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Eigenvalue Constraints for the Stability of T-S Fuzzy Models

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Abstract— **This paper deals with the stability issue of T-S fuzzy models by eigenvalue analysis. Based on Lyapunov's direct method, the stability of T-S fuzzy models can be reduced to finding a common positive definite matrix. Conditions for the existence of such a matrix are discussed in terms of eigenvalue distributions. Then, the relaxed eigenvalue constraints for the the stabilization of T-S fuzzy models are given via state feedback controllers.**

I. INTRODUCTION

Fuzzy dynamic models proposed by Takagi and Sugeno are known as T-S fuzzy models. The basic idea of fuzzy modeling for T-S fuzzy models is to decompose the input space into a number of fuzzy regions in which the system behavior is approximated by a local linear model. The overall fuzzy model is then a fuzzy blending of the local models interconnected by a set of membership functions [9]. Since T-S fuzzy models can be finally formulated in terms of differential or difference equations, they can be taken as conventional nonlinear systems as well. Thereby, most of the stability analysis approaches for nonlinear systems can also be applied to the study of T-S fuzzy models. By Lyapunov's direct method the stability of fuzzy T-S models can be reduced to finding a common positive definite matrix [7]. In order to find such a common positive definite matrix, a lot of numerical approaches have been presented in the literature, such as gradient algorithm [3], genetic approach [1], LMI approach [5], etc.. Theoretical results are also reported e.g. in [7], [8] and [2] with respect to the necessary conditions for the existence of such a matrix. However, the necessary and sufficient conditions for the existence of such a matrix remain open. We show first the eigenvalue locations for the existence of such a common matrix. Then, by employing fuzzy state feedback controllers, extended stabilization conditions for T-S fuzzy models are presented in terms of eigenvalue constraints.

II. EIGENVALUE-BASED STABILITY CONDITIONS

In discrete case, T-S fuzzy models can be described by the following fuzzy rules ([6], [9]):

Plant rules: If $x_1(k)$ is M_1^i and ... and $x_n(k)$ is M_n^i, then
$$x(k+1) = A_i x(k) + B_i u(k) \ (i = 1, 2, ..., r)$$
where r is the number of fuzzy rules, M_j^i stands for the fuzzy set of the j-th antecedent variable in the i-th fuzzy rule, $u(k) = [u_1(k), u_2(k), ..., u_m(k)]^T$ is the control input, and $x(k) = [x_1(k), x_2(k), ..., x_n(k)]^T$ is the state variable. By the singleton fuzzifier, product inference and the center average defuzzifier, the final outputs of the fuzzy systems can be represented by:

$$x(k+1) = \sum_{i=1}^{r} \alpha_i(x(k))(A_i x(k) + B_i u(k)) \tag{1}$$

where $\alpha_i(x(k)) = \omega_i(x(k)) / \sum_{j=1}^{r} \omega_j(x(k))$, $\omega_i(x(k)) = \prod_{j=1}^{n} \mu_{M_j^i}(x(k))$ and $\sum_{i=1}^{r} \omega_i(x(k)) \neq 0$ for all $k \geqslant 0$. Obviously it holds: $0 \leqslant \alpha_i(x(k)) \leqslant 1 \quad (i = 1, 2, ..., r)$ and $\sum_{i=1}^{r} \alpha_i(x(k)) = 1$. In general, $\alpha_i(x(t))$ can be regarded as the matching degree between the state variable and the antecedent of the i-th fuzzy rule.

Specially, the undriven (i.e. $u(k) \equiv 0$) discrete T-S fuzzy models can be formulated as:

$$x(k+1) = \sum_{i=1}^{r} \alpha_i(x(k)) A_i x(k). \tag{2}$$

According to Theorem 4.2 in [7], the open loop model (2) is globally asymptotically stable if there is a common positive definite matrix P such that $A_i^T P A_i - P < 0$ $(i = 1, 2, ..., r)$. If all matrices A_i are non-singular, then the necessary condition for the existence of such a common positive definite matrix P is that $A_i A_j$ is stable for all $i, j = 1, 2, ..., r$ (Theorem 4.3, [7]). We will show that the non-singular condition of A_i is unnecessary. The results in [7] can be extended to:

Lemma 1: For discrete T-S fuzzy model (2), the following sufficient stability conditions are equivalent:

1. There is a positive symmetric matrix P such that $A_i^T P A_i - P < 0$ $(i = 1, 2, ..., r)$.
2. $A_{i_k}^T A_{i_{k-1}}^T ... A_{i_1}^T P A_{i_1} ... A_{i_{k-1}} A_{i_k} - P < 0$ for all $A_{i_j} \in \{A_1, A_2, ..., A_r\}$.
3. $(\frac{A_{i_1}+A_{i_2}+...+A_{i_k}}{k})^T P (\frac{A_{i_1}+A_{i_2}+...+A_{i_k}}{k}) - P < 0$ for all $A_{i_j} \in \{A_1, A_2, ..., A_r\}$ and all $k \in N$.

Proof: (1⇒2) Since $A_i^T P A_i - P < 0$, we have:

$$A_{i_1}^T P A_{i_1} - P =: -Q_1 < 0 \tag{3}$$

$$A_{i_2}^T P A_{i_2} - P =: -Q_2 < 0 \tag{4}$$

...

$$A_{i_k}^T P A_{i_k} - P =: -Q_k < 0.$$

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Multiplying $A_{i_2}^T$ to the left side and A_{i_2} to the right side of (3), we have:

$$A_{i_2}^T A_{i_1}^T P A_{i_1} A_{i_2} - A_{i_2}^T P A_{i_2} =: -A_{i_2}^T Q_1 A_{i_2} \leqslant 0. \quad (5)$$

Then, from (4) and (5) it yields:

$$A_{i_2}^T A_{i_1}^T P A_{i_1} A_{i_2} - P = -Q_2 - A_{i_2}^T Q_1 A_{i_2} < 0.$$

Continuing the procedure we obtain:

$$A_{i_k}^T A_{i_{k-1}}^T ... A_{i_1}^T P A_{i_1} ... A_{i_{k-1}} A_{i_k} - P < 0.$$

(1⇒3) $(\frac{A_{i_1}+A_{i_2}+...+A_{i_k}}{k})^T P (\frac{A_{i_1}+A_{i_2}+...+A_{i_k}}{k}) - P$

$$= \frac{1}{k^2}(\sum_{j=1}^{k} A_{i_j}^T P A_{i_j} + \sum_{1\leqslant s<t\leqslant k}(A_{i_s}^T P A_{i_t} + A_{i_t}^T P A_{i_s})) - P$$

$$\leqslant \frac{1}{k^2}(\sum_{j=1}^{k} A_{i_j}^T P A_{i_j} + \sum_{1\leqslant s<t\leqslant k}(A_{i_s}^T P A_{i_s} + A_{i_t}^T P A_{i_t})) - P$$

$$= \frac{1}{k}\sum_{j=1}^{k}(A_{i_j}^T P A_{i_j} - P) < 0.$$

(2⇒1) and (3⇒1) are obvious. ■

Theorem 1: If there exists $P > 0$ such that $A_i^T P A_i - P < 0$ for all $i = 1, 2, ..., r$, then

1). the eigenvalues of the product of any number of A_i ($i = 1, 2, ..., r$) must be located strictly in the unit circle,

2). the eigenvalues of the average of any number of A_i ($i = 1, 2, ..., r$) must be located strictly in the unit circle,

Proof: It follows from Lemma 1 directly. ■

The above result holds true without additional conditions for non-singularity of A_i, which can be taken as a generalization of Theorem 4.3 in [7]. It is easy to see that the necessary conditions in Theorem 1 are satisfied if all the spectral norms $||A_i|| < 1$. Moreover, we have:

$$||A_i|| < 1 \ (i = 1, 2, ..., r)$$
$$\Rightarrow \exists P > 0, \ s.t. \ A_i^T P A_i - P < 0 \ (i = 1, 2, ..., r)$$
$$\Rightarrow |\lambda(A_i)| < 1 \ (i = 1, 2, ..., r).$$

That is, the eigenvalue constraint on A_i for the existence of a common $P > 0$ such that $A_i^T P A_i - P < 0$ for $i = 1, 2, ..., r$ expresses just a region, which is included in the unit circle and contains $\{\lambda_{A_i} : ||A_i|| < 1\}$.

We consider next the stabilization of (1) using fuzzy state feedback controllers. Based on the parallel distributed compensation (PDC), the fuzzy control law for (1) can be expressed by the following fuzzy rules:

Controller rules: If $x_1(k)$ is M_1^i and ... and $x_n(k)$ is M_n^i, then

$$u(k) = F_i x(k) \ (i = 1, 2, ..., r),$$

where $F_i \in \Re^{m\times n}$ are the state feedback gains to be designed. Thereby, the overall state feedback fuzzy controller is of the form:

$$u(k) = \sum_{i=1}^{r} \alpha_i(x(k)) F_i x(k). \quad (6)$$

For brevity, we denote:

$$H_{ij} := A_i + B_i F_j, \ G_{ij} := \frac{H_{ij}+H_{ji}}{2},$$
$$\lambda_{ij} := \lambda_{\max}(G_{ij}^T P G_{ij} - P), \ \alpha_i := \alpha_i(x(k)),$$
$$q := \max_{x(k)} |\{\alpha_i(x(k)) : \alpha_i(x(k)) \neq 0, \ i = 1, 2, ..., r\}|,$$

where q stands for the maximum number of the fired rules for all $k \geqslant 0$. Then we can prove:

Theorem 2: The fuzzy system described by (1) is globally asymptotically stabilized via fuzzy control law (6), if there exists a matrix $P > 0$ and $F_i \in \Re^{m\times n}$ such that $\lambda_{ii} < 0$ for $i = 1, 2, ..., r$ and $\lambda_{ij} < \frac{\sqrt{\lambda_{ii}\lambda_{jj}}}{q-1}$ for $1 \leqslant i < j \leqslant r$ excepting the pairs (i, j) such that $\alpha_i(x(k))\alpha_j(x(k)) \equiv 0$.

Proof: Let $V(x(k)) = x^T(k) P x(k)$. Then:

$$\Delta V(x(k)) = V(x(k+1)) - V(x(k))$$
$$= \sum_{i,j,s,t} \alpha_i\alpha_j\alpha_s\alpha_t x^T(k) H_{ij}^T P H_{st} x(k) - x^T(k) P x(k)$$
$$\leqslant \sum_{i,j} \alpha_i\alpha_j x^T(k) G_{ij}^T P G_{ij} x(k) - x^T(k) P x(k)$$
$$= \sum_{i} \alpha_i^2 x^T(k)(H_{ii}^T P H_{ii} - P) x(k)$$
$$+ \sum_{i<j} 2\alpha_i\alpha_j x^T(k)(G_{ij}^T P G_{ij} - P) x(k)$$
$$\leqslant \sum_{i} \alpha_i^2 x^T(k)\lambda_{ii} x(k) + \sum_{i<j} 2\alpha_i\alpha_j x^T(k)\lambda_{ij} x(k)$$
$$= \frac{1}{q-1}\sum_{i}(q-1)\alpha_i^2\lambda_{ii}|x(k)|^2 + \sum_{i<j} 2\alpha_i\alpha_j\lambda_{ij}|x(k)|^2$$
$$\leqslant \sum_{i<j} 2\alpha_i\alpha_j(\lambda_{ij} - \frac{\sqrt{\lambda_{ii}\lambda_{jj}}}{q-1})|x(k)|^2.$$

Thereby $\Delta V(x(k)) < 0$ holds for all $x(k) \neq 0$, if the given conditions are satisfied, which completes the proof. ■

Based on the eigenvalue constraints in the theorem, the desired feedback gains F_i can be solved by the following exploratory procedures:

(1). Set $\varepsilon = 0$, $N = 0$.

(2). Find $0 < P \in \Re^{n\times n}$ and $F_i \in \Re^{m\times n}$, such that $\lambda_{ii} + \varepsilon < 0$ for $i = 1, 2, ..., r$.

(3). Verify the inequalities $\lambda_{ij} < \frac{\sqrt{\lambda_{ii}\lambda_{jj}}}{q-1}$ for all $i < j$.

(4). If the inequalities in (3) are not satisfied, set $N = N + 1$, $\varepsilon = N \times \varepsilon_0$, then go to (2).

In the above solution procedures, step (2) can be solved by employing the linear matrix inequality (LMI) tools in Matlab. The parameter ε_0 can be chosen to be a sufficiently small positive scalar such that step (2) is always feasible.

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