2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Finite-Time Stabilization of Port-Controlled Hamiltonian Systems with Application to Nonlinear Affine Systems

Yuzhen Wang and Gang Feng

Abstract— **The finite-time stabilization of nonlinear Port-Controlled Hamiltonian (PCH) systems is investigated in this paper, and a number of approaches to the finite-time control design are proposed. Based on a finite-time stability criterion and the so-called "energy shaping plus damping injection" technique, the continuous finite-time stabilization problem is studied for the PCH systems, and several global stabilization results are obtained. Via Hamiltonian realization, the results obtained for the Hamiltonian systems are applied to investigate continuous finite-time stabilization of nonlinear affine systems, and several global control design results are presented. Study on several examples shows that the control design approaches developed in this paper work very well.**

I. INTRODUCTION

The Port-Controlled Hamiltonian (PCH) system, proposed by [9], [17], has been well investigated in a series of recent works, see, e.g. [3], [4], [10], [13], [18], [20]. A new passivity-based control theory, known as the interconnection and damping assignment (IDA-PBC) methodology, was developed in [13] and then many stabilization results were obtained for both PCH systems and mechanical systems by using the new methodology [13], [14]. It is well worth pointing out that [14] proposed a new control design procedure, called "energy shaping plus damping injection", for the PCH system. This procedure is very important because it can not only provide the PCH system a better Hamiltonian formulation but also set up an easy way to the stabilization of the PCH system. It is noted that the Hamiltonian function, the sum of potential and kinetic energies in physical systems, is a good candidate of Lyapunov functions for many physical systems. Due to this and its nice structure with clear physical meaning, the PCH system has some distinctive advantages in control designs and has found its wide use in many practical control problems. Particularly, it has been successfully applied to the control of power systems [5], [12], [16], [21] and mechanical systems [14], [19], respectively.

It is also noted that, in general, an asymptotical stable controller cannot guarantee that the system under study achieves the control performance of fast convergence, and a finite-time controller often has to be designed in practice for some special control problems. As indicated in [8], a finite-time controller possesses not only fast convergence but also better robustness and disturbance attenuation properties. During the past two decades, the finite-time control problem has drawn an increasing attention and many results have been obtained for several classes of dynamic systems, see, e.g. [1], [2], [6]–[8], [11], [15]. Different versions of Lyapunov stability theorem were proposed in [1], [6] for analyzing the finite-time stability of nonlinear systems. These results are very important in the sense that they have provided a basic tool for the finite-time stability analysis and control design of nonlinear control systems. It should be pointed out that designing a continuous finite-time feedback controller is challenging because such a controller necessarily involves non-Lipschitz closed-loop dynamics [1]. As a result, there are few works on the continuous finite-time control design for general nonlinear systems. In particular, there are, to the authors' best knowledge, few results on the finite-time control of nonlinear Hamiltonian systems.

In this paper, we investigate the global finite-time stabilization of nonlinear PCH systems, and propose a number of approaches to their finite-time control design. Based on an obtained stability criterion and the so-called "energy shaping plus damping injection" technique [14], the continuous finite-time stabilization problem is studied for the PCH systems, and several global stabilization results are obtained. Via Hamiltonian realization, we apply the stability results obtained for the Hamiltonian systems to investigate continuous finite-time stabilization of nonlinear affine systems, and present several global control design results. Study on several examples shows that the control design approaches proposed for both the PCH systems and nonlinear affine systems in this paper work very well.

The remainder of the paper is organized as follows. Section II presents a result on the finite-time stability of a class of Hamiltonian systems. In Section III, the continuous finite-time stabilization problem is investigated for the PCH system. Section IV is the application to nonlinear affine systems, which is followed by the conclusion in Section V.

II. FINITE-TIME STABILITY OF A CLASS OF HAMILTONIAN SYSTEMS

This section studies the finite-time stability of a class of Hamiltonian systems, and presents a global stability criterion.

Consider a dissipative Hamiltonian system described as follows [10], [22]

$$\dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x}, \quad x(t_0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$, $J^T(x) = -J(x) \in \mathbb{R}^{n\times n}$, $0 \leqslant R(x) \in \mathbb{R}^{n\times n}$, and $H(x)$ is the Hamiltonian function with $x = 0$ as

This work is supported by the National Natural Science Foundation of China [G60774009], the Natural Science Foundation of Shandong Province of China [Y2006G10] and the Grant from the Research Grants Council of the Hong Kong Special Administrative Region of China [Project No.: CityU 112907].

Yuzhen Wang is with the School of Control Science and Engineering, Shandong University, P.R. China. yzwang@sdu.edu.cn.

Gang Feng is with the Department of Manufacturing Engineering and Engineering Management, City University of Hong Kong, Hong Kong. megfeng@cityu.edu.hk.

its minimum point.

To analyze the finite-time stability of the system (1), we give two lemmas first.

Lemma 1: (Jensen's inequality [7])

$$\Big(\sum_{i=1}^{n}|x_i|^{a_2}\Big)^{\frac{1}{a_2}} \leqslant \Big(\sum_{i=1}^{n}|x_i|^{a_1}\Big)^{\frac{1}{a_1}}, \quad 0 < a_1 \leqslant a_2, \qquad (2)$$

where a_1, a_2 and x_i, $i = 1, 2, ..., n$, are all real numbers.

In Lemma 1, let $a_2 = 1$ and $a_1 = \frac{1}{p}$, then we obtain the following inequality

$$\sum_{i=1}^{n}|x_i|^{\frac{1}{p}} \geqslant \Big(\sum_{i=1}^{n}|x_i|\Big)^{\frac{1}{p}}, \qquad p \geqslant 1. \qquad (3)$$

Lemma 2: ([1]) Consider a dynamic system

$$\dot{x} = f(x), \qquad f(0) = 0, \;\; x(t_0) = x_0, \;\; x \in \mathbb{R}^n. \qquad (4)$$

If there exist a real number $\beta > 1$ and a C^1 radially unbounded Lyapunov function, $V(x)$, of the system such that

$$\dot{V} \leqslant -kV^{\frac{1}{\beta}}(x(t)), \qquad k > 0 \qquad (5)$$

holds along the trajectories of the system starting from any $x_0 \in \mathbb{R}^n$, then the origin is a global finite-time stable equilibrium of the system (4). Furthermore, the settling time of the system (4) with respect to x_0 satisfies

$$T(x_0) \leqslant t_0 + \frac{\beta}{k(\beta-1)}V^{\frac{\beta-1}{\beta}}(x_0), \quad \forall\, x_0 \in \mathbb{R}^n. \qquad (6)$$

Now we are ready to investigate the global finite-time stability of the system (1). By studying the Hamiltonian structural properties of the system, a proper form of the Hamiltonian function is obtained (see (7) below), which leads to the following result.

Theorem 1: Consider the Hamiltonian system (1). If

(1) the Hamiltonian function is given as

$$H(x) = \sum_{i=1}^{n}\left(x_i^2\right)^{\frac{\alpha}{2\alpha-1}}, \qquad (7)$$

where $\alpha > 1$ is a real number, and

(2)

$$k := \min_{1\leqslant i\leqslant n}\left\{\inf_{x\in\mathbb{R}^n}\left\{\sigma_i^R(x)\right\}\right\} > 0, \qquad (8)$$

where $\sigma_i^R(x)$, $i = 1, 2, ..., n$, denote the eigenvalues of $R(x)$,

then the system (1) is globally finite-time stable at the origin. Furthermore, the settling time with respect to x_0 satisfies

$$T(x_0) \leqslant t_0 + \frac{(2\alpha-1)^2}{4k\alpha(\alpha-1)}H^{\frac{\alpha-1}{\alpha}}(x_0). \qquad (9)$$

Proof: It is easy to see that $H(x)$ is a C^1 radially unbounded positive definite function. For any $x_0 \in \mathbb{R}^n$, let $x(t) := x(t; t_0, x_0)$ be the trajectory of the system (1) starting from x_0. Choose $H(x)$ as a Lyapunov function candidate, then along $x(t)$ we obtain

$$\begin{aligned}\dot{H} &= \nabla^T H[J(x) - R(x)]\nabla H = -\nabla^T H R(x)\nabla H \\ &\leqslant -k\nabla^T H \cdot \nabla H = -k\Big(\frac{2\alpha}{2\alpha-1}\Big)^2 \sum_{i=1}^{n}\left(x_i^2\right)^{\frac{1}{2\alpha-1}}.\end{aligned}$$

Since $\alpha > 1$, it follows from (3) that

$$\begin{aligned}\dot{H} &\leqslant -k\Big(\frac{2\alpha}{2\alpha-1}\Big)^2 \sum_{i=1}^{n}\Big[\left(x_i^2\right)^{\frac{\alpha}{2\alpha-1}}\Big]^{\frac{1}{\alpha}} \\ &\leqslant -k\Big(\frac{2\alpha}{2\alpha-1}\Big)^2 \Big[\sum_{i=1}^{n}\left(x_i^2\right)^{\frac{\alpha}{2\alpha-1}}\Big]^{\frac{1}{\alpha}},\end{aligned}$$

that is,

$$\dot{H} \leqslant -k\Big(\frac{2\alpha}{2\alpha-1}\Big)^2 H^{\frac{1}{\alpha}}(x(t)). \qquad (10)$$

By Lemma 2, it can be concluded that the origin is a global finite-time stable equilibrium of the system (1) and moreover, the settling time $T(x_0)$ satisfies (9). □

Remark 1: When $R(x)$ is a constant positive definite matrix, Condition (8) holds naturally.

With Theorem 1, we have the following corollary.

Corollary 1: Consider the Hamiltonian system (1). If the Hamiltonian function is given as

$$H(x) = \sum_{i=1}^{n}|x_i|^{\beta}, \qquad 1 < \beta < 2 \qquad (11)$$

and meanwhile, the dissipative matrix $R(x)$ satisfies (8), then the system (1) is globally finite-time stable at the origin and moreover, the settling time with respect to x_0 satisfies

$$T(x_0) \leqslant t_0 + \frac{1}{k\beta(2-\beta)}H^{\frac{2-\beta}{\beta}}(x_0). \qquad (12)$$

Proof: It follows from (11) that

$$H(x) = \sum_{i=1}^{n}\left(x_i^2\right)^{\frac{\beta}{2}} = \sum_{i=1}^{n}\left(x_i^2\right)^{\frac{\alpha}{2\alpha-1}},$$

where $\alpha := \frac{\beta}{2\beta-2}$. Since $1 < \beta < 2$, it is easy to show that $\alpha > 1$. Thus, all the conditions of Theorem 1 are satisfied, and it can be concluded that the system (1) is globally finite-time stable at the origin. Moreover, substituting $\alpha = \frac{\beta}{2\beta-2}$ into (9), we can obtain (12). Therefore, the proof is completed. □

Remark 2: It is noted from the proof of Corollary 1 that the Hamiltonian function given in (11) can be rewritten as the form of (7). On the other hand, since $\alpha > 1 \Rightarrow 1 < \frac{2\alpha}{2\alpha-1} < 2$, the Hamiltonian function given in (7) can be also expressed as the form of (11). Thus, the form of the Hamiltonian function given in (7) is equivalent to that given in (11).

Remark 3: It is noted that the stability criterion proposed in Theorem 1 or Corollary 1 can reduce to the case of conventional asymptotical stability, as long as one takes $\alpha = 1$ in (7) or $\beta \geqslant 2$ in (11). In this sense, the stability results obtained in this paper can be regarded as an extension of some existing results on conventional asymptotical stability [22].

III. FINITE-TIME STABILIZATION OF PCH SYSTEMS

In this section, we utilize the results obtained in Section 2 to investigate continuous finite-time stabilization of nonlinear PCH systems, and propose several global stabilization results.

Consider the following PCH system [9], [17]

$$\dot{x}=[J(x)-R(x)]\frac{\partial H}{\partial x}+g(x)u, \quad x(t_0)=x_0, \tag{13}$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$, $J^T(x)=-J(x)\in\mathbb{R}^{n\times n}$, $0\leqslant R(x)\in\mathbb{R}^{n\times n}$, and $H(x)$ is the Hamiltonian function with $x=0$ as its minimum point.

Our aim is to design a continuous feedback law $u=u(x)$ such that the closed-loop system consisting of the system (13) and the control $u(x)$ is globally finite-time stable at the origin.

In the following, we use Theorem 1 and the "energy shaping plus damping injection" technique [14] to design the continuous finite-time control law. We will design a continuous feedback law $u=u(x)$ such that the closed-loop system can be expressed as

$$\dot{x}=[\bar{J}(x)-\bar{R}(x)]\frac{\partial \bar{H}}{\partial x}, \tag{14}$$

where $\bar{J}(x)$ is some skew-symmetric matrix, $\bar{R}(x)$ is a positive definite one with

$$k:=\min_{1\leqslant i\leqslant n}\left\{\inf_{x\in\mathbb{R}^n}\left\{\sigma_i^{\bar{R}}(x)\right\}\right\}>0, \tag{15}$$

$\sigma_i^{\bar{R}}(x)$, $i=1,2,...,n$, stand for the eigenvalues of $\bar{R}(x)$, and

$$\bar{H}=\sum_{i=1}^{n}\left(x_i^2\right)^{\frac{\alpha}{2\alpha-1}}, \quad \alpha>1. \tag{16}$$

Remark 4: In the above design, the Hamiltonian function (the system's total energy) is shaped from $H(x)\to\bar{H}(x)$, which is called "energy shaping", while the dissipative part of the structural matrix is altered from $R(x)\to\bar{R}(x):=R(x)+\Delta R(x)$, which is called "damping injection", where $\Delta R(x)$ can be regarded as the injected damping [14].

Let

$$H_a(x):=\bar{H}(x)-H(x), \tag{17}$$

and recall $g(x)\in\mathbb{R}^{n\times m}$. When $m<n$, without loss of generality, we assume that $g(x)$ has full column rank. In this case, to obtain the Hamiltonian structure given in (14), the control law $u(x)$ should be designed such that

$$g(x)u(x)=[J(x)-R(x)]\nabla H_a+[J_a(x)-R_a(x)]\nabla\bar{H}, \tag{18}$$

where $J_a(x)$ is a skew-symmetric matrix to be determined, $R_a(x)$ is a symmetric one to be determined such that $R_a(x)+R(x):=\bar{R}(x)$ satisfies (15), and $\nabla H:=\frac{\partial H}{\partial x}$. Since $g(x)$ is not invertible and only has full column rank, $u(x)$ can only influence the terms in the range space of $g(x)$. This leads to the following constraint equation

$$g^{\perp}(x)\Big([J(x)-R(x)]\nabla H_a+[J_a(x)-R_a(x)]\nabla\bar{H}\Big)=0 \tag{19}$$

for any choice of $u(x)$ such that (18) holds, where $g^{\perp}(x)$ is a full rank left annihilator satisfying $g^{\perp}(x)\cdot g(x)=0$.

Equation (19), called *the matching condition*, is a set of algebraic equations with respect to $J_a(x)$ and $R_a(x)$. For many PCH systems, especially some typical physical systems, their particular structure can help us obtain such a solution pair $(J_a(x),R_a(x))$.

If a solution pair $(J_a(x),R_a(x))$ of (19) is obtained, then $[J(x)-R(x)]\nabla H_a+[J_a(x)-R_a(x)]\nabla\bar{H}$ with this pair $(J_a(x),R_a(x))$ can be expressed as

$$[J(x)-R(x)]\nabla H_a+[J_a(x)-R_a(x)]\nabla\bar{H}=g(x)\tau(x). \tag{20}$$

Choosing

$$u=\tau(x), \tag{21}$$

which is a continuous feedback controller, we can show that $u=\tau(x)$ is a global finite-time stabilizer of the system (13).

Moreover, it follows from Theorem 1 that the settling time of the closed-loop system satisfies

$$T(x_0)\leqslant t_0+\frac{(2\alpha-1)^2}{4k\alpha(\alpha-1)}\bar{H}^{\frac{\alpha-1}{\alpha}}(x_0). \tag{22}$$

For the convenience of description, the continuous feedback law $u=\tau(x)$ given in (21) is called *the derived control* of Equation (19) with respect to the solution pair $(J_a(x),R_a(x))$.

Summarizing the above leads to the following result.

Theorem 2: Consider the system (13) with $\operatorname{rank}g(x)=m<n$. If there exist a real number $\alpha>1$, a symmetric matrix $R_a(x)\in\mathbb{R}^{n\times n}$ and a skew-symmetric matrix $J_a(x)\in\mathbb{R}^{n\times n}$ such that (19) holds and meanwhile, $\bar{R}(x):=R_a(x)+R(x)$ satisfies (15), then the system (13) can be globally finite-time stabilized by the continuous feedback law $u=\tau(x)$, which is the derived control of Equation (19) with respect to the solution pair $(J_a(x),R_a(x))$. Furthermore, the settling time of the closed-loop system satisfies (22).

Remark 5: The control law $u=\tau(x)$ given in Theorem 2 is continuous but, in general, non-smooth.

Remark 6: $R_a(x)$ in Theorem 2 is only required to be symmetric, not positive (semi-)definite, which can provide us more choices of $(J_a(x),R_a(x))$ such that Equation (19) holds.

In the following, we give an example to show how to apply Theorem 2 to design continuous finite-time stabilizers for PCH systems.

Example 1: Design a finite-time controller to stabilize the following PCH system

$$\dot{x}=[J(x)-R(x)]\frac{\partial H}{\partial x}+g(x)u, \quad x(t_0)=x_0, \tag{23}$$

where $x=(x_1,x_2,x_3)^T\in\mathbb{R}^3$, $u=(u_1,u_2)^T\in\mathbb{R}^2$, and

$$J(x)=\begin{bmatrix}0&1&0\\-1&0&-1\\0&1&0\end{bmatrix},\quad g(x)=\begin{bmatrix}1&0\\0&1\\1&0\end{bmatrix},$$

$$R(x)=\mathrm{Diag}\{0,1,2\},\quad H(x)=\frac{1}{2}x_1^2+\frac{1}{2}x_2^2+x_3^{\frac{4}{3}}.$$

Let

$$\bar{H}(x)=\left(x_1^2\right)^{\frac{2}{3}}+\left(x_2^2\right)^{\frac{2}{3}}+\left(x_3^2\right)^{\frac{2}{3}},\quad H_a(x):=\bar{H}(x)-H(x),$$

where $\alpha=2$ in (16). Then, it can be checked that

$$g^{\perp}(x)\Big([J(x)-R(x)]\nabla H_a+[J_a(x)-R_a(x)]\nabla\bar{H}\Big)=0 \tag{24}$$

has a solution pair $(J_a(x),R_a(x))$ as follows

$$J_a(x)=\begin{bmatrix}0&-1&1\\1&0&1\\-1&-1&0\end{bmatrix},\ R_a(x)=\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}.$$

Moreover, it is easy to see that $\bar{R}(x)=R_a(x)+R(x)$ ($=\mathrm{Diag}\{1,1,1\}$) satisfies (15). Thus, it follows from Theorem

2 that the system (23) can be globally finite-time stabilized by a continuous feedback law.

On the other hand, a straightforward computation shows that

$$[J(x) - R(x)]\nabla H_a + [J_a(x) - R_a(x)]\nabla \bar{H} = g(x)u(x),$$

where the derived control with respect to $(J_a(x), R_a(x))$ is given as

$$u(x) = \begin{bmatrix} -x_2 - \frac{4}{3}x_1^{\frac{1}{3}} + \frac{4}{3}x_3^{\frac{1}{3}} \\ x_1 + x_2 - \frac{4}{3}x_2^{\frac{1}{3}} + \frac{4}{3}x_3^{\frac{1}{3}} \end{bmatrix}, \tag{25}$$

which is a global continuous finite-time stabilizer of the system (23).

Furthermore, noticing that $\alpha = 2$ and $k = 1$ in this example, it is easy to know from (22) that the settling time of the closed-loop system consisting of the system (23) and the controller (25) satisfies

$$T(x_0) \leqslant t_0 + \frac{9}{8}\bar{H}^{\frac{1}{2}}(x_0).$$

In what follows, we provide a more general result on the finite-time stabilization of the system (13).

Theorem 3: Consider the system (13). If there exist a real number $\alpha > 1$ and an $n \times n$ symmetric matrix $R_a(x)$ such that

(1)
$$f(x) \in \text{Span}_0\{g(x)\} + \text{Ker}\{d\bar{H}\}, \tag{26}$$
where $f(x) := [J(x) - R(x)]\nabla H_a - R_a(x)\nabla \bar{H}$, $\bar{H}(x)$ and $H_a(x)$ are given in (16) and (17), respectively, $\text{Span}_0\{\cdot\}$ stands for the generated space with C^0 scalar functions as the coefficients and $d\bar{H}$ is the differential one-form of $\bar{H}$, and

(2)
$$k := \min_{1 \leqslant i \leqslant n}\left\{\inf_{x \in \mathbb{R}^n}\left\{\sigma_i^{R+R_a}(x)\right\}\right\} > 0, \tag{27}$$
where $\sigma_i^{R+R_a}(x)$, $i = 1, 2, ..., n$, denote the eigenvalues of $R(x) + R_a(x)$,

then the system (13) can be globally finite-time stabilized by a continuous feedback law $u(x)$. Furthermore, the settling time of the resulting closed-loop system satisfies (22).

Proof: It can be seen from (26) that there exist a continuous vector field $\eta(x) \in \mathbb{R}^m$ and $\xi(x) \in \text{Ker}\{d\bar{H}\}$ such that

$$f(x) = g(x)\eta(x) + \xi(x). \tag{28}$$

Choosing $u = \eta(x)$ and substituting it into the system (13) lead to the following closed-loop system

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\nabla H + f(x) - \xi(x) \\ &= [J(x) - R(x)]\nabla H + [J(x) - R(x)]\nabla H_a \\ &\quad - R_a(x)\nabla \bar{H} - \xi(x) \\ &= [J(x) - R(x)]\nabla \bar{H} - R_a(x)\nabla \bar{H} - \xi(x) \\ &= [J(x) - (R(x) + R_a(x))]\nabla \bar{H} - \xi(x). \end{aligned} \tag{29}$$

On the other hand, since $\xi(x) \in \text{Ker}\{d\bar{H}\}$, $d\bar{H} \cdot \xi(x) = L_\xi \bar{H} = 0$. It thus follows from [22] that $\xi(x)$ can be expressed as

$$\xi(x) = J_a(x)\nabla \bar{H}, \tag{30}$$

where $J_a(x)$ is an $n \times n$ skew-symmetric matrix.

Substituting (30) into (29) leads to

$$\dot{x} = \left[\Big(J(x) - J_a(x)\Big) - \Big(R(x) + R_a(x)\Big)\right]\frac{\partial \bar{H}}{\partial x}. \tag{31}$$

Noticing that $J(x) - J_a(x)$ is skew-symmetric and $R(x) + R_a(x)$ satisfies (27), it follows from Theorem 1 that the closed-loop system (31) is globally finite-time stable at the origin, and moreover, the settling time of the closed-loop system satisfies (22). Thus, the proof is completed. □

Remark 7: For the convenience of description in the sequel, the control $u = \eta(x)$ given in the above proof is called *the derived control* of (26) with respect to $f(x)$.

Remark 8: It can be seen from Corollary 1 that Theorems 2 and 3 still hold when $\bar{H}(x)$ given in (16) is replaced by $\bar{H}(x) = \sum_{i=1}^n |x_i|^\beta$ $(1 < \beta < 2)$.

IV. FINITE-TIME STABILIZATION OF NONLINEAR AFFINE SYSTEMS

In this section, we apply the results obtained for Hamiltonian systems to study continuous finite-time stabilization of nonlinear affine systems, and propose several control design results for the systems.

Consider the following nonlinear affine system

$$\dot{x} = f(x) + g(x)u, \quad f(0) = 0, \quad x(t_0) = x_0, \tag{32}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

If the system (32) has a dissipative Hamiltonian realization as follows

$$\dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x} + g(x)u$$

with $J(x)$ skew-symmetric and $R(x)$ positive semi-definite, then we can directly utilize Theorem 2 or 3 to design its continuous finite-time stabilizer. However, a dissipative Hamiltonian realization is very difficult to obtain for most nonlinear systems. It is thus necessary to develop a new way based on general Hamiltonian structures to handle the finite-time stabilization problem for the system (32).

In the following, we apply the orthogonal decomposition method [22] to provide the system (32) a Hamiltonian structure first. Then, based on the obtained Hamiltonian structure, we propose several approaches to the finite-time control design for the system (32).

Choose

$$H(x) = \sum_{i=1}^n \left(x_i^2\right)^{\frac{\alpha}{2\alpha - 1}}, \quad \alpha > 1, \tag{33}$$

which is a regular positive definite function [22]. By the orthogonal decomposition Hamiltonian realization [22], with this $H(x)$, the system (32) can be expressed as

$$\dot{x} = [J(x) + S(x)]\frac{\partial H}{\partial x} + g(x)u, \tag{34}$$

where

$$J(x) = \begin{cases} \frac{1}{\|\nabla H\|^2}[f_{td}(x)\nabla H^T - \nabla H f_{td}^T(x)], & x \neq 0 \\ 0, & x = 0 \end{cases} \tag{35}$$

is skew-symmetric, $f_{td}(x) = f(x) - f_{gd}(x)$, $f_{gd}(x) = \frac{L_f H}{\|\nabla H\|^2}\nabla H$, and

$$S(x) = \begin{cases} \frac{L_f H}{\|\nabla H\|^2} I_n, & x \neq 0 \\ 0, & x = 0 \end{cases} \tag{36}$$

is symmetric.

Remark 9: A nonlinear affine system always has the orthogonal decomposition Hamiltonian realization given in (34) [22].

Notice that $S(x)$ given in (36) is not a negative definite matrix as desired. In the following, we use the idea of "damping injection" to design a continuous feedback law $u(x)$ such that the closed-loop system is globally finite-time stable at the origin.

Based on the Hamiltonian realization (34), we have the following result.

Theorem 4: Consider the system (32) with $\text{rank}\{g(x)\} = m < n$. If there exist a real number $\alpha > 1$, a symmetric matrix $R_a(x) \in \mathbb{R}^{n\times n}$ and a skew-symmetric matrix $J_a(x) \in \mathbb{R}^{n\times n}$ such that

(1) the equation
$$g^{\perp}(x)\Big([J_a(x) - R_a(x)]\nabla H\Big) = 0 \tag{37}$$
holds, i.e., Equation (37) has a solution pair $(J_a(x), R_a(x))$, where $H(x)$ is given in (33), and

(2)
$$k := \min_{1\leqslant i\leqslant n}\left\{\inf_{x\in\mathbb{R}^n}\left\{\sigma_i^{R_a-S}(x)\right\}\right\} > 0, \tag{38}$$
where $\sigma_i^{R_a-S}(x)$, $i = 1, 2, ..., n$, denote the eigenvalues of $R_a(x) - S(x)$, and $S(x)$ is given in (36),

then the system (32) can be globally finite-time stabilized by the derived control $u = \tau(x)$ of Equation (37) with respect to the solution pair $(J_a(x), R_a(x))$. Furthermore, the settling time of the closed-loop system consisting of the system (32) and the control $u = \tau(x)$ satisfies
$$T(x_0) \leqslant t_0 + \frac{(2\alpha-1)^2}{4k\alpha(\alpha-1)}H^{\frac{\alpha-1}{\alpha}}(x_0). \tag{39}$$

Proof: Since Equation (37) holds for $(J_a(x), R_a(x))$, there exists a continuous vector field $\tau(x) \in \mathbb{R}^m$ such that
$$[J_a(x) - R_a(x)]\nabla H = g(x)\tau(x).$$
Choosing $u(x) = \tau(x)$, which is the so-called derived control, and substituting it into (34), we obtain
$$\dot{x} = \Big[\bar{J}(x) - \Big(R_a(x) - S(x)\Big)\Big]\frac{\partial H}{\partial x}, \tag{40}$$
where $\bar{J}(x) := J(x) + J_a(x)$ is skew-symmetric.

On the other hand, Condition (2) holds. It follows from Theorem 1 that the system (40) is globally finite-time stable at the origin, and meanwhile, the settling time satisfies (39). Thus, the proof is completed. □

Remark 10: It is noted that $R_a(x)$ in Theorem 4 is only required to be symmetric, not positive (semi-) definite, and this would allow for more choices of $(J_a(x), R_a(x))$ such that Condition (37) holds.

In the following, we give an example to show how to apply Theorem 4 to design continuous finite-time stabilizers for nonlinear affine systems.

Example 2: Consider the following affine system
$$\dot{x} = \begin{bmatrix} x_2 \\ -x_1 - x_2 \\ x_2 + \frac{4}{3}x_1^{\frac{1}{3}} - \frac{4}{3}x_3^{\frac{1}{3}} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u := f(x) + g(x)u. \tag{41}$$

Let
$$H(x) = \left(x_1^2\right)^{\frac{2}{3}} + \left(x_2^2\right)^{\frac{2}{3}} + \left(x_3^2\right)^{\frac{2}{3}}$$
with $\alpha = 2$, then the system (41) can be expressed as
$$\dot{x} = [J(x) + S(x)]\nabla H + g(x)u,$$
where
$$J(x) = \begin{cases} \frac{1}{\|\nabla H\|^2}[f_{td}(x)\nabla H^T - \nabla H f_{td}^T(x)], & x \neq 0 \\ 0, & x = 0, \end{cases}$$
$f_{td}(x) = f(x) - f_{gd}(x)$, $f_{gd}(x) = \frac{L_f H}{\|\nabla H\|^2}\nabla H$, and
$$S(x) = \begin{cases} \frac{L_f H}{\|\nabla H\|^2} I_3, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Now, consider the following equation
$$g^{\perp}(x)\Big([J_a(x) - R_a(x)]\nabla H\Big) = 0. \tag{42}$$
A straightforward computation shows that (42) has a solution pair as follows
$$J_a(x) = -J(x), \quad R_a(x) = I_3 + S(x),$$
and furthermore,
$$\min_{1\leqslant i\leqslant 3}\left\{\inf_{x\in\mathbb{R}^3}\{\sigma_i^{R_a-S}(x)\}\right\} = 1 > 0.$$
Thus, it follows from Theorem 4 that the system (41) can be globally finite-time stabilized by the derived control of (42) with respect to the solution pair $(J_a(x), R_a(x))$.

Next, we find the derived control. Through a straightforward computation, we know that
$$[J_a(x) - R_a(x)]\nabla H = g(x)\cdot\begin{bmatrix} -x_2 - \frac{4}{3}x_1^{\frac{1}{3}} \\ x_1 + x_2 - \frac{4}{3}x_2^{\frac{1}{3}} \end{bmatrix}.$$
Therefore, the derived control of (42) is given as
$$u(x) = \begin{bmatrix} -x_2 - \frac{4}{3}x_1^{\frac{1}{3}} \\ x_1 + x_2 - \frac{4}{3}x_2^{\frac{1}{3}} \end{bmatrix}. \tag{43}$$

Moreover, noticing that $\alpha = 2$ and $k = 1$ in this example, it follows from Theorem 4 that the settling time of the closed-loop system consisting of the system (41) and the control (43) satisfies
$$T(x_0) \leqslant t_0 + \frac{9}{8}H^{\frac{1}{2}}(x_0),$$
where $x(t_0) = x_0$ is the system's initial condition. □

Similar to the case of Theorem 3, we have the following result on the finite-time control design of the system (32).

Theorem 5: Consider the system (32) with the Hamiltonian function (33). If there exist a real number $\alpha > 1$ and a symmetric matrix $R_a(x) \in \mathbb{R}^{n\times n}$ such that both
$$R_a(x)\nabla H \in \text{Span}_0\{g(x)\} + \text{Ker}\{dH\} \tag{44}$$
and (38) hold, then the system (32) can be globally finite-time stabilized by $u = -\eta(x)$, where $\eta(x)$ is the derived control of (44) with respect to $R_a(x)\nabla H$. Moreover, the settling time of the closed-loop system consisting of the system (32) and the control $u = -\eta(x)$ satisfies (39).

Proof: The proof is similar to those of Theorems 3 and 4, and thus omitted. □

Remark 11: Since $R_a(x)$ in Theorem 5 is only required to be symmetric, not positive (semi-)definite, it is easy to see that there are more choices of $R_a(x)$ such that (44) holds.

Next, we present another result on the finite-time stabilization of the system (32) by using Hamiltonian structure obtained with the Jacobian matrix of $f(x)$.

Letting J_f denote the Jacobian matrix of $f(x)$, we have the following result.

Theorem 6: Consider the system (32) with J_f non-singular. If there exist a real number $\alpha > 1$ and a symmetric matrix $R_a(x) \in \mathbb{R}^{n\times n}$ such that

(1)

$$h(x) \in \text{Span}_0\{g(x)\} + \text{Ker}\{d\bar{H}\}, \tag{45}$$

where $h(x) := J_f^{-T}\nabla H_a - R_a(x)\nabla\bar{H}$, $\bar{H}(x) = \sum_{i=1}^{n}\left(x_i^2\right)^{\frac{\alpha}{2\alpha-1}}$, $H_a(x) := \bar{H}(x) - H(x)$ and

$$H(x) = \frac{1}{2}f^T(x)f(x); \tag{46}$$

(2)

$$k := \min_{1\leqslant i\leqslant n}\left\{\inf_{x\in\mathbb{R}^n}\left\{\sigma_i^{\bar{R}}(x)\right\}\right\} > 0, \tag{47}$$

where $\bar{R}(x) := R_a(x) - \frac{1}{2}(J_f^T + J_f^{-T})$,

then the system (32) can be globally finite-time stabilized by the derived control $u = \eta(x)$ of (45) with respect to $h(x)$.

Proof: Since (45) holds, similar to the proof of Theorem 3, it follows that there exist a continuous vector field $\eta(x) \in \mathbb{R}^m$ and a skew-symmetric matrix $J_a(x) \in \mathbb{R}^{n\times n}$ such that

$$h(x) = g(x)\eta(x) + J_a(x)\nabla\bar{H}. \tag{48}$$

On the other hand, since J_f is non-singular, it thus follows from [22] that the system (32) has a Hamiltonian realization as follows

$$\dot{x} = J_f^{-T}\nabla H + g(x)u, \tag{49}$$

where $H(x)$ is given in (46).

Choose $u = \eta(x)$, which is the so-called derived control of (45). Substituting $u = \eta(x)$ into the system (49) and using (48) lead to

$$\begin{aligned}\dot{x} &= J_f^{-T}\nabla H + h(x) - J_a(x)\nabla\bar{H} \\ &= \Big[\Big\{\frac{1}{2}(J_f^{-T} - J_f^{-1}) - J_a(x)\Big\} \\ &\quad -\Big\{R_a(x) - \frac{1}{2}(J_f^{-T} + J_f^{-1})\Big\}\Big]\nabla\bar{H} \\ &= [\bar{J}(x) - \bar{R}(x)]\nabla\bar{H},\end{aligned} \tag{50}$$

where $\bar{J}(x) := \frac{1}{2}(J_f^{-T} - J_f^{-1}) - J_a(x)$ is skew-symmetric.

Noticing that (47) holds true, it follows from Theorem 1 that the closed-loop system (50) is globally finite-time stable at the origin, which implies that the system (32) can be globally finite-time stabilized by the derived control of (45). Thus, the proof is completed. □

V. CONCLUSION

We have investigated the finite-time stabilization of nonlinear PCH systems in this paper, and proposed a number of approaches to the finite-time control design. Via Hamiltonian realization, the results obtained for the Hamiltonian systems have been applied to the global continuous finite-time stabilization of nonlinear affine systems, and a number of control design approaches have been obtained. Study on several examples has shown that the control design approaches proposed for both the PCH systems and nonlinear affine systems in this paper work very well.

REFERENCES

[1] S. P. Bhat and D. S. Bernstein, 1998. Continuous finite-time stabilization of the translational and rotational double integrators. *IEEE Trans. Automatic Control*, 43(5): 678-682.

[2] S. P. Bhat and D. S. Bernstein, 2000. Finite-time stability of continuous autonomous systems, *SIAM J. Control Optim.*, 38(3): 751-766.

[3] M. Dalsmo and A. J. van der Schaft, 1999. On representations and integrability of mathematical structures in energy-conserving physical systems, *SIAM J Contr Optim.*, 37(1): 54-91.

[4] K. Fujimoto, K. Sakurama and T. Sugie, 2003. Trajectory tracking control of port-controlled Hamiltonian systems via generalized canonical transformations. *Automatica*, 39(12): 2059-2069.

[5] M. Galaz, R. Ortega, A. S. Bazanella and A. M. Stankovic, 2003. An energy-shaping approach to the design of excitation control of synchronous generators. *Automatica*, 39(1): 111-119.

[6] V. T. Haimo, 1986. Finite-time controllers. *SIAM J. Contr Optim*, 24: 760-770.

[7] Y. Hong, and Z. Jiang, 2006. Finite-time stabilization of nonlinear systems with parametric and dynamic uncertainties. *IEEE Trans. Automatic Control*, 51(12): 1950-1956.

[8] Y. Hong, J. Huang, and Y. Xu, 2001. On an output feedback finite-time stabilization problem. *IEEE Trans. Automatic Control*, 46(2): 305-309.

[9] B. M. Maschke and A. J. van der Schaft, 1992. Port-controlled Hamiltonian systems: modeling origins and system theoretic properties. *Proc of the IFAC Symposium on NOLCOS*, Bordeaux, France: pp. 282-288.

[10] B. M. Maschke, R. Ortega and A. J. van der Schaft, 2000. Energy-based Lyapunov functions for forced Hamiltonian systems with dissipation. *IEEE Trans. on Autom. Contr.*, 45(8): 1498-1502.

[11] Y. Orlov, 2005. Finite time stabilityand robust control synthesis of uncertain switched systems, *SIAM J. Control Optim.*, 43(4): 1253-1271.

[12] R. Ortega, M. Galaz, A. Astolfi, Y. Sun, and T. Shen, 2005. Transient stabilization of multi-machine power systems with nontrivial transfer conductances, *IEEE Trans. on Automatic Control,* 50(1): 60-75.

[13] R. Ortega, A. J. van der Schaft, B. Maschke and G. Escobar, 2002. Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems, *Automatica*, 38(4): 585-596.

[14] R. Ortega, M. Spong, F. Gomez-Estern and G. Blankenstein, 2002. Stabilization of underactuated mechanical systems via interconnection and damping assignment, *IEEE Trans. Autom. Control*, 47(8): 1218-1233.

[15] C. Qian and J. Li, 2005. Global finite-time stabilization by output feedback for planar systems without observable linearization, *IEEE Trans. Autom. Control*, 50(6): 885-890.

[16] T. Shen, R. Ortega, Q. Lu, S. Mei, and K. Tamura, 2003. Adaptive L_2 disturbance attenuation of Hamiltonian systems with parameter perturbations and application to power systems. *Asian J. of Control*, 5(1): 143-152.

[17] A. J. van der Schaft and B. M. Maschke, 1995. The Hamiltonian formulation of energy conserving physical systems with external ports, *Archive für Elektronik und Übertragungstechnik*, 49: 362-371.

[18] A. J. van der Schaft, 1999, *L_2-gain and Passivity Techniques in Nonlinear Control*, Berlin: Springer.

[19] G. Viola, R. Ortega, R. Banavar, J. A. Acosta and A. Astofi, 2007. Total energy shaping control of mechanical systems: Simplifying the matching equations via coordinate changes. *IEEE Trans Automatic Contr*, 52(6): 1093-1099.

[20] Y. Wang, G. Feng, and D. Cheng, 2007. Simultaneous stabilization of a set of nonlinear port-controlled Hamiltonian systems. *Automatica*, 43(3): 403-415.

[21] Y. Wang, G. Feng, D. Cheng and Y. Liu, 2006. Adaptive L_2 disturbance attenuation control of multimachine power systems with SMES units. *Automatica*, 42(7): 1121-1132.

[22] Y. Wang, C. Li and D. Cheng, D., 2003. Generalized Hamiltonian realization of time-invariant nonlinear systems. *Automatica*, 39(8), 1437-1443.

[23] Z. Xi, D. Cheng, Q. Lu and S. Mei, 2002. Nonlinear decentralized controller design for multi-machine power systems using Hamiltonian function method. *Automatica*, 38(3): 527-534.