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Sector conditions for the global stability of high-speed communication networks

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Abstract— **This paper considers the design of TCP-like congestion control protocols for deployment in high-speed communication networks. A basic problem in this area is to design congestion control strategies that probe more aggressively than standard TCP, but which co-exist with each other and result in globally stable and equitable network behaviour. Under the assumption of player synchronisation, conditions for global network stability are derived in the form of growth conditions on the local nonlinear probing functions.**

I. INTRODUCTION

Traffic generated by the *Transmission Control Protocol* (TCP) accounts for 85% to 95% of all traffic in today's internet [2]. TCP, in congestion avoidance mode, is based primarily on the Chiu and Jain's [3] *Additive-Increase Multiplicative-Decrease* (AIMD) paradigm for decentralized allocation of a shared resource (e.g., bandwidth) among competing users. Recently, in the context of designing high speed communication networks, several authors have suggested basic modifications to the AIMD algorithm; for example, see [4], [7]. One idea underlying these modifications is to replace the constant growth rate of window size in standard TCP with an increasing rate. The rationale for this change is that such protocols probe more agressively for available bandwidth as network capacity increases. These algorithms, which we refer to as nonlinear AIMD (NAIMD), result in networks with different dynamic properties than those employing the basic (linear) AIMD; see [5]. A basic question in the design of NAIMD networks is how to choose the probing action so that the resulting network exhibits desirable properties. Remarkably, despite increasing deployment of these algorithms (e.g., a high-speed TCP algorithm is implemented as part of the Linux operating system), little work has been carried out in this area and basic questions concerning the existence and nature of network equilibria have yet to be addressed.

Our objective in this paper is to study basic convergence and stability properties of a class of NAIMD congestion control protocols. In this preliminary study, we summarise recent findings in [1]. We restrict attention to deterministic networks in which all sources (players) are informed of network congestion simultaneously, and where all sources employ NAIMD protocols whose growth rates depend only on their most recent congestion window size. The study of synchronized networks is in general restrictive, however it is important for a number of reasons. Firstly, it has been observed by many researchers that source synchronisation is a feature of high-speed networks, and in the context of such networks, the assumption of synchronisation is not overly conservative. Secondly, the study of synchronised networks represents an important first step toward the study of more realistic network types in which sources are informed of congestion asynchronously.

Our basic finding in this paper is that NAIMD protocols are stable and lead to a reasonably equitable allocation of resources as long as the growth rate functions of all users in the network increase more slowly than linearly with window size.

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II. BACKGROUND

Throughout, we use notation $\mathbb{R}$ for the real numbers, $\mathbb{R}^n$ for the n-dimensional real Euclidean space and $\mathbb{R}^{n\times n}$ for the space of $n \times n$ matrices with real entries.

(a) The AIMD algorithm : The AIMD algorithm of Chiu and Jain [3] is a decentralised resource allocation algorithm. The algorithm consists of two operational modes. The first mode, a backoff mode, is invoked whenever a source (player) is informed that the network is congested. Users respond to this notification by down-scaling their utilization-rate in a multiplicative manner. This mode is called the *Multiplicative Decrease* (MD) phase. After reducing their utilisation rates, users probe for available capacity until congestion is reached again, at which point the mode of the next cycle is entered. This second mode of operation is called the *Additive Increase* (AI) phase.

(b) A model of NAIMD : The evolution of the network states over one cycle of operation may be conveniently written as follows. Let $w(k) = [w_1(k), \ldots, w_n(k)]^T$ denote the share of network capacity allocated to each user at the time of the k^{th} congestion event $t(k)$. The capacity constraint requires that $\sum_{i=1}^n w_i(k) = C$, with C as the total capacity of the resource available to the entire system. After the next cycle of multiplicative decrease and additive increase, the utilization-rate of player i becomes

$$w_i(k+1) = \beta_i w_i(k) + a_i(w_i(k))\,(t(k+1) - t(k)), \qquad (1)$$

where β_i is a constant in the open interval $(0,1)$, and $a_i(\cdot)$ is the growth rate function of user i. It follows that the time between congestion events $t(k+1) - t(k)$ is a function of

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$w(k)$, that is $t(k+1)-t(k) = T(w(k))$ where

$$T(w(k)) = \frac{C - \sum_{j=1}^{n} \beta_j w_j(k)}{\sum_{j=1}^{n} a_j(w_j(k))} = \frac{\sum_{j=1}^{n} (1-\beta_j) w_j(k)}{\sum_{j=1}^{n} a_j(w_j(k))}. \tag{2}$$

(c) Growth assumptions: The key assumptions concern the growth rate functions $a_i(\cdot)$. We will always assume that $a_i(x)$ is defined and positive for all $x > 0$, and is continuous and non-decreasing. In addition, and by a slight abuse of a conventional definition, we will say that a_i is *sector bounded* if the function $f_i(x)$ is strictly increasing, where

$$f_i(x) = \frac{x}{a_i(x)} \tag{3}$$

In the case where a_i is differentiable, the non-decreasing and sector bounded properties can be expressed compactly as the inequalities

$$0 \leq \frac{d \log a_i(x)}{d \log x} < 1 \tag{4}$$

(d) Matrix formulation: We define a new set of network states $z(k) = [z_1(k), \ldots, z_n(k)]^T$ as follows:

$$z_i(k) = (1-\beta_i) \frac{w_i(k)}{a_i(w_i(k))} \tag{5}$$

In terms of these variables the network dynamics (1) becomes

$$z_i(k+1) = \frac{a_i(w_i(k))}{a_i(w_i(k+1))} \times \Big(\beta_i z_i(k) + (1-\beta_i) \sum_{j=1}^{n} z_j(k) p_j(w(k))\Big)$$

where $p_j(w(k))$=$\Big(\sum_{l=1}^{n} a_l(w_l(k))\Big)^{-1} a_j(w_j(k))$. Gathering all i we get the vector equation

$$z(k+1) = R(k) z(k) \tag{6}$$

where the matrix $R(k)$ is defined by

$$R(k)_{ij} = \frac{a_i(w_i(k))}{a_i(w_i(k+1))} \Big(\beta_i \delta_{ij} + (1-\beta_i)\, p_j(w(k))\Big) \tag{7}$$

The second part of this expression $\Big(\beta_i \delta_{ij} + (1 - \beta_i)\, p_j(w(k))\Big)$ defines a row stochastic matrix. As is well-known, a row stochastic matrix is a contraction on vectors in the positive orthant $\mathbb{R}_+^n$ which are orthogonal to $[1, \ldots, 1]^T$. As the following Lemma shows, $R(k)$ asymptotically shares this property in the limit $k \to \infty$. We write $z_{\max}$ and $z_{\min}$ to denote the largest and smallest components of the vector z.

Lemma 2.1: Consider the dynamic system defined in Equation (6), and suppose that all growth functions $a_i(x)$ are sector bounded. Then

(i) $z_{\max}(k+1) \leq z_{\max}(k)$, $z_{\min}(k+1) \geq z_{\min}(k)$ for all $k \geq 0$

(ii) there is $r < 1$ and a sequence η_k converging to zero, such that for all $k \geq 0$

$$z_{\max}(k+1) - z_{\min}(k+1) \leq r\Big(z_{\max}(k) - z_{\min}(k)\Big) + \eta_k \tag{8}$$

Lemma 2.1 is proved in [1]. The main idea of the proof is to show that for large k the ratio $\frac{a_i(w_i(k))}{a_i(w_i(k+1))}$ in (7) becomes close to one, in which case $R(k)$ is approximately row stochastic, leading to the contractive behavior in (8). We will use this Lemma in the proof of our main result to show that the nonlinear system (6) behaves essentially as a linear system with row stochastic matrices, and that the solution $z(k)$ is driven to the fixed point $z = [1, \ldots, 1]^T$.

III. Main result

Referring to (1), a *fixed point* of the network is defined to be a vector $w^* \in \tilde{\Sigma}$ satisfying $w_i^* = \beta_i w_i^* + a_i(w_i^*) T(w^*)$ and $\sum w_i^* = C$, where

$$\tilde{\Sigma} = \{w \in \mathbb{R}^n \, : \, w_i > 0 \text{ for } i = 1, \ldots, n\} \tag{9}$$

Theorem 3.1: Consider the network of NAIMD sources described in the preamble. Then the following properties hold.

(i) If all $a_i(\cdot)$ are sector bounded, then for all $C > 0$ the network has a unique fixed point which is globally asymptotically stable.

(ii) Suppose that a_i is differentiable for all i. If at least two of the $a_i(\cdot)$ are not sector bounded, then any fixed point is unstable.

Idea of the proof: the full proof is presented in [1]. Here we summarise the ideas: for Part (i) of the Theorem, the existence of a fixed point follows by using the condition $\sum_{j=1}^{n} w_j^* = C$ to derive an equation for $T(w^*)$, and then using the assumption that the functions f_i are all increasing and continuous to show that the equation has a unique solution. The convergence to the fixed point follows from Lemma 2.1, which implies that $z(k)$ converges to the fixed point $(1, \ldots, 1)^T$, and hence that $w(k)$ converges to w^*. Part (ii) of the Theorem follows from a linear analysis around the fixed point.

IV. Conclusions

In this paper we have studied stability properties of heterogeneous networks in which different users implement different versions of nonlinear AIMD algorithms and with different levels of aggressiveness. Conditions for global stability are given in the form of sector conditions on the network growth functions a_i.

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