mathbb{R}$, the point-to-set distance $\|\xi\|_\Gamma$ is defined as $\|\xi\|_\Gamma := \inf_{\eta\in\Gamma} \|\xi - \eta\|$. Denote by $\phi(t, x_0, u(t))$ the unique solution to a smooth differential equation $\dot{x} = f(x) + g(x)u(t)$, with initial condition x_0 and piecewise continuous control input signal $u(t)$. We use the standard notation $L_f V$ to denote the Lie derivative of a C^1 function V along a vector field f.

Consider the control-affine system,

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i \\ y &= h(x) \end{aligned} \tag{1}$$

with state space $\mathcal{X} \subset \mathbb{R}^n$, set of input values $\mathcal{U} \subset \mathbb{R}^m$ and set of output values $\mathcal{Y} \subset \mathbb{R}^m$. The set $\mathcal{X}$ is assumed to be either an open subset or a smooth submanifold of $\mathbb{R}^n$, in which case $\mathcal{X}$ inherits a metric from $\mathbb{R}^n$. We assume that f and g_i, $i = 1, \dots m$, are smooth vector fields, and that

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h is a smooth mapping. The system (1) is *passive* if there exists a C^r ($r \geq 1$) nonnegative function $V : \mathcal{X} \to \mathbb{R}$, called the storage function, such that for all piecewise-continuous functions $u : [0, \infty) \to \mathcal{U}$, for all $x_0 \in \mathcal{X}$, and for all t in the maximal interval of existence of $\phi(\cdot, x_0, u)$,

$$V(x(t)) - V(x_0) \leq \int_0^t u(\tau)^\top y(\tau) d\tau,$$

where $x(t) = \phi(t, x_0, u(t))$ and $y(t) = h(x(t))$. The passivity property is equivalent to the two conditions $L_f V(x) \leq 0$, $L_{g_i} V(x) = h_i(x)^\top$, $i = 1, \ldots, m$. Let $\Gamma \subset \mathcal{X}$ be a closed invariant set for a system $\Sigma : \dot{x} = f(x)$, $x \in \mathcal{X}$.

Definition II.1 (Set Stability, [20]) *(i)* Γ *is* uniformly stable *for* Σ *if* $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x_0 \in \mathcal{X})$ $(\|x_0\|_\Gamma < \delta \Rightarrow (\forall t \geq 0) \|x(t)\|_\Gamma < \varepsilon)$.

(ii) Γ *is an* attractor *for* Σ *if* $(\exists \delta > 0)$ $(\forall x_0 \in \mathcal{X})$ $\|x_0\|_\Gamma < \delta \Rightarrow \lim_{t \to \infty} \|x(t)\|_\Gamma = 0$. *It is a* global attractor *if* $\delta = \infty$.

(iii) Γ *is* asymptotically stable *with respect to* Σ *if it is a uniformly stable attractor for* Σ. *It is* globally asymptotically stable *if it is a uniformly stable global attractor.*

Definition II.2 (Relative Set Stability) *Let* $\Xi \subset \mathcal{X}$ *be such that* $\Xi \cap \Gamma \neq \emptyset$. Γ *is* uniformly stable relative to Ξ *for* Σ *if, for any* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for all* $\|x_0\|_\Gamma < \delta$ *and* $x_0 \in \Xi$*, one has that* $\|x(t)\|_\Gamma < \varepsilon$ *for all* $t \geq 0$.

In [18] we pose the following set stabilization problem: assume that system (1) is passive with a positive semi-definite storage function V. Given a closed set $\Gamma \subseteq V^{-1}(0) = \{x \in \mathcal{X} : V(x) = 0\}$, find conditions under which a passivity-based feedback makes Γ an attractor or an asymptotically stable set for the closed-loop system. We refer to Γ as the *goal set.*

This problem was investigated by Shiriaev in [19] in the special case when Γ is compact and $\Gamma = V^{-1}(0)$. His main condition for the existence of a stabilizing passivity-based feedback is the notion of V-detectability, a generalization of the well-known notion of zero-state detectability by Byrnes, Isidori, and Willems in [21]. In [18], we extended and generalized these results by considering non-compact Γ which is not necessarily equal to $V^{-1}(0)$. There, we addressed the cases of bounded and unbounded trajectories. For the case of bounded trajectories, pertinent to the unicycle problems presented here, we replaced the notion of V-detectability by that of Γ-detectability and using this notion we gave sufficient conditions (see [18, Theorem III.1]) for which a passivity based controller makes Γ attractive[1] for the closed-loop system. We repeat our definition of Γ-detectability below.

Definition II.3 (Γ-Detectability [18]) *System* (1) *is* locally Γ-detectable *if there exists a neighborhood* $\mathcal{N}(\Gamma)$ *of* Γ *such that, for all* $x_0 \in \mathcal{N}(\Gamma)$*, the open-loop solution* $x(t) = \phi(t, x_0, 0)$ *satisfies*

$$y(t) = 0 \;\; \forall t \geq 0 \implies \lim_{t \to \infty} \|x(t)\|_\Gamma = 0.$$

The system is Γ-detectable *if it is locally* Γ*-detectable with* $\mathcal{N}(\Gamma) = \mathcal{X}$.

We now present a new result concerning the (global) asymptotic stabilization of a compact Γ by means of a passivity-based feedback. The proof of this result is omitted and will be reported elsewhere.

Theorem II.4 *Consider system* (1) *with a passivity-based feedback* $u = -\varphi(h(x))$*, where* $\varphi(\cdot) : \mathcal{Y} \to \mathcal{U}$ *is a locally Lipschitz function such that* $\varphi(0) = 0$ *and* $y^\top \varphi(y) > 0$ *for all* $y \neq 0$*. Let* $\Gamma \subset V^{-1}(0)$ *be a compact set which is open-loop invariant. Then,* Γ *is asymptotically stable for the closed-loop system if, and only if, system* (1) *is locally* Γ*-detectable and* Γ *is uniformly stable relative to* $V^{-1}(0)$*. If, in addition, all trajectories of the closed-loop system are bounded, then* Γ *is globally asymptotically stable for the closed-loop system if, and only if, system* (1) *is* Γ*-detectable and* Γ *is uniformly stable relative to* $V^{-1}(0)$.

In the next two sections we will use this result to design global and almost-global stabilizers for path following and maneuvering of the unicycle.

III. Circular path following

Consider the following model of the kinematic unicycle

$$\begin{aligned} \dot{x}_1 &= u_1 \cos(x_3) \\ \dot{x}_2 &= u_1 \sin(x_3) \\ \dot{x}_3 &= u_2, \end{aligned} \tag{2}$$

with state space $\mathcal{X} = \mathbb{R}^2 \times S^1$, where $(x_1, x_2) \in \mathbb{R}^2$ are the Cartesian coordinates of the unicycle, $x_3 \in S^1$ is the heading, and $u = (u_1, u_2)$ is the control input, with u_1 the linear velocity and u_2 the angular velocity.

The *Circular Path Following* problem is as follows: find a feedback for (2) such that the unicycle approaches and follows a circle of radius r centred at the origin of the (x_1, x_2) plane. Moreover, it is required that the unicycle traverses the circle in a prescribed direction of rotation with a desired velocity.

The motivation for the passivity-based controller that we present for this problem comes from the following observation. Consider the open-loop control $u_1 = v$ and $u_2 = v/r$ (or $-v/r$), where $v > 0$ is the desired velocity of the unicycle on the circle. With these controls and for any initial condition, the unicycle rotates in a circle of radius r and linear velocity v in the counter-clockwise (when $u_2 = v/r$) or clockwise (when $u_2 = -v/r$) direction, see Figure 1. What is noticed here is that for any initial state there is a preserved quantity which is the vector $z(t)$ (when $u_2 = v/r$) or $z'(t)$ (when $u_2 = -v/r$) representing the coordinates of the centre of the circle of rotation as shown in the

[1] Actually, *semi-attractive* when Γ is unbounded. See [18] for the definition of a semi-attractor.

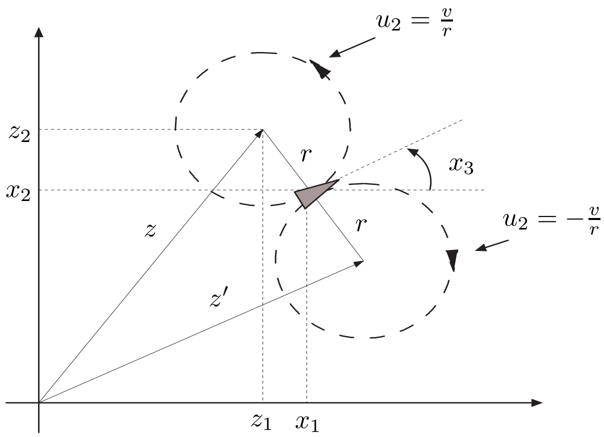


Fig. 1. Unicycle Circular Motion

figure. These vectors can be expressed in term of the states (x_1, x_2, x_3) as follows

$$\begin{aligned} z &= (x_1 - r \sin x_3,\ x_2 + r \cos x_3) \\ z' &= (x_1 + r \sin x_3,\ x_2 - r \cos x_3). \end{aligned}$$

Consider the sets $\Gamma = \{x \in \mathbb{R}^2 \times S^1 : x_1 = r \sin x_3, x_2 = -r \cos x_3\}$ and $\Gamma' = \{x \in \mathbb{R}^2 \times S^1 : x_1 = -r \sin x_3, x_2 = r \cos x_3\}$, and notice that, when $x \in \Gamma$ ($x \in \Gamma'$) the unicycle is on the circle of radius r centred at the origin with heading tangent to the circle in the counter-clockwise (clockwise) direction. Hence, solving the circular path following problem is equivalent to stabilizing Γ for counter-clockwise motion, or Γ' for clockwise motion. Notice also that Γ and Γ' can be expressed as $\Gamma = \{x : z_1 = z_2 = 0\}$, $\Gamma' = \{x : z'_1 = z'_2 = 0\}$ so, referring to Figure 1, the physical incentive behind the stabilization of Γ or Γ' is to force the vectors z or z' to zero so that, in steady-state, the unicycle travels around a circle of radius r in the counter-clockwise (clockwise) direction, centred at $z = 0$ ($z' = 0$). For the rest of this section we will consider the case of counter-clockwise rotation, focusing on the stabilization of the set Γ. The development for the clockwise case is identical. We will use the vector z to obtain a storage function for the system and to design a passivity-based controller. Let $v > 0$ denote the desired velocity of the unicycle on the circle, and consider the following choice of control inputs

$$\begin{aligned} u_1 &= v \\ u_2 &= \frac{v}{r} + u_c, \end{aligned}$$

where u_c is a feedback to be designed later. By substituting the control above into (2) we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} v \cos x_3 \\ v \sin x_3 \\ v/r \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_c. \qquad (3)$$

Proposition III.1 *System (3) with output function*

$$y = -r\,(x_1 \cos x_3 + x_2 \sin x_3) \qquad (4)$$

is passive with storage function

$$V_c = \frac{1}{2}\left((x_1 - r \sin x_3)^2 + (x_2 + r \cos x_3)^2\right).$$

Let $\varphi(\cdot) : \mathbb{R} \to \mathbb{R}$ be any function such that $\varphi(0) = 0$ and $y\varphi(y) > 0$ for $y \neq 0$. The control

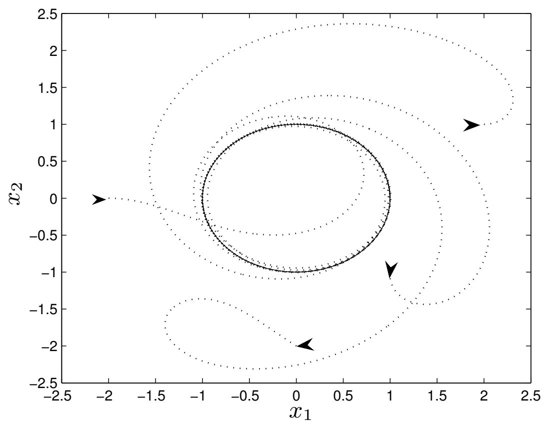
$$\begin{aligned} u_1 &= v \\ u_2 &= \frac{v}{r} + \varphi(r x_1 \cos x_3 + r x_2 \sin x_3) \end{aligned} \qquad (5)$$

globally asymptotically stabilizes the set $\Gamma = \{x \in \mathcal{X} : x_1 = r \sin x_3, x_2 = -r \cos x_3\}$ and thus globally solves the circular path following problem.

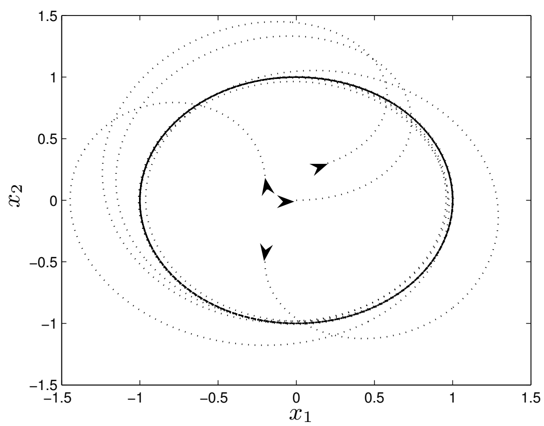
Proof: Rewriting system (3) in the form $\dot{x} = f(x) + g(x)u_c$, $y = h(x)$, with $f = [v\cos(x_3)\ \ v\sin(x_3)\ \ v/r]^\top$, $g = [0\ 0\ 1]^\top$, and $h(x) = -r(x_1 \cos x_3 + x_2 \sin x_3)$, by straightforward calculations we get $L_f V_c = 0$ and $L_g V_c = h$, and so the system is passive. The storage function V_c is positive semi-definite and its zero level set is the goal set Γ and so, trivially, Γ is uniformly stable relative to $V^{-1}(0)$. Since $x_3 \in S^1$ and S^1 is a compact set, all level sets of V are compact, and therefore all trajectories of the closed-loop system are bounded. The system (3)-(4) is Γ-detectable. This can be shown as follows. The zero level set of the output y is given by $h^{-1}(0) = \{x : x_1 \cos(x_3) = -x_2 \sin(x_3))\}$, so $x \in h^{-1}(0)$ if either $x_1 = x_2 = 0$, or $x_3 = \arg(x_2 - i x_1)$, or $x_3 = \arg(-x_2 + i x_1)$. Thus, on $h^{-1}(0)$, the unicycle is either at the origin with any heading, or at any other position with heading perpendicular to the position vector (x_1, x_2). Now suppose that, for the closed-loop system, $y(t) \equiv 0$ and $u_c(t) \equiv 0$. Then, the unicycle must travel along a circle of radius r centred at $(z_1(t), z_2(t)) \equiv (z_1(0), z_2(0))$, with heading perpendicular to the vector $(x_1(t), x_2(t))$. This can only occur if the centre of the circle is at the origin or, equivalently, $z_1(t) \equiv z_2(t) \equiv 0$, hence Γ-detectability. Using the feedback (5) corresponds to setting $u_c = \varphi(-y) = \varphi(r x_1 \cos x_3 + r x_2 \sin x_3)$. The feedback above can equivalently be expressed as $u_c = -\bar{\varphi}(y)$, where $\bar{\varphi}(y) = -\varphi(-y)$, and $y\bar{\varphi}(y) = -y\varphi(-y) > 0$ for all $y \neq 0$. The result follows directly from Theorem II.4. ■

Remark 1 *An important advantage of the passivity-based controller* (5) *is that it can be made to be compatible with any input saturation constraint. For, if the controller is subject to saturation constraints $|u_1| \leq U_1$, $|u_2| \leq U_2$, one can choose v small enough that $v < U_1$ and $v/r < U_2$ and choose φ so that $v/r + \sup_{\mathbb{R}} |\varphi(\cdot)| \leq U_2$.*

Now we present simulation results for controlling the circular motion of the unicycle (2) using control (5) with $\varphi(y) = ky$, $v = 1$, and $r = 1$. Figure 2 presents results for a number of initial states with control gain $k = 1$. Figure 3 shows results for initial state $(5, 5, 0)$ and different values of the control gain k. The first part of Figure 3 shows the control effort $u_2(t)$ during transient and steady-state, while the second part shows the distance of the unicycle from the unit circle as a function of time. Notice how lower gains lead



(a) Initial States: $(2, 1, 0)$, $(-2, 0, 0)$, $(0, -2, \pi)$, $(1, -1, -\frac{2\pi}{3})$



(b) Initial States: $(0, 0, 0)$, $(-.2, .2, \frac{2\pi}{3})$, $(.2, .3, \frac{\pi}{6})$, $(-.2, -.5, \frac{\pi}{2})$

Fig. 2. Controlling the Circular Motion

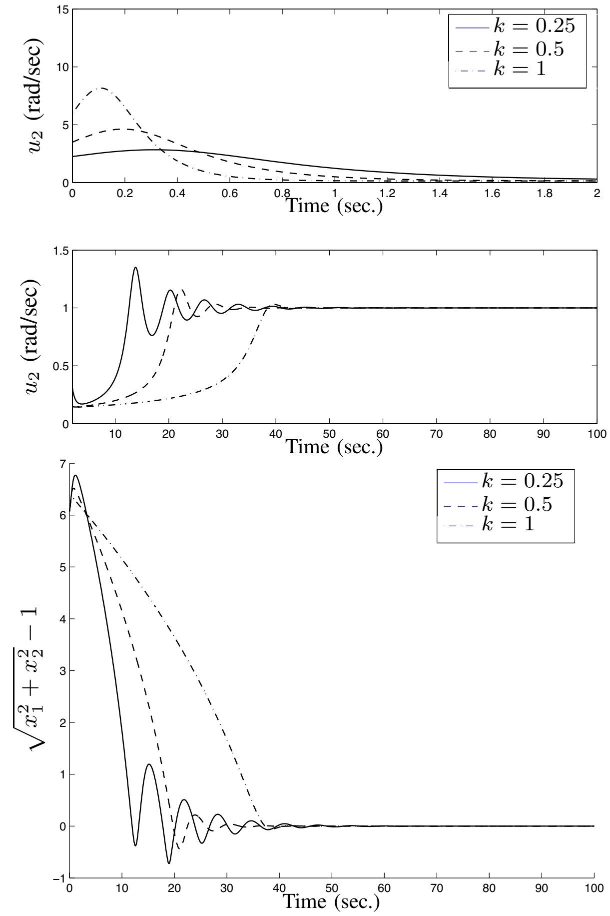


Fig. 3. Effects of different control gains

to faster convergence but increasing oscillations. Obviously, convergence can be sped up by increasing v and excessive control effort can be avoided by means of saturation.

IV. STABILIZING THE UNIT CIRCLE WITH DESIRED HEADING

In this section we address the following maneuvering problem: stabilize the unicycle to a circle of radius r centred at the origin, and make it assume a desired heading on the circle. In other words, it is required to stabilize the set

$$\Gamma = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 = r^2, x_3 = a\}, \qquad (6)$$

where a is the desired reference heading. The orientation requirement adds challenge to the problem. Due to the nonholonomic constraint in the unicycle, the only way that the unicycle can remain on the circle with constant heading is that the unicycle stands still on the circle. Hence, this problem is of an intrinsically different nature than the circular path following problem solved in Section III. For the rest of the section we take the radius $r = 1$. Note that for system (2), $f = [0\ 0\ 0]^\top$, and so the system is passive with any storage function $V(x_1, x_2, x_3)$, and output $y = L_g V$. Consider the storage function $V_1 = (x_1^2 + x_2^2 - 1)^2/2 + (x_3 - a)^2/2$. Note that $V^{-1}(0)$ is the goal set (6), and $L_g V_1 = \text{col}(2(x_1^2 + x_2^2 - 1)(x_1 \cos x_3 + x_2 \sin x_3), x_3 - a)$. Using as control $u = -L_g V_1$ fails to stabilize Γ in this case because system (2) with output $L_g V_1$ is not Γ-detectable. For, suppose that $u(t) \equiv 0$ and $L_g V_1(t) \equiv 0$. Then, $(x_1(t), x_2(t)) \equiv (\bar{x}_1, \bar{x}_2)$ and $x_3(t) \equiv a$, and so the unicycle dynamics are stationary. Moreover, it must be that $(\bar{x}_1^2 + \bar{x}_2^2 - 1)(\bar{x}_1 \cos(a) + \bar{x}_2 \sin(a)) \equiv 0$. Figure 4 illustrates this set of positions with heading $\bar{x}_3 = a$. It is clear that this set strictly contains the goal set Γ in (6), and so the system is not Γ-detectable. In general, if we choose for the system a storage function $V(e_1, e_2)$, where e_1 and e_2 are the position and heading errors $e_1 = x_1^2 + x_2^2 - 1$, $e_2 = x_3 - a$, and $(e_1, e_2) \mapsto V(e_1, e_2)$ is positive definite, then the first component of $L_g V$, $(L_g V)_1$, is given by $(L_g V)_1 = (\partial V / \partial e_1)(x_1 \cos(x_3) + x_2 \sin(x_3))$, which gives an obstruction to Γ-detectability. The above suggest that if one wants to solve the maneuvering problem using a passivity-based approach, one should *not* attempt to find a storage V with the property that $V^{-1}(0) = \Gamma$. Guided by this principle, we choose the simplest storage V such that $V^{-1}(0) \supsetneq \Gamma$, namely $V(x) = 1/4(x_1^2 + x_2^2 - 1)^2$. Our design approach is this: design u_2 to insure Γ-detectability and stability of Γ relative to $V^{-1}(0)$ for the system with input u_1 and output $L_{g_1} V$. Then, let u_1 be a standard passivity-

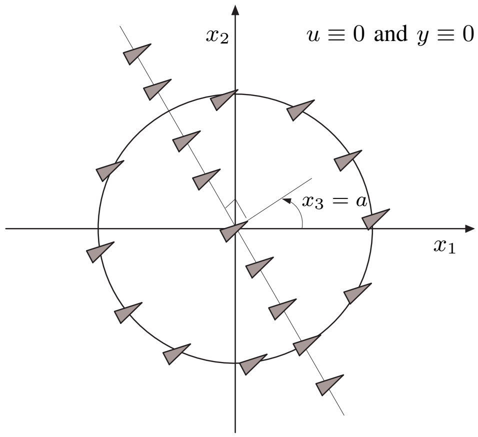


Fig. 4. Failure of Γ-detectability.

based feedback.

Proposition IV.1 *Let $e = x_1^2 + x_2^2 - 1$ and let $f(e)$ be a Lipschitz continuous positive definite function. The control*

$$\begin{aligned} u_1 &= -\varphi_1\big(e(x_1\cos x_3 + x_2 \sin x_3)\big) \\ u_2 &= -\varphi_2(x_3 - a) + f(e)\sin t \end{aligned} \tag{7}$$

where $\varphi_{1,2}(\cdot)$ are smooth functions such that $\varphi_{1,2}(0) = 0$ and $y\varphi_{1,2}(y) > 0$ for all $y \neq 0$, renders both sets

$$\begin{aligned} G &= \{(x_1, x_2, x_3) : x_1^2 + x_2^2 = 1\} \\ \Gamma &= \{(x_1, x_2, x_3) : x_1^2 + x_2^2 = 1, x_3 = a\}, \end{aligned}$$

asymptotically stable for the unicycle with domain of attraction $\mathcal{X}\backslash\{(x_1, x_2, x_3) : x_1 = x_2 = 0\}$. Hence, G and Γ are almost globally asymptotically stable for the closed-loop system.

Proof: Replacing the expression for u_2 in (2) and augmenting (2) with the equation $\dot\theta = 1$, $\theta \in S^1$, we obtain an autonomous control system with input u_1, state (x_1, x_2, x_3, θ), and state space $\mathcal{X}_e = \mathbb{R}^2 \times S^1 \times S^1$,

$$\begin{aligned} \dot x_1 &= u_1 \cos x_3 \\ \dot x_2 &= u_1 \sin x_3 \\ \dot x_3 &= -\varphi_2(x_3 - a) + f(e)\sin\theta \\ \dot\theta &= 1. \end{aligned} \tag{8}$$

Consider the lift of the sets Γ and G to the state space $\mathcal{X}_e$, $\Gamma_e = \Gamma \times S^1$, $G_e = G \times S^1$. Notice that $\|(x,\theta)\|_{\Gamma_e} = \|x\|_\Gamma$ and $\|(x,\theta)\|_{G_e} = \|x\|_G$, and so showing that Γ_e and G_e are asymptotically stable for (8) with u_1 as in (7), is equivalent to showing that Γ and G are asymptotically stable for the original non-autonomous closed-loop system. Consider the storage function $V = \frac{1}{4}e^2$, with $e = x_1^2 + x_2^2 - 1$. Since $L_f V = 0$, the system is passive with output

$$y = L_g V = e(x_1 \cos x_3 + x_2 \sin x_3). \tag{9}$$

Notice that V is proper in $\mathcal{X}_e$, and thus all trajectories of the closed-loop system are bounded. System (8)-(9) is locally Γ_e-detectable. This is shown as follows. If $u_1 \equiv 0$, the system dynamics are given by

$$\begin{aligned} &\dot x_1 = 0,\ \dot x_2 = 0 \\ &\dot x_3 = -\varphi_2(x_3 - a) + f(e)\sin\theta \\ &\dot\theta = 1, \end{aligned}$$

and so we have

$$\begin{aligned} &x_1(t) = \bar x_1,\ x_2(t) = \bar x_2 \\ &\dot x_3 = -\varphi_2(x_3 - a) + f(e)\sin(\theta_0 + t). \end{aligned}$$

If in addition we have $y(t) \equiv 0$, we get either (1) $\bar x_1 = \bar x_2 = 0$, or (2) $e(t) \equiv 0$, or $\bar x_1 \cos(x_3) + \bar x_2 \sin(x_3) = 0$, which implies (3) $x_3(t) \equiv \bar x_3$. In case (3), $\dot x_3(t) \equiv 0$ and this can only be satisfied if $f(e(t)) \equiv 0$ (i.e., $e(t) \equiv 0$), and $x_3(t) \equiv a$. In case (2) we have $e(t) \equiv 0$, and so $\dot x_3 = -\varphi_2(x_3 - a)$, thus $x_3(t) \to a$. The only problematic situation is case (1). Since any point $(x_1, x_2, x_3, \theta) = (0, 0, \star, \star)$ is a local maximum of V, and V is nonincreasing along closed-loop solutions, any solution originating in $\mathcal{X}_2\backslash 0\times 0\times S^1 \times S^1$ never enters the set $0\times 0\times S^1\times S^1$. In conclusion, system (8) is locally Γ_e-detectable with $\mathcal{N}(\Gamma_e) = (\mathbb{R}^2 \times S^1 \times S^1)\backslash(0\times 0\times S^1\times S^1)$ as in Definition II.3. As for the other conditions of Theorem II.4, we have $\Gamma_e \subset G_e = V^{-1}(0)$. On the set $V^{-1}(0)$, the open-loop system dynamics take the form

$$\dot x_1 = 0,\ \dot x_2 = 0,\ \dot x_3 = -\varphi_2(x_3 - a),\ \dot\theta = 1$$

and so the set Γ_e is asymptotically stable relative to $V^{-1}(0)$ for the open-loop system. Since $u_1 = -\varphi_1(y)$, by Theorem II.4 we get asymptotic stability of Γ_e, and in fact almost-global asymptotic stability of Γ_e since every point in $(0\times 0\times S^1\times S^1)$ is a local maximum for the storage function V. We now turn our attention to the stability of G_e. Since $\Gamma_e \subset G_e$ and since (8)-(9) is almost globally Γ_e-detectable, this also implies that it is almost globally G_e-detectable. Since $G_e = V^{-1}(0)$ and all trajectories of the closed-loop system are bounded, Theorem II.4 once again implies that the control $u_1 = -\varphi_1(y)$ asymptotically stabilizes G_e, and in fact almost globally stabilizes G_e by the same reason explained above. ■

Once again the feedback (7) can be made to satisfy any saturation limits on the inputs u_1 and u_2 by appropriate choices of φ_1, φ_2, and f. From a practical viewpoint, the property that both Γ and G are asymptotically stable is particularly useful: if the unicycle is initialized on the circle with *any* heading, then the unicycle adjusts its heading without leaving the circle. Similarly, if the unicycle is close to the circle, it adjusts its heading while remaining close to it. We present simulation results for the controller (7) used to stabilize the position of the unicycle to a unit circle with a reference heading $a = \pi/6$. We choose $\varphi_1(y) = \varphi_2(y) = y$, and $f(e) = |e|$. Figure 5 presents results for a number of different initial states, while Figure 6 presents the corresponding error signals $e_1 = x_1^2 + x_2^2 - 1$, $e_2 = x_3 - a$.

V. Conclusions

In this paper we have investigated two set stabilization problems for the kinematic unicycle. First, we provided

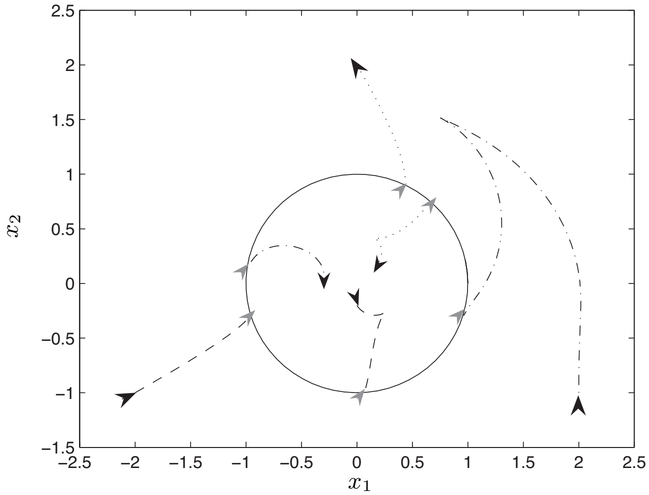


Fig. 5. Control for Circular reference with desired heading

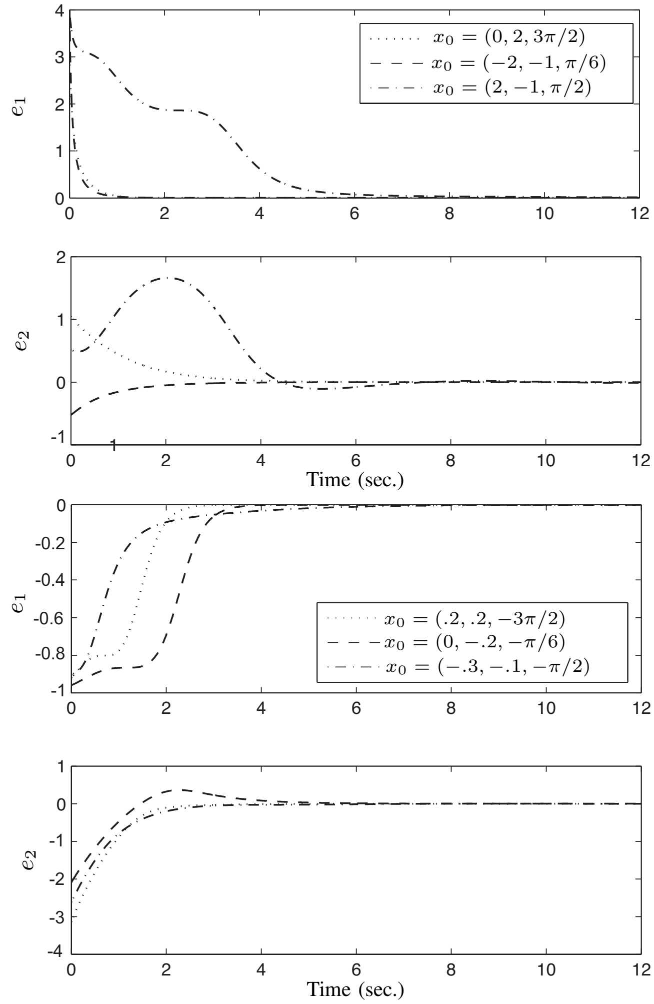


Fig. 6. Error signals

a smooth and time-invariant state feedback that globally asymptotically stabilizes the circular motion in a desired direction. Second, we solved a maneuvering problem whereby it is required to stabilize the unicycle to a circle and to have a desired constant heading on it. For this latter problem we provided an almost globally stabilizing passivity-based time-varying feedback. Both designs are based on an extension to recent results on passivity-based set stabilization.

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