2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008



# A minimal dimension observer for global frequency estimation

Daniele Carnevale and Alessandro Astolfi

***Abstract*— A minimum dimension observer for on-line global estimation of the frequencies of a signal resulting from the sum of a bias and $n$ sinusoids with unknown amplitudes, frequencies and phases is proposed. The dimension of the observer is $3n-1$ when no bias is considered, $3n$ otherwise.**

## I. Introduction

The problem of estimating the unknown frequencies and bias of the signal

$$y(t) = E_0 + \sum_{i=1}^{n} E_i \sin(\omega_i t + \phi_i), \tag{1}$$

with known $n \geq 1$ and unknown bias $E_0$, angular frequencies $\omega_i > 0$, amplitudes $E_i$ and phases $\phi_i$, for $i = 1, .., n$, has been widely studied in systems theory, since the problem arises in different engineering fields as telecommunication, image processing, identification and control.

The first solutions have been proposed from a signal processing point of view [1], using classical Fourier analysis performed off-line on batch processing data. Such an approach is not suitable when on-line estimation of the unknown frequencies is needed, as in control applications, where compensation or simply identification of disturbances may benefit from a frequencies estimator with suitable asymptotic properties (see [8]). As an example, the *extremum seeking* feedback schema in [3], where the probing sinusoidal signal with unknown frequency is reconstructed simply filtering the system output, would considerably benefit from a robust frequency estimator (in this case with $n = 1$, it is straightforward to retrieve $E_1$ and $\phi_1$ when an estimate of $\omega_1$ is available).

An on-line estimator, employing infinite impulse response filters or notch filters, has been proposed in [2]. This estimator has been enhanced with an adaptation mechanism in [4], where the problem of single frequency estimation has been solved globally for the first time. Since then, many authors have, almost simultaneously, provided global solutions to the estimation problem for $n$ unknown frequencies.

The first on-line global estimators for $n \geq 1$ make use of $3n$ dimensional adaptive observers and resort to the adaptive identifiers structure [10], [6]. A different approach is proposed in [7], where a linear adaptive observer of dimension $5n - 1$, exploiting a filtered transformation of coordinates, has been proposed. The latter usually provides improved convergence properties when compared to the formers. In [11] a revised extended Kalman filter is proposed for the case $n = 1$ and in the presence of additive broad-band noise. Finally, an adaptive observer of dimension $5n$ for the estimation of the amplitudes $E_i$, exploiting adaptive identifiers for the estimation of the $n$ unknown frequencies $\omega_i$, has been introduced in [9].

The solution that we present for the global estimation of $n$ unknown angular frequencies is based on the ideas developed in [13], [14]. The observer design problem is cast into the problem of defining an appropriate manifold in the extended state space of the observer and of a system generating the signal (1). The estimation error is identically equal to zero for all trajectories on the manifold, which has, therefore, to be rendered invariant and attractive. As discussed in [14], the determination of the above manifold requires (in general) the solution of a set of PDEs. However, in the present context, we show that the manifold, with the desired properties, can be directly determined.

The reduced-order observer that we propose is of dimension $3n - 1$ ($3n$ if the bias $E_0$ is considered), and it is of smaller dimension than the previous ones. Furthermore, such a dimension is the smallest possible (hence the observer is minimal in this respect), since $2n-1$ states are needed for the estimation of the states of the linear time invariant system generating the measured signal $y(t)$, and the remaining $n$ states are used for the estimation of the $n$ unknown frequencies ($n + 1$ states are necessary if the bias is considered).

The paper is organized as follows. To introduce our approach we present in Section II the global frequency estimator for signals having one frequency, *i.e.* $n = 1$, and a constant bias. The generalization to the case $n > 1$ is presented in Section III, together with a robustness analysis. Simulation results are shown in Section IV to prove the effectiveness of the proposed approach and conclusions, in Section V, summarize the results.

## II. Global Estimator of a Single Frequency

The measured signal $y(t) = E_1 \sin(\omega_1 t + \phi_1)$ may be regarded as the output of the system

$$\begin{aligned} \dot{y} &= x, \\ \dot{x} &= -\theta_1 y, \end{aligned} \tag{2}$$

with $\theta_1 = \omega_1^2 > 0$. Define the estimation error $z = [z_x, z_\theta] = \beta(y, \xi) - h(\xi, x, \theta_1)$, with $h(\cdot)$ left invertible with respect to $x$ and $\theta_1$, and $\xi = [\xi_1, \xi_2]^\top \in \mathbb{R}^2$ an auxiliary variable, the dynamics of which will be selected in the sequel. In particular, let

$$z_x = \beta_1(y, \xi) - h_1(\xi, \theta_1, x), \tag{3}$$
$$z_\theta = \beta_2(y, \xi) - h_2(\xi, \theta_1), \tag{4}$$

with $h_1(\cdot)$ and $h_2(\cdot)$ invertible with respect to $x$ and $\theta_1$, respectively.

978-1-4244-2079-7/08/$25.00 ©2008 AACC.

Consider now the variables $y$, $x$ and $\xi$ and the set $z = 0$, which describes a manifold, parameterized by $\theta_1$ in the space $(y, x, \xi)$. Note that the estimation problem is solved if this manifold is rendered invariant and attractive by some selection of the functions $\beta_i(\cdot)$ and $h_i(\cdot)$, and of the $\xi$ dynamics. In fact, on the manifold, one can express the unknown variables $x$ and $\theta_1$ as a function of the known variable $\xi$, exploiting the assumed invertibility conditions.

To this aim, consider the selection

$$\begin{aligned} \beta_1(y) &= k_1 y, \\ \beta_2(y,\xi) &= \gamma_2 y \xi_1 + \xi_2, \\ h_1(\xi_1,\theta,x) &= -\gamma_1 \xi_1 (k_1^2+\theta_1) + x, \\ h_2(\theta_1) &= \gamma_1 \theta_1, \end{aligned}$$

with $k_1$ and $\gamma_i$ positive constants. The given choice for $\beta_i(\cdot)$ and $h_i(\cdot)$ yields

$$z_x = k_1 y + \gamma_1 \xi_1 \left(k_1^2 + \theta_1\right) - x, \tag{5}$$

$$z_\theta = \gamma_2 y \xi_1 + \xi_2 - \gamma_1 \theta_1, \tag{6}$$

hence

$$\dot{z}_x = k_1 x + \gamma_1 \dot{\xi}_1 \left(k_1^2 + \theta_1\right) + \theta_1 y, \tag{7}$$

$$\dot{z}_\theta = \gamma_2 \left(\xi_1 x + y \dot{\xi}_1\right) + \dot{\xi}_2. \tag{8}$$

Substituting $x$ in $\dot{z}_x$, using (5), yields

$$\dot{z}_x = -k_1 z_x - \Delta_1(y,\xi_1)(k_1^2+\theta_1), \tag{9}$$

where

$$\Delta_1(y,\xi_1) = (\dot{\xi}_1 + k_1 \xi_1)\gamma_1 + y.$$

It is now evident that the selection of $h_1(\cdot)$, in which the free variable $\xi_1$ is multiplied by the unknown parameter $\theta_1$, allows to render asymptotically stable and independent from $z_\theta$ (*i.e.* the unknown parameter $\theta_1$) the dynamics of (9) selecting $\dot{\xi}_1$ such that $\Delta_1(y,\xi_1) = 0$, *i.e.*

$$\dot{\xi}_1 = -k_1 \xi_1 - \frac{y}{\gamma_1}.$$

Substituting $\theta_1$ in (8), using (6), yields

$$\dot{z}_\theta = -\gamma_2 \xi_1 z_x - \gamma_2 \xi_1^2 z_\theta + \Delta_2(y,\xi),$$

where

$$\begin{aligned} \Delta_2(y,\xi) \;=\; & \gamma_2 \xi_1 \left(k_1 y + \xi_1 \gamma_1 \left(k_1^2 + \tfrac{\xi_2 + \gamma_2 \xi_1 y}{\gamma_1}\right)\right) + \\ & \gamma_2 y \dot{\xi}_1 + \dot{\xi}_2. \end{aligned}$$

Note now that selecting

$$\dot{\xi}_2 = -\gamma_2 \xi_1^2 \left(\gamma_1 k_1^2 + \xi_2 + \gamma_2 \xi_1 y\right) + \frac{\gamma_2}{\gamma_1} y^2,$$

yields $\Delta_2(y,\xi) = 0$, hence the dynamics of $z$ are given by

$$\dot{z}_x = -k_1 z_x, \tag{10}$$

$$\dot{z}_\theta = -\gamma_2 \xi_1 z_x - \gamma_2 \xi_1^2 z_\theta. \tag{11}$$

This system can be regarded as the cascade of the LTI and the LTV systems (10) and (11), respectively. Global exponential convergence of the estimation error to zero can be easily proved noting that $\xi_1(t) = 0$ no more then two times in each time interval $2\pi/\omega_1$. As a result, the following fact holds, the proof of which is omitted since it is *contained* in the proof of Proposition 2.

*Proposition 1:* Consider the signal $y(t) = E_1 \sin(\omega_1 t + \phi_1)$, with $\omega_1 \geq 0$ and unknown, and $E_1$ and $\phi_1$ unknown.

Consider the dynamical system

$$\begin{aligned} \dot{\xi}_1 &= -k_1 \xi_1 - \frac{y}{\gamma_1}, \\ \dot{\xi}_2 &= -\gamma_2 \xi_1^2 \left(\gamma_1 k_1^2 + \xi_2 + \gamma_2 \xi_1 y\right) + \frac{\gamma_2}{\gamma_1} y^2, \end{aligned}$$

with output

$$\begin{aligned} \hat{\omega}_1 &= \sqrt{\gamma_1^{-1} \left|\gamma_2 y \xi_1 + \xi_2\right|}, \\ \hat{x} &= k_1 y + \xi_1 \left(\gamma_1 k_1^2 + \gamma_2 y \xi_1 + \xi_2\right). \end{aligned}$$

Then

$$\hat{\omega}_1(t) - \omega_1 \in \mathcal{L}_\infty,$$

$$\lim_{t \to +\infty} \hat{x}(t) - \dot{y}(t) = 0,$$

exponentially,

$$\xi_1(t) z_\theta(t) \in \mathcal{L}_2$$

and

$$\lim_{t \to +\infty} \xi_1(t) z_\theta(t) = 0.$$

In addition, if $E_1 \neq 0$ and $\omega_1 > 0$ then

$$\lim_{t \to +\infty} \hat{\omega}_1(t) = \omega_1,$$

exponentially. □

When the measured signal $y(t)$ has a constant bias of unknown magnitude $E_0$, *i.e.* $y(t) = E_0 + E_1 \sin(\omega_1 t + \phi_1)$, which may be modeled as the output $y$ of the dynamical system

$$\begin{aligned} \dot{y} &= x, \\ \dot{x} &= -\theta_1 y + \theta_0, \end{aligned} \tag{12}$$

with $\theta_0 = E_0 \theta_1$, the proposed observer has to be modified introducing the vectors $\Theta = [\theta_1, \theta_0]^\top \in \mathbb{R}^2$ and $R(\xi_1) = [\xi_1, 1/(\gamma_1 k_1)] \in \mathbb{R}^{1\times 2}$, leading to the definition

$$\begin{aligned} z_x &= k_1 y + \gamma_1 \left(k_1^2 \xi_1 + R(\xi_1)\Theta\right) - x, \\ z_\Theta &= \gamma_2 R(\xi_1)^\top y + \xi_2 - \gamma_1 \Theta. \end{aligned} \tag{13}$$

The dynamics of the observer states $\xi_1 \in \mathbb{R}$ and $\xi_2 \in \mathbb{R}^2$ are selected, similarly to the previous case, as

$$\begin{aligned} \dot{\xi}_1 &= -k_1 \xi_1 - \frac{y}{\gamma_1}, \\ \dot{\xi}_2 &= -\Big[\gamma_2 R(\xi_1)^\top \Big(k_1 y + \gamma_1 \big(k_1^2 \xi_1 + \\ &\qquad R(\xi_1) \tfrac{\xi_2 + \gamma_2 R(\xi_1)^\top y}{\gamma_1}\big)\Big) + \gamma_2 y \dot{R}^\top(\xi_1)\Big], \end{aligned} \tag{14}$$

yielding

$$\begin{aligned} \dot{z}_x &= -k_1 z_x, \\ \dot{z}_\Theta &= -\gamma_2 R(\xi_1)^\top z_x - \gamma_2 R(\xi_1)^\top R(\xi_1) z_\Theta. \end{aligned} \tag{15}$$

Note the cascaded structure of the system (15). We now briefly analyze the properties of system (15), and in particular show that its zero equilibrium is globally exponentially stable, provided $E_1 \neq 0$ and $\omega_1 > 0$. To this end, we first prove that the LTV system (15) with $z_x = 0$, *i.e.*

$$\dot{z}_\Theta = -\gamma_2 \begin{bmatrix} \xi_1^2 & \frac{\xi_1}{\gamma_1 k_1} \\ \frac{\xi_1}{\gamma_1 k_1} & \frac{1}{\gamma_1^2 k_1^2} \end{bmatrix} = -\gamma_2 F(t) z_\Theta, \qquad (16)$$

is uniformly globally asymptotically stable (UGES). To this end, since at steady state $\xi_1(t) = c_0 + c_1 \sin(w_1 t + c_2)$ for some constants $c_i$, with $c_1 \neq 0$, the vector $R(\xi)$ is persistently exciting (PE), *i.e.* there exist $\delta_t > 0$, $\sigma_1 > 0$, and $\sigma_2 > 0$ such that, for all $t_0 \geq 0$,

$$\sigma_1 I \leq \int_{t_0}^{t_0+\delta_t} R(\xi_1(\tau))^\top R(\xi_1(\tau)) d\tau \leq \sigma_2 I.$$

This yields (see [5]) that the origin $z_\Theta = 0$ of the system (16) is UGES. Moreover, since $\xi_1(t)$ is bounded and continuous, so is $F(t)$ and by [12, Theorem 4.12] there exists a positive definite, symmetric, smooth and bounded matrix $P(t)$ solution of the time varying Lyapunov equation

$$-\dot{P}(t) = P(t)F(t) + F(t)^\top P(t) + Q(t),$$

with $Q(t)$ any continuous, bounded, positive definite, and symmetric matrix. This implies the existence of a Lyapunov function $V(t, z_\Theta) = z_\Theta^\top P(t) z_\Theta$ and proves that system (15) is ISS with respect to $z_x$. Then, exponential convergence to zero of the estimation error $z = [z_x, z_\Theta]$ trivially follows from the exponential convergence to zero of $z_x$ for any strictly positive constants $k_1$ and $\gamma_i$.

As a result, claims similar to those in Proposition 1 hold with the $\xi$ dynamics as in equation (14) and

$$\hat{\Theta} = \gamma_1^{-1} \left( \gamma_2 R(\xi_1)^\top y + \xi_2 \right).$$

## III. The general case

Consider now the case in which the measured signal is given by 1 with[1] $E_0 = 0$, known $n \geq 2$, unknown angular frequencies $w_i > 0$, amplitudes $E_i$ and phases $\phi_i$. This signal can be regarded as the output $y$ of a time invariant neutrally stable linear system in observer canonical form described by the equations

$$\bar{\Sigma} : \left\{ \begin{array}{rcl} \dot{\eta} & = & \bar{A}\eta, \\ y & = & \bar{C}\eta, \end{array} \right.$$

with $y \in \mathbb{R}$, $\eta \in \mathbb{R}^{2n}$,

$$\bar{A} = \left[ \begin{array}{c|ccccc} 0 & 1 & 0 & \dots & 0 \\ \hline -\theta_1 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \dots & \dots & \dots & \vdots \\ 0 & \dots & \dots & 0 & 1 \\ -\theta_n & 0 & \dots & 0 & 0 \end{array} \right] \in \mathbb{R}^{2n \times 2n},$$

$$\bar{C} = \left[ \begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \end{array} \right] \in \mathbb{R}^{1 \times 2n},$$

[1] We present the result for an unbiased signal since the case $E_0 \neq 0$ can be easily derived mimicking the case $n = 1$.

and where the unknown parameters $\theta_i$, $i = 1, .., n$, are related to the angular frequencies $w_i$ by the relation

$$\prod_{k=1}^{n} \left( s^2 + \omega_k^2 \right) = s^{2n} + \theta_1 s^{2(n-1)} + \cdots + \theta_n. \qquad (17)$$

System $\bar{\Sigma}$ can be written, in compact form, as

$$\Sigma : \left\{ \begin{array}{rcl} \dot{y} & = & Cx, \\ \dot{x} & = & Ax - M\Theta y, \end{array} \right. \qquad (18)$$

with $x \in \mathbb{R}^{2n-1}$, $A \in \mathbb{R}^{(2n-1)\times(2n-1)}$, $C \in \mathbb{R}^{1\times(2n-1)}$, and where $C$ and $A$ are obtained eliminating the last column and the first row and column from $\bar{C}$ and $\bar{A}$, respectively, $\Theta = [\theta_1, \dots, \theta_n]^\top \in \mathbb{R}^n$, and $M \in \mathbb{R}^{(2n-1)\times n}$ has non-zero elements $m_{2j-1,j} = 1$, for $j = 1, \dots, n$.

Note that the signal $y(t)$ and its derivatives $y^{(j)}(t)$, $j = 1, .., 2n-1$, are related to $x(t)$ by

$$x(t) = T_\Theta Y(t), \qquad (19)$$

where $T_\Theta \in \mathbb{R}^{(2n-1)\times 2n}$ is defined as

$$T_\Theta = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ \theta_1 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & \theta_1 & 0 & 1 & 0 & \dots & 0 \\ \theta_2 & 0 & \theta_1 & 0 & 1 & \dots & 0 \\ 0 & \theta_2 & 0 & \theta_1 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \ddots & \vdots \\ 0 & \theta_{n-1} & 0 & \theta_{n-2} & \dots & 0 & 1 \end{bmatrix}$$

and

$$Y(t) = [y(t), y^{(1)}(t), y^{(2)}(t), \dots, y^{(2n-1)}(t)]^\top \in \mathbb{R}^{2n}.$$

As in the previous section, define the estimation error $z = [z_x^\top, z_\theta^\top]^\top \in \mathbb{R}^{3n-1}$, as $z = \beta(y, \xi) - h(\xi, x, \Theta)$, with $\xi = [\xi_1^\top \ \xi_2^\top]^\top \in \mathbb{R}^{3n-1}$, and

$$\begin{array}{rcl} \beta(y, \xi) & = & \left[ \begin{array}{c} Ky \\ \gamma_2 (CR(\xi_1))^\top y + \xi_2 \end{array} \right] \\ h(\xi, x, \Theta) & = & \left[ \begin{array}{c} x - \gamma_1 (G(\xi_1) + R(\xi_1)\Theta) \\ \gamma_1 \Theta \end{array} \right], \end{array}$$

yielding

$$\begin{aligned} z_x &= Ky + \gamma_1 (G(\xi_1) + R(\xi_1)\Theta) - x, \\ z_\theta &= \gamma_2 (CR(\xi_1))^\top y + \xi_2 - \gamma_1 \Theta, \end{aligned}$$

with $K = [k_1, \dots, k_{2n-1}]^\top \in \mathbb{R}^{2n-1}$, where the $k_i$'s are the coefficients of the Hurwitz polynomial

$$s^{2n-1} + k_1 s^{2n-2} + \cdots + k_{2n-1},$$

$\gamma_i > 0$ for all $i$, and the column vector $G(\cdot) \in \mathbb{R}^{2n-1}$ and the matrix $R(\cdot) \in \mathbb{R}^{2n-1 \times n}$ have entries given by

$$G_i(\xi_1) = \left\{ \begin{array}{ll} -\sum_{j=1}^{2n-1} \xi_{1,j} k_{2n-j} k_i & \text{if } i \geq j, \\ \sum_{j=1}^{2n-1} \xi_{1,j} (k_{i-j+2n} - k_i k_{2n-j}) & \text{if } i < j, \end{array} \right. \qquad (20)$$

$$R_{ij}(\xi_1) = \left\{ \begin{array}{ll} -\sum_{h=1}^{i} k_{i-h} \xi_{1,p+h} & \text{if } i < 2j, \\ \sum_{h=0}^{2n-1-i} k_{i+h} \xi_{1,p-h} & \text{if } i \geq 2j, \end{array} \right. \qquad (21)$$

where $p = 2(n-j)$, $k_0 = 1$, and $\xi_{1,j}$ stands for the $j$-th element of the vector $\xi_1$. With these definitions we are able to state the following result.

*Proposition 2:* Consider the signal (1) with unknown $\omega_i \geq 0$, for $i = 1, \cdots, n$, and unknown $E_i$ and $\phi_i$, for $i = 1, \cdots, n$ with $E_0 = 0$. Consider the dynamical system

$$\dot{\xi}_1 = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \dots & \dots & \vdots \\ 0 & \dots & \dots & 1 \\ -k_{2n-1} & -k_{2n-2} & \dots & -k_1 \end{bmatrix} \xi_1 + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \gamma_1^{-1} \end{bmatrix} y, \quad (22)$$

$$\begin{aligned} \dot{\xi}_2 = &-\gamma_2 R(\xi_1)^\top C^\top C \Big( R(\xi_1)(\gamma_2 y R(\xi_1)^\top C^\top + \xi_2) \\ &+ Ky + G(\xi_1) \Big) - \gamma_2 y \dot{R}(\xi_1)^\top C^\top \end{aligned} \quad (23)$$

with input $y$ and output

$$\begin{aligned} \hat{\Theta} &= \tfrac{1}{\gamma_1}\left(\gamma_2 y R(\xi_1)^\top C^\top + \xi_2\right), \\ \hat{x} &= Ky + \gamma_1 \left(G(\xi_1) + R(\xi_1)\hat{\Theta}\right), \end{aligned} \quad (24)$$

where $G(\xi_1)$ and $R(\xi_1)$ are defined as in equations (20) and (21), respectively.

Then $\xi(t) \in \mathcal{L}_\infty$,

$$\hat{\Theta}(t) - \Theta \in \mathcal{L}_\infty,$$

$$\lim_{t\to+\infty} \hat{x}(t) - T_\Theta Y(t) = 0$$

exponentially, with $T_\Theta$ as in equation (19),

$$C^\top C R(\xi_1(t)) z_\Theta(t) \in \mathcal{L}_2$$

and

$$\lim_{t\to\infty} C^\top C R(\xi_1(t)) z_\Theta(t) = 0. \quad (25)$$

In addition, if $\omega_i \neq \omega_j$, for all $i \neq j$, $\omega_i > 0$, for all $i$, and $E_i \neq 0$, for all $i$, then

$$\lim_{t\to+\infty} \hat{\Theta}(t) = \Theta,$$

exponentially. □

*Proof:* To begin with note that the time derivative of the variable $z$ is given by

$$\begin{aligned} \dot{z}_x &= (A - KC)z_x + \Delta_1(y,\xi_1) z_\theta + \Delta_2(y,\xi_1), \\ \dot{z}_\theta &= -\gamma_2 R(\xi_1)^\top C^\top C z_x + \Delta_3(y,\xi) + \dot{\xi}_2 + \\ &\quad -\gamma_2 R(\xi_1)^\top C^\top C R(\xi_1) z_\theta, \end{aligned} \quad (26)$$

where

$$\begin{aligned} \Delta_1(y,\xi_1) &= \frac{1}{\gamma_1}\Big(\gamma_1(KC - A)R(\xi_1) + \gamma_1 \dot{R}(\xi_1) + yM\Big), \\ \Delta_2(y,\xi) &= \Delta_1(y,\xi_1)\Big(\gamma_2 y (CR(\xi_1))^\top + \xi_2\Big) + \\ &\quad (KC - A)(Ky + \gamma_1 G(\xi_1)) + \gamma_1 \dot{G}(\xi_1), \quad (27) \\ \Delta_3(y,\xi) &= \gamma_2 R(\xi_1)^\top C^\top C \Big( R(\xi_1)(\gamma_2 y R(\xi_1)^\top C^\top + \xi_2) \\ &\quad + Ky + G(\xi_1) \Big) + \gamma_2 y \dot{R}(\xi_1)^\top C^\top. \end{aligned}$$

The selection of $R(\xi_1)$ and $G(\xi_1)$ in (20) and (21), respectively, is such that

$$\begin{aligned} (KC - A)R(\xi_1) + \dot{R}(\xi_1) + \frac{y}{\gamma_1} M &= 0, \\ (KC - A)G(\xi_1) + \dot{G}(\xi_1) + \frac{y}{\gamma_1}(KC - A)K &= 0, \end{aligned}$$

hence $\Delta_1(y,\xi_1) = \Delta_2(y,\xi) = 0$. This, together with the selection of $\dot{\xi}_2$ given in equation (23), yields

$$\dot{z} = F(t) z, \quad (28)$$

where

$$F(t) = \begin{bmatrix} A - KC & 0 \\ -\gamma_2 R(\xi_1)^\top C^\top C & -\gamma_2 R(\xi_1)^\top C^\top C R(\xi_1) \end{bmatrix}. \quad (29)$$

Note now that, as in the previous section, the system (28) has a cascaded structure.

To prove the first claims, consider the Lyapunov function

$$V(z_x, z_\Theta) = \frac{1}{2} z_x^\top P z_x + \frac{1}{2\gamma_2} z_\Theta^\top z_\Theta,$$

with $P$ positive definite and such that

$$(A - KC)^\top P + P(A - KC) = -I,$$

and note that, along the trajectories of (28),

$$\dot{V} \leq -\frac{1}{2}\|z_x\|^2 - \frac{1}{2}\|C^\top C R(\xi_1) z_\Theta\|^2,$$

where we have used the fact $C^\top C = (C^\top C)(C^\top C)$. As a result, $z_x(t) \in \mathcal{L}_\infty$, $z_\Theta(t) \in \mathcal{L}_\infty$ and $C^\top C R(\xi_1(t)) z_\Theta(t) \in \mathcal{L}_2$, and by Barbalat's Lemma

$$\lim_{t\to\infty} C^\top C R(\xi_1(t)) z_\Theta(t) = 0.$$

Moreover, by the cascaded structure of system (28) and stability of the matrix $A - KC$ we infer that $\lim_{t\to\infty} z_x(t) = 0$, exponentially, hence $\lim_{t\to\infty} \hat{x}(t) - T_\Theta Y(t) = 0$, exponentially. Finally, $\xi_1$ is the state of an asymptotically stable linear system with bounded input, hence $\xi_1(t) \in \mathcal{L}_\infty$ and from the definition of $z_\Theta$ we conclude that $\xi_2(t) \in \mathcal{L}_\infty$ and $\hat{\Theta}(t) \in \mathcal{L}_\infty$, which completes the proof of the first claims. To prove the second claim note that $CR(\xi_1(t)) = -S\xi_1(t)$, where each element of $S$ is such that $S_{i,2n-i} = 1$, zero otherwise, *i.e.* $CR(\xi_1(t))$ is a vector with the components the states (with inverted order with respect to $\xi_1(t)$) of the filter (22), that can be also written, in terms of Laplace transforms, as

$$CR(\xi_1(s)) = \frac{1}{s^{2n-1} + \cdots + k_{2n-2}s + k_{2n-1}} \begin{bmatrix} s^{2(n-1)} \\ \vdots \\ s^2 \\ 1 \end{bmatrix} y(s).$$

As a result, if $\omega_i \neq \omega_j$, for all $i \neq j$, $\omega_i > 0$, for all $i$, and $E_i \neq 0$, for all $i$, by [5, Lemma 2.6.7], $CR(\xi_1(t))$ is PE. This implies, by [5, Theorem 2.5.1], that the equilibrium $z_\theta = 0$ of the system

$$\dot{z}_\Theta = -R(\xi_1)^\top C^\top C R(\xi_1) z_\Theta \quad (30)$$

is UGES. Finally, by [12, Theorem 4.12], the equilibrium $z = 0$ of system (28) is UGES, which proves the claim. ■

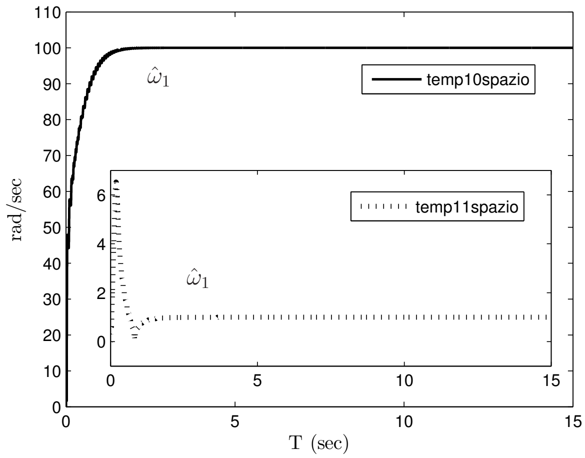


Fig. 1. Estimation of two single-frequency signals.

*Remark 1:* We stress that the zero equilibrium of system (28) is globally stable even if $CR(\xi_1(t))$ is not PE. This implies that even if some of the $E_i$'s are zero or $\omega_i = \omega_j$ for some $i$ and $j$, then it is possible to partially estimate the angular frequencies associated with nonzero $E_i$ and to estimate a subset of the angular frequencies $\omega_i$, as implied by equation (25).

*Remark 2:* When $n$ is large the effectiveness of the proposed on-line method, and of all other available on-line methods, may be compromised since the estimates of the $\omega_i$'s have to be evaluated by means of numeric procedures to find the zeros of a polynomial.

*Remark 3:* Consider the case in which the system generating the signal $y(t)$ is affected by additive bounded disturbances $(v_1(t), v_2(t))$, *i.e.* it is described by equations of the form

$$\begin{array}{rcl} \dot{y} &=& Cx + v_1, \\ \dot{x} &=& Ax - M\theta\, y + v_2. \end{array} \tag{31}$$

Defining the variable $z$ as in the previous section, yields

$$\dot{z} = F(t)z + v(t), \tag{32}$$

where

$$v(t) = \begin{bmatrix} Kv_1 - v_2 \\ \gamma_2 R(\xi_1)^\top C^\top v_1 \end{bmatrix}$$

and the matrix $F(t)$ is as in equation (29). If the signals $v_i(t)$ are such that all solutions of (31) exist and are bounded for all time, and if the resulting signal $CR(\xi_1)$ is PE (this is the case if $v(t)$ is sufficiently small) then [12, Theorem 4.12] implies input-to-state-stability of system (32) with respect to $v(t)$. This implies that approximate frequency estimation can be achieved even in the presence of a class of (small) perturbations.

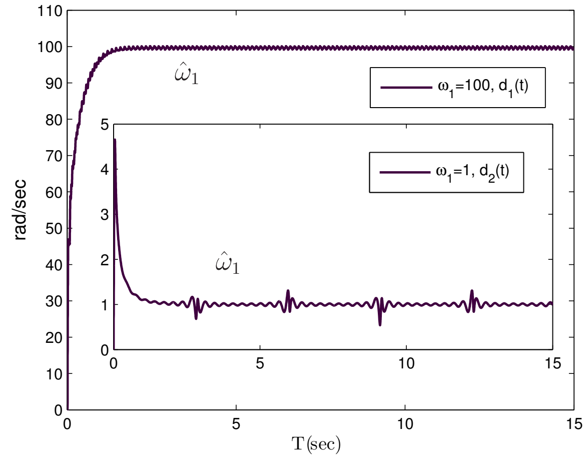


Fig. 2. Estimation of single-frequency signals with noise.

## IV. NUMERICAL EXAMPLES

In the first example we show the observer effectiveness by choosing two signals with $n = 1$ and with sensibly different frequencies, namely $y_1(t) = \sin(100t)$ and $y_2(t) = \sin(t)$. The observer dynamics are chosen as in Section II, with observer gains $[\gamma_1,\ \gamma_2] = [0.01, 10]$, and $k_1 = 2$. The initial values $\xi_i(0)$ are set to zero. Figure 1 displays the frequency estimates for the signals $y_1(t)$ and $y_2(t)$, whereas Figure 2 shows simulations results in the presence of the disturbances $v_1 = A_d \sin(50t)$ and $v_2 = 0$. It can be seen that in the case $\omega_1 = 100$ the estimation is not affected by the noise $v_1(t)$ with $A_d = 0.05$, whereas for $\omega_1 = 1$ a noise $v_1(t)$ with $A_d = 0.001$ leads to a sensible degradation of the estimate, mainly due to the low value of $\omega_1 = 1$ (better estimate can be obtained decreasing $\gamma_2$).

The observer of Section III is implemented in the case $n = 2$, for the signal $y(t) = \sin(2t) + \sin(5t)$, as in [7]. The estimates of the vector $\Theta$ and of the state $x$ are defined as in equation (24), and $R(\xi_1)$ and $G(\xi_1)$ are selected according to equations (21) and (20), namely

$$R(\xi_1) = \begin{bmatrix} -\xi_{1,3} & -\xi_{1,1} \\ k_3\xi_{1,1} + k_2\xi_{1,2} & -k_1\xi_{1,1} - \xi_{1,2} \\ k_3\xi_{1,2} & -k_2\xi_{1,1} - k_1\,\xi_{1,2} - \xi_{1,3} \end{bmatrix},$$

$$G(\xi_1) = \begin{bmatrix} -k_3k_1\xi_{1,1} + (k_3 - k_2k_1)\xi_{1,2} + (k_2 - k_1^2)\xi_{1,3} \\ -k_3k_2\xi_{1,1} - k_2^2\xi_{1,2} + (k_3 - k_2k_1)\xi_{1,3} \\ -k_3^2\xi_{1,1} - k_3k_2\xi_{1,2} - k_3k_1\xi_{1,3} \end{bmatrix}.$$

The derivatives $\dot{\xi}_1$ and $\dot{\xi}_2$ are defined as in equation (23), and $\xi_1(0)$ and $\xi_2(0)$ have been set to zero. In the simulation results, shown in Figure 3, the vector $K$ has been chosen as $K = [k_1,\ k_2,\ k_3] = [6, 12, 8]$ yielding the polynomial $k_3 + k_2\lambda + k_1\lambda^2 + \lambda^3 = (\lambda + 2)^3$, and $\gamma_1 = 0.01$ and $\gamma_2 = 10$ as in the previous example. The unknown frequencies $w_1$ and $w_2$ are evaluated from $\hat{\theta}_1$ and $\hat{\theta}_2$ using the equation

$$\hat{w}_{1,2} = \sqrt{\frac{\left|\hat{\theta}_1 \pm \sqrt{\left|\hat{\theta}_1^2 - 4\hat{\theta}_2\right|}\right|}{2}}.$$

The same observer has been applied to the signal $y(t) = 2\sin(t) + 4\sin(1.2t)$ (as in [9]). The peculiarity of this signal stands in the closeness of the two frequencies $\omega_1$ and $\omega_2$, resulting in a weaker persistency of excitation property, that may lead to slower convergence of the estimates (see the adaptive identifier simulation results in [9]). However, as illustrated in Figure 4, even in this case the estimates converge to the correct values, with a good convergence rate. Finally, the same observer is tested on a single frequency signal, namely $y(t) = 2\sin(3t + \pi/8)$, that could be re-interpreted as the signal (1) with $n = 2$ and $E_0 = E_2 = 0$, *i.e.*, an overestimation of the frequencies of $y(t)$. In this case, illustrated in Figure 5, the frequency $\omega_1 = 3$ is correctly estimated by $\hat{\omega}_1$, while $\hat{\omega}_2$ remains bounded.

## V. CONCLUSIONS

We have presented a minimal dimension observer for the global estimation of $n$ unknown frequencies. The asymptotic properties and the robustness of the proposed observer have been proved by means of classical analysis tools. Simulations results show the effectiveness of the observer.

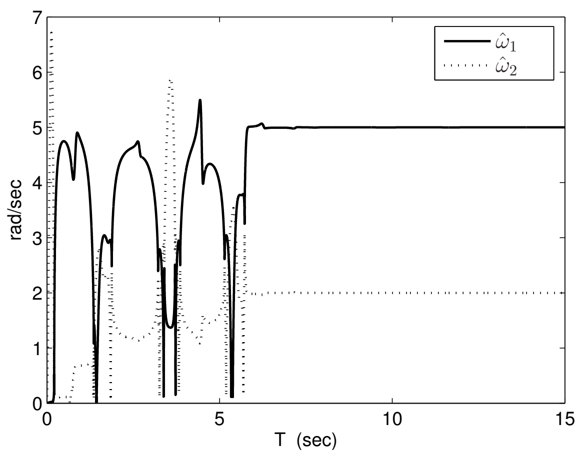


Fig. 3. Frequencies estimation for the signal $y(t) = \sin(2t) + \sin(5t)$.

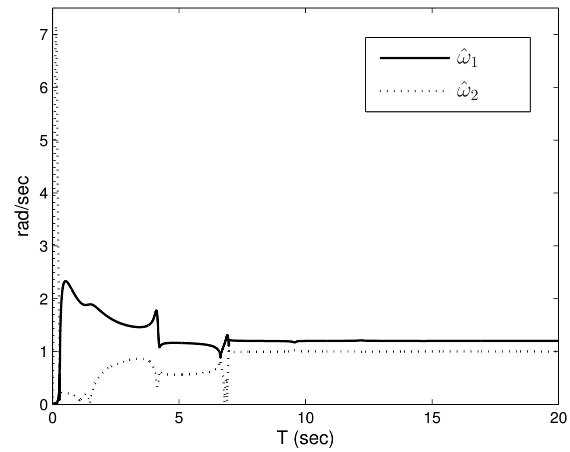


Fig. 4. Frequencies estimation for the signal $y(t) = 2\sin(t)+4\sin(1.2t)$.

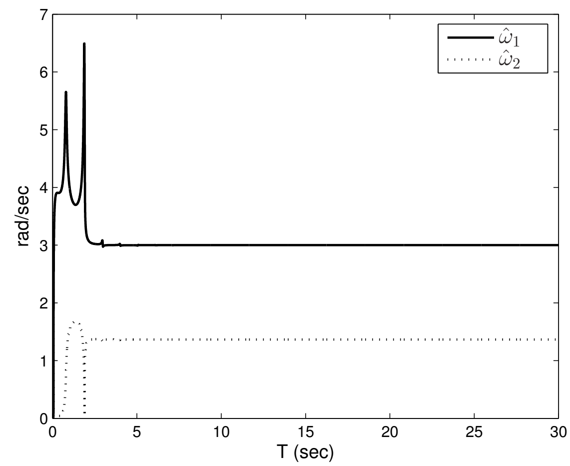


Fig. 5. Frequency estimation when $n = 2$ and $E_2 = 0$.

## REFERENCES

[1] S. M. Kay and S. L. Marple, "Spectrum analysis – A modern perspective", *Proc. IEEE*, vol. 69, pp. 1380–1419, Nov. 1981.

[2] P. Regalia, "An improved lattice-based adaptive IIR notch filter", *IEEE Trans. Signal Processing*, vol. 39, pp. 2124–2128, Sept. 1991.

[3] C. Centioli, F. Iannone, G. Mazza, M. Panella, L. Pangione, S. Podda, A. Tuccillo, V. Vitale, and L. Zaccarian,"Extremum seeking applied to the plasma control system of the Frascati Tokamak Upgrade" in *Proc. 44th IEEE Conf. Decision and Control, Seville, Spain*, 2005, pp. 8227-8232.

[4] L. Hsu, R. Ortega, and G. Damm, "A globally convergent frequency estimator" *IEEE Trans. Autom. Control*, vol. 44, no. 4, pp. 698–713, 1999.

[5] S. S. Sastry and M. Bodson, "Adaptive Control: Stability, Convergence and Robustness", Upper Saddle River, NJ, Prentice–Hall, 1989.

[6] X. Xia, "Global frequency estimation using adaptive identifiers," *IEEE Trans. Autom. Control*, vol. 47, no. 7, pp. 1188–1191, 2002.

[7] R. Marino and P. Tomei,"Global estimation of $n$ unknown frequencies", *IEEE Trans. Autom. Control*, vol. 47, no. 8, pp. 1324–1328, 2002.

[8] R. Marino, G. L. Santosuosso and P. Tomei,"Robust adaptive compensation of biased sinusoidal disturbances with unknown frequency", *Automatica*, vol. 39, pp. 1755–1761, 2003.

[9] M. Hou,"Estimation of sinusoidal frequencies and amplitude using adaptive identifier and observer", *IEEE Trans. Autom. Control*, vol. 52, no. 3, pp. 493–499, 2007.

[10] G. Obregón-Pulido, B. Castillo-Toledo, and A.Loukianov, "A globally convergent estimator for $n$-frequencies ", *IEEE Trans. Autom. Control*, vol. 47, no. 5, pp. 857–863, 2002.

[11] S. Bittanti and S. M. Savaresi, "On the parameterization and design of an extended kalman filter frequency tracker", *IEEE Trans. Autom. Control*, vol. 45, no. 9, pp. 1718–1715, 2000.

[12] H. K. Khalil, "Nonlinear Systems", 3rd ed., Upper Saddle River, NJ:Prentice–Hall, 2002.

[13] A. Astolfi and R. Ortega, "Immersion and invariance: a new tool for stabilization and adaptive control of nonlinear systems", *IEEE Trans. Autom. Control*, vol. 48, no. 4, pp. 590–606, 2002.

[14] D. Karagiannis and A. Astolfi, "Nonlinear observer design using invariant manifolds and applications," in *Proc. 44th IEEE Conf. Decision and Control and European Control Conference, Seville, Spain*, 2005, pp. 7775–7780.

[15] D. Carnevale, D. Karagiannis, and A. Astolfi, "Reduced-order observer design for systems with non-monotonic nonlinearities", in *Proc. 45th IEEE Conf. Decision and Control, San Diego, California*, 2006, pp. 5269–5274.

[16] D. Carnevale, D. Karagiannis, and A. Astolfi, "Reduced-order observer design for nonlinear systems" in *Proc. ECC, Kos, Greece*, 2007, pp. 559-564.