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Robust Delay-Dependent Stability and Stabilization of Polytopic Systems With Time-Delay and Its Application to Flight Control

Chao Shen, Ying Ban, Georgi M. Dimirovski, *Senior Member, IEEE* and Yuan-wei Jing

Abstract—**This paper concerns the problem of robust stability analysis and control synthesis for a polytopic system with time-varying delays via parameter-dependent Lyapunov functions. By a relaxation approach with slack matrices and a descriptor model transformation, a new robust delay dependent stability criterion is expressed as a set of linear matrix inequalities (LMIs) with less computational burden. This criterion can be employed for robust controller synthesis. The obtained stabilizability criterion is applied to design a flight controller for aircraft dynamic systems with multiple operating points. The simulation results illustrate the effectiveness of the proposed approach.**

I. INTRODUCTION

DURING the past years, the robust stability analysis of linear systems subject to time-invariant uncertainties has attracted considerable attention in robust control literature. For convex polytopic uncertainty the Edge theorem and related works provide stability conditions. Undoubtedly, the Lyapunov theory is one of the main approaches to deal with such systems. However, the quadratic stability, which uses a single or parameter-independent Lyapunov function for testing the stability over the whole uncertain domain, may lead to conservative results in the case where the uncertain parameters are time-invariant. Motivated by this fact, Lyapunov functions depending on the uncertain parameters have been proposed to reduce quadratic stability conservatism. Sufficient LMI conditions for the existence of an affine parameter-dependent Lyapunov function have been introduced in [1] for affine parameter dependence as well as in [2-4] for polytopic uncertainty. In [2], sufficient conditions for the robust stability of a polytopic system are proposed based on a set of constraints which make the derivative of a Lyapunov function a convex function with respect to the uncertain vector. The auxiliary constraints, however, may produce conservativeness. In [3], by replacing the unity matrix with a positive definite matrix a less conservative result is presented as the modification of that of [2]. But unfortunately, the modified constraints give rise to conservativeness likewise. Besides, the results of [1-3] are not applicable for robust controller synthesis, which means that for each vertex there is a product between the vertex matrix and corresponding positive definite matrix and thus the control gain can not be obtained. By introducing a slack variable, [4] recently established a new and less conservative condition, in terms of LMIs, for robust stability of linear systems with parametric uncertainties. Beside the reduced conservatism, the conditions do not involve any product of the matrices in the parameter dependent Lyapunov function and the system matrices. As such, this robust stability condition can be adapted for controller synthesis.

Time-delay is a source of performance degradation and instability in many cases. Therefore, the stability problem of time-delay systems is of theoretical and practical importance. Several results on robust control of time-delay systems subject to polytopic uncertainties have been reported in [5]-[9]. In [5], the authors examine the problems of robust stabilization and H_∞ control for uncertainty systems with a constant time delay. The obtained results can be easily extended to polytopic systems. Although simulation examples are presented to demonstrate the potentials of the proposed methods, the result derived remains conservative. In [6]-[8], the problems of robust stability and robust stabilization for linear systems with a constant time-delay and subject to convex polytopic uncertainty are considered based on parameter-dependent Lyapunov functionals. However, the proposed criterion depends on extra and positive scalar parameters, which increases computational burden and produces conservativeness. In [9], a sufficient condition is proposed for the stability of polytopic systems, which ensures a larger upper bound for time-varying delays affecting the state vector. But unfortunately, a nonlinear matrix inequality is obtained when this condition is employed for controller synthesis. Consequently, extra scalar parameters that must be positive are introduced to secure a stabilizability condition in terms of LMIs, which causes conservativeness likewise. Recently, a descriptor system approach was proposed for time-delayed systems. It reduced significantly the over-design compared to the traditional methods and fewer terms needed to be bounded in the derivation[10]-[11]. This approach was also applicable for polytopic systems.

So, in this paper, the problems of robust stability analysis and state feedback synthesis for linear systems with polytopic type of uncertainties and time-varying delays are investigated by means of parameter-dependent Lyapunov functions. With

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Chao Shen, Ying Ban and Yuanwei Jing are with Faculty of Information Science and Engineering, Northeastern University, 110004, Shenyang, Liaoning, P.R. of China. (e-mail: shenchao0728@163.com).

Georgi M. Dimirovski is with Faculty of Engineering, Computer Engg. Dept, Dogus University of Istanbul, TR-347222 Istanbul, Rep. of Turkey (e-mall: gdimirovski@dogus.edu.tr).

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the introduction of a slack variable, a descriptor system approach is adopted to obtain a new robust delay-dependent stability criterion in terms of LMIs. It is shown that this criterion includes the delay-dependent/rate-independent and delay-independent/rate-dependent criteria as special cases and reduces the computational burden involved in solving LMIs. In the derivative of the Lyapunov functional, with the introduction of the augmented vector $\xi(t) \triangleq \begin{bmatrix} x^{\mathrm{T}}(t) & \dot{x}^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}$, $h^2\dot{x}^{\mathrm{T}}(t)P_2(\lambda)\dot{x}(t)$ is formulated as $\xi^{\mathrm{T}}(t)\begin{bmatrix} 0 & 0 \\ 0 & h^2P_2(\lambda) \end{bmatrix}\xi(t)$, which avoids replacing $\dot{x}(t)$ in the term $h^2\dot{x}^{\mathrm{T}}(t)P_2(\lambda)\dot{x}(t)$ with the state equation. In consequence, the Lyapunov matrix P_2, which handles time delay, is not involved in any product terms with the system matrices A and A_d. Then the criterion can be readily extended to provide a criterion for robust stabilization via state feedback. Furthermore, the results are applied to the longitudinal dynamics in the flight envelope containing different operating points. Finally, the performance of the obtained controller is presented based on simulation results.

II. ROBUST STABILITY

Consider the following system with a time-varying delay

$$\begin{aligned} \dot{x}(t) &= A(\lambda)x(t) + A_d(\lambda)x(t-\tau(t)) \\ x(t) &= \phi(t), t \in [-h, 0] \end{aligned}, \tag{1}$$

where $x(t) \in R^n$ is the state vector and the initial vector ϕ is a continuously differentiable function from $[-h, 0]$ to R^n. We assume that $\tau(t)$ is a differentiable function, satisfying for all $t \geq 0$

$$0 \leq \tau(t) \leq h, \ \dot{\tau}(t) \leq d < 1 . \tag{2a,b}$$

Suppose that the system matrices $A(\lambda)$ and $A_d(\lambda)$ are not precisely known, but belong to a polytopic uncertainties domain Ω_1. In this case, system matrices $(A(\lambda), A_d(\lambda))$ in the uncertainties domain Ω_1 can be written as a convex combination of the polytope vertices (A_i, A_{di}), $i = 1, \cdots, N$, that is,

$$(A(\lambda), A_d(\lambda)) = \sum_{i=1}^{N} \lambda_i (A_i, A_{di}) \in \Omega_1 , \tag{3}$$

where $\lambda \triangleq [\lambda_1, \cdots, \lambda_N]^{\mathrm{T}} \in R^N$ denotes a vector of uncertain and time-invariant real parameters satisfying

$$\sum_{i=1}^{N} \lambda_i = 1, \lambda_i \geq 0 . \tag{4}$$

For the convenience of proof, we first introduce the following inequality which will be used in the proof of our results.

Lemma 1 [12]**.** For any constant matrix $P > 0$ and differentiable vector function $x(t)$ with appropriate dimensions, we have

$$[\int_{t-\tau(t)}^{t} \dot{x}(s)ds]^{\mathrm{T}} P [\int_{t-\tau(t)}^{t} \dot{x}(s)ds]$$
$$\leq \tau(t) \cdot \int_{t-\tau(t)}^{t} \dot{x}^{\mathrm{T}}(s)P\dot{x}(s)ds \leq h \cdot \int_{t-h}^{t} \dot{x}^{\mathrm{T}}(s)P\dot{x}(s)ds . \tag{5}$$

We represent system (1) in the equivalent descriptor form

$$\begin{aligned} \dot{x}(t) &= \eta(t) \\ \eta(t) &= A(\lambda)x(t) + A_d(\lambda)x(t-\tau(t)) \end{aligned}. \tag{6}$$

Now, the following theorem presents a new delay-dependent and rate-dependent robust stability result.

Theorem 1. System (1) with parameter uncertainty (3) and time-varying delay $\tau(t)$ satisfying (2) is robustly asymptotically stable if there exist symmetric positive definite matrices P_{0i}, P_{1i} P_{2i} and a matrix P_3 such that

$$\begin{bmatrix} \Delta_{11} & P_{0i} - P_3^{\mathrm{T}} + A_i^{\mathrm{T}}P_3 & P_3^{\mathrm{T}}A_{di} + P_{2i} \\ * & -P_3^{\mathrm{T}} - P_3 + h^2P_{2i} & P_3^{\mathrm{T}}A_{di} \\ * & * & -\bar{d}P_{1i} - P_{2i} \end{bmatrix} < 0 ,$$
$$i = 1, \cdots, N \tag{7}$$

where $\Delta_{11} = P_3^{\mathrm{T}}A_i + A_i^{\mathrm{T}}P_3 + P_{1i} - P_{2i}$, $\bar{d} = 1 - d$.

Proof: Define the following Lyapunov–Krasovskii functional

$$V(t,\lambda) = x^{\mathrm{T}}(t)P_0(\lambda)x(t) + \int_{t-\tau(t)}^{t} x^{\mathrm{T}}(s)P_1(\lambda)x(s)ds$$
$$+h \cdot \int_{t-h}^{t} (s-(t-h))\dot{x}^{\mathrm{T}}(s)P_2(\lambda)\dot{x}(s)ds , \tag{8}$$

where

$$P_0(\lambda) = \sum_{i=1}^{N} \lambda_i P_{0i} , \ P_1(\lambda) = \sum_{i=1}^{N} \lambda_i P_{1i} , \ P_2(\lambda) = \sum_{i=1}^{N} \lambda_i P_{2i} , \tag{9}$$

and $P_{0i} > 0$, $P_{1i} > 0$, $P_{2i} > 0$ are matrices to be determined. Then, along the solution of system (1), the time derivative of $V(t,\lambda)$ is given by

$$\begin{aligned} \dot{V}(t,\lambda) = &\, x^{\mathrm{T}}(t)P_0(\lambda)\dot{x}(t) + \dot{x}^{\mathrm{T}}(t)P_0(\lambda)x(t) \\ &-(1-\dot{\tau}(t))x^{\mathrm{T}}(t-\tau(t))P_1(\lambda)x(t-\tau(t)) \\ &+x^{\mathrm{T}}(t)P_1(\lambda)x(t) + h^2\dot{x}^{\mathrm{T}}(t)P_2(\lambda)\dot{x}(t) \\ &-h\int_{t-h}^{t} \dot{x}^{\mathrm{T}}(s)P_2(\lambda)\dot{x}(s)ds . \end{aligned} \tag{10}$$

From (10), lemma 1 and Leibniz–Newton formula, we have

$$\begin{aligned} \dot{V}(t,\lambda) \leq &\, x^{\mathrm{T}}(t)P_0(\lambda)\dot{x}(t) + \dot{x}^{\mathrm{T}}(t)P_0(\lambda)x(t) \\ &-(1-d)x^{\mathrm{T}}(t-\tau(t))P_1(\lambda)x(t-\tau(t)) \\ &+x^{\mathrm{T}}(t)P_1(\lambda)x(t) + h^2\dot{x}^{\mathrm{T}}(t)P_2(\lambda)\dot{x}(t) \\ &-[\int_{t-\tau(t)}^{t} \dot{x}(s)ds]^{\mathrm{T}}P_2(\lambda)[\int_{t-\tau(t)}^{t} \dot{x}(s)ds] \\ = &\begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_0(\lambda) & P_3^{\mathrm{T}} \\ 0 & P_3^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix} \\ &+ \begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_0(\lambda) & 0 \\ P_3 & P_3 \end{bmatrix} \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix} \end{aligned}$$

$$+\begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_1(\lambda)-P_2(\lambda) & P_3^{\mathrm{T}} \\ 0 & P_3^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x(t) \\ 0 \end{bmatrix}$$

$$-x^{\mathrm{T}}(t-\tau(t))(\bar{d}P_1(\lambda)+P_2(\lambda))$$

$$\times x(t-\tau(t))+\begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ 0 & h^2 P_2(\lambda) \end{bmatrix}$$

$$\times \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_2(\lambda) \\ 0 \end{bmatrix} x(t-\tau(t))$$

$$+x^{\mathrm{T}}(t-\tau(t))\begin{bmatrix} P_2(\lambda) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix}. \tag{11}$$

Note that one can obtain

$$\begin{bmatrix} x(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix}, \tag{12}$$

$$\begin{bmatrix} \eta(t) \\ -\eta(t)+A(\lambda)x(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ A(\lambda) & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix}. \tag{13}$$

Substituting (6) into (11) and from (12) and (13), we obtain

$$\dot{V}(t,\lambda) \le \xi^{\mathrm{T}}(t) \begin{bmatrix} P_0(\lambda) & P_3^{\mathrm{T}} \\ 0 & P_3^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \eta(t) \\ -\eta(t)+A(\lambda)x(t) \end{bmatrix}$$

$$+\xi^{\mathrm{T}}(t) \begin{bmatrix} P_0(\lambda) & P_3^{\mathrm{T}} \\ 0 & P_3^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} 0 \\ A_d(\lambda)x(t-\tau(t)) \end{bmatrix}$$

$$+\begin{bmatrix} \eta(t) \\ -\eta(t)+A(\lambda)x(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_0(\lambda) & 0 \\ P_3 & P_3 \end{bmatrix} \xi(t)$$

$$+\begin{bmatrix} 0 \\ A_d(\lambda)x(t-\tau(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_0(\lambda) & 0 \\ P_3 & P_3 \end{bmatrix} \xi(t)$$

$$+\xi^{\mathrm{T}}(t) \begin{bmatrix} P_1(\lambda)-P_2(\lambda) & 0 \\ 0 & 0 \end{bmatrix} \xi(t)$$

$$-x^{\mathrm{T}}(t-\tau(t))(\bar{d}P_1(\lambda)+P_2(\lambda))$$

$$\times x(t-\tau(t))+\xi^{\mathrm{T}}(t) \begin{bmatrix} 0 & 0 \\ 0 & h^2 P_2(\lambda) \end{bmatrix} \xi(t)$$

$$+\xi^{\mathrm{T}}(t) \begin{bmatrix} P_2(\lambda) \\ 0 \end{bmatrix} x(t-\tau(t))$$

$$+x^{\mathrm{T}}(t-\tau(t))\begin{bmatrix} P_2(\lambda) & 0 \end{bmatrix} \xi(t)$$

$$=\xi^{\mathrm{T}}(t) \begin{bmatrix} P_3^{\mathrm{T}}A(\lambda) & P_0(\lambda)-P_3^{\mathrm{T}} \\ P_3^{\mathrm{T}}A(\lambda) & -P_3^{\mathrm{T}} \end{bmatrix} \xi(t)$$

$$+\xi^{\mathrm{T}}(t) \begin{bmatrix} P_3^{\mathrm{T}}A_d(\lambda) \\ P_3^{\mathrm{T}}A_d(\lambda) \end{bmatrix} x(t-\tau(t))$$

$$+\xi^{\mathrm{T}}(t) \begin{bmatrix} A^{\mathrm{T}}(\lambda)P_3 & A^{\mathrm{T}}(\lambda)P_3 \\ P_0(\lambda)-P_3 & -P_3 \end{bmatrix} \xi(t)$$

$$+x^{\mathrm{T}}(t-\tau(t)) \begin{bmatrix} P_3^{\mathrm{T}}A_d(\lambda) \\ P_3^{\mathrm{T}}A_d(\lambda) \end{bmatrix}^{\mathrm{T}} \xi(t)$$

$$+\xi^{\mathrm{T}}(t) \begin{bmatrix} P_1(\lambda)-P_2(\lambda) & 0 \\ 0 & h^2 P_2(\lambda) \end{bmatrix} \xi(t)$$

$$-x^{\mathrm{T}}(t-\tau(t))(\bar{d}P_1(\lambda)+P_2(\lambda))$$

$$\times x(t-\tau(t))+\xi^{\mathrm{T}}(t) \begin{bmatrix} P_2(\lambda) \\ 0 \end{bmatrix} x(t-\tau(t))$$

$$+x^{\mathrm{T}}(t-\tau(t))\begin{bmatrix} P_2(\lambda) & 0 \end{bmatrix} \xi(t)$$

$$=\begin{bmatrix} \xi(t) \\ x(t-\tau(t)) \end{bmatrix}^{\mathrm{T}} \Xi(\lambda) \begin{bmatrix} \xi(t) \\ x(t-\tau(t)) \end{bmatrix}, \tag{14}$$

where

$$\xi(t) \triangleq \begin{bmatrix} x^{\mathrm{T}}(t) & \eta^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}},$$

$$\Xi(\lambda) = \begin{bmatrix} \Gamma(\lambda) & \begin{bmatrix} P_3^{\mathrm{T}}A_d(\lambda)+P_2(\lambda) \\ P_3^{\mathrm{T}}A_d(\lambda) \end{bmatrix} \\ * & -\bar{d}P_1(\lambda)-P_2(\lambda) \end{bmatrix},$$

$$\Gamma(\lambda) = \begin{bmatrix} P_3^{\mathrm{T}}A(\lambda)+A^{\mathrm{T}}(\lambda)P_3 \\ +P_1(\lambda)-P_2(\lambda) & P_0(\lambda)-P_3^{\mathrm{T}}+A^{\mathrm{T}}(\lambda)P_3 \\ * & -P_3^{\mathrm{T}}-P_3+h^2P_2(\lambda) \end{bmatrix}. \tag{15}$$

According to (7) and (15), we have

$$\Xi(\lambda) = \sum_{i=1}^{N} \lambda_i \begin{bmatrix} \Gamma_i & \begin{bmatrix} P_3^{\mathrm{T}}A_{di}+P_{2i} \\ P_3^{\mathrm{T}}A_{di} \end{bmatrix} \\ * & -\bar{d}P_{1i}-P_{2i} \end{bmatrix} < 0, \tag{16}$$

where

$$\Gamma_i = \begin{bmatrix} P_3^{\mathrm{T}}A_i+A_i^{\mathrm{T}}P_3 \\ +P_{1i}-P_{2i} & P_{0i}-P_3^{\mathrm{T}}+A_i^{\mathrm{T}}P_3 \\ * & -P_3^{\mathrm{T}}-P_3+h^2P_{2i} \end{bmatrix}. \tag{17}$$

From (14) and (16), we get $\dot{V}(t,\lambda)<0$. Then, according to the Lyapunov theory, system (1) with parameter uncertainty (3) and time-varying delay $\tau(t)$ satisfying (2) is robustly asymptotically stable for all uncertain parameter λ. This proof is completed.

Remark 1. In Theorem 1, with the introduction of the slack variable P_3 and the corresponding augmented vector $\xi(t) \triangleq \begin{bmatrix} x^{\mathrm{T}}(t) & \eta^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}$, the delay-dependent stability criterion (7) does not involve the product between the Lyapunov matrix P_0 and the system dynamic matrices A and A_d. Hence, for stability criterion (7), P_{0i} are not required to be the same, but the slack variables P_3 is. So, it is expected to lead to a less conservative stability condition, as there are no other constraints imposed on P_3. Besides, the augmented vector can be used to formulate $h^2\dot{x}^{\mathrm{T}}(t)P_2(\lambda)\dot{x}(t)$ as $\xi^{\mathrm{T}}(t)\begin{bmatrix} 0 & 0 \\ 0 & h^2P_2(\lambda) \end{bmatrix}\xi(t)$, which avoids replacing $\dot{x}(t)$ in the term $h^2\dot{x}^{\mathrm{T}}(t)P_2(\lambda)\dot{x}(t)$ with the state equation and so eliminates the product between the Lyapunov matrix P_2 and the system matrices A and A_d.

Remark 2. A stability criterion was also given in Theorem 1 of [9]. This criterion, however, requires more matrix variables.

Consequently, the dimension of LMI (12a) in [9] becomes higher than that of (7) in this paper (the corresponding dimensions of (12a), (7) are $7n$ and $3n$, respectively). The computational burden is increased accordingly, which can be verified through the simulation analysis.

It is worth mentioning that with the introduction of another slack variable P_4, Theorem 1 can be extended to a stability condition in a more general case. That is, by replacing the transformation matrix $\begin{bmatrix} P_0(\lambda) & P_3^{\mathrm{T}} \\ 0 & P_3^{\mathrm{T}} \end{bmatrix}$ with $\begin{bmatrix} P_0(\lambda) & P_3^{\mathrm{T}} \\ 0 & P_4^{\mathrm{T}} \end{bmatrix}$, Theorem 1 will become in the following form.

Corollary 1: System (1) with parameter uncertainty (3) and time-varying delay $\tau(t)$ satisfying (2) is robustly asymptotically stable if there exist symmetric positive definite matrices P_{0i}, P_{1i} P_{2i} and matrices P_3, P_4 such that

$$\begin{bmatrix} \Delta_{11} & P_{0i} - P_3^{\mathrm{T}} + A_i^{\mathrm{T}} P_4 & P_3^{\mathrm{T}} A_{di} + P_{2i} \\ * & -P_4^{\mathrm{T}} - P_4 + h^2 P_{2i} & P_4^{\mathrm{T}} A_{di} \\ * & * & -\bar{d} P_{1i} - P_{2i} \end{bmatrix} < 0, \ i = 1, \cdots, N,$$

where $\Delta_{11} = P_3^{\mathrm{T}} A_i + A_i^{\mathrm{T}} P_3 + P_{1i} - P_{2i}$, $\bar{d} = 1 - d$.

However, this stability condition is not applicable for the design of robust controller. The reason is that matrices by which the system matrix A is multiplied are not the same one such that a single and fixed gain matrix can not be gotten.

By following similar lines as in the proof of Theorem 1, we can obtain the following delay-dependent/rate-independent robust stability condition, as well as a delay-independent robust stability condition.

Corollary 2: Polytopic system (1) with $\tau(t)$ satisfying (2a) is robustly asymptotically stable if there exist symmetric positive definite matrices P_{0i}, P_{2i} and a matrix P_3 such that

$$\begin{bmatrix} \Delta_{11} & P_{0i} - P_3^{\mathrm{T}} + A_i^{\mathrm{T}} P_3 & P_3^{\mathrm{T}} A_{di} + P_{2i} \\ * & -P_3^{\mathrm{T}} - P_3 + h^2 P_{2i} & P_3^{\mathrm{T}} A_{di} \\ * & * & -P_{2i} \end{bmatrix} < 0, \ i = 1, \cdots, N,$$

where $\Delta_{11} = P_3^{\mathrm{T}} A_i + A_i^{\mathrm{T}} P_3 - P_{2i}$.

Corollary 3: Polytopic system (1) with $\tau(t)$ satisfying (2b) is robustly asymptotically stable if there exist symmetric positive definite matrices P_{0i}, P_{1i} and a matrix P_3 such that

$$\begin{bmatrix} \Delta_{11} & P_{0i} - P_3^{\mathrm{T}} + A_i^{\mathrm{T}} P_3 & P_3^{\mathrm{T}} A_{di} \\ * & -P_3^{\mathrm{T}} - P_3 & P_3^{\mathrm{T}} A_{di} \\ * & * & -\bar{d} P_{1i} \end{bmatrix} < 0, \ i = 1, \cdots, N,$$

where $\Delta_{11} = P_3^{\mathrm{T}} A_i + A_i^{\mathrm{T}} P_3 + P_{1i}$, $\bar{d} = 1 - d$.

III. Stabilization

In this section, we apply the results of section 2 to a robust control problem. Consider the uncertain system described by

$$\begin{aligned} \dot{x}(t) &= A(\lambda)x(t) + A_d(\lambda)x(t - \tau(t)) + B(\lambda)u(t) \\ x(t) &= \phi(t), t \in [-h, 0] \end{aligned}, \quad (18)$$

where $x(t)$ and $\phi(t)$ are defined in the previous section, $u(t) \in R^m$ is the control input, $(A(\lambda), A_d(\lambda), B(\lambda))$ satisfies

$$(A(\lambda), A_d(\lambda), B(\lambda)) = \sum_{i=1}^{N} \lambda_i (A_i, A_{di}, B_i) \in \Omega_2. \quad (19)$$

Consider system (18) with the state feedback control

$$u(t) = Kx(t). \quad (20)$$

The closed-loop system is given by

$$\dot{x}(t) = (A(\lambda) + B(\lambda)K)x(t) + A_d(\lambda)x(t - \tau(t)). \quad (21)$$

Applying Theorem 1, the robust controller design method for system (21) is presented in the next theorem.

Theorem 2: The closed-loop system (21) with delay $\tau(t)$ is robustly stable if there exist symmetric positive definite matrices X_{0i}, X_{1i} X_{2i} and matrices X_3, Y such that

$$\begin{bmatrix} \bar{\Delta}_{11} & X_{0i} - X_3 + X_3^{\mathrm{T}} A_i^{\mathrm{T}} + Y^{\mathrm{T}} B_i^{\mathrm{T}} & A_{di} X_3 + X_{2i} \\ * & -X_3 - X_3^{\mathrm{T}} + h^2 X_{2i} & A_{di} X_3 \\ * & * & -\bar{d} X_{1i} - X_{2i} \end{bmatrix} < 0,$$

$$i = 1, \cdots, N, \quad (22)$$

where $\bar{\Delta}_{11} = A_i X_3 + X_3^{\mathrm{T}} A_i^{\mathrm{T}} + B_i Y + Y^{\mathrm{T}} B_i^{\mathrm{T}} + X_{1i} - X_{2i}$. The state feedback gain is given by $K = Y X_3^{-1}$.

Proof: Defining $P_{0i} = P_3^{\mathrm{T}} X_{0i} P_3 > 0$, $P_{1i} = P_3^{\mathrm{T}} X_{1i} P_3 > 0$, $P_{2i} = P_3^{\mathrm{T}} X_{2i} P_3 > 0$, $P_3 = X_3^{-1}$, $Y = K X_3$ multiplying (22) by $\mathrm{diag}\{P_3^{\mathrm{T}}, P_3^{\mathrm{T}}, P_3^{\mathrm{T}}\}$ and $\mathrm{diag}\{P_3, P_3, P_3\}$, on the left and the right, respectively, we obtain

$$\begin{bmatrix} \Pi_{11} & P_{0i} - P_3^{\mathrm{T}} + A_{cli}^{\mathrm{T}} P_3 & P_3^{\mathrm{T}} A_{di} + P_{2i} \\ * & -P_3^{\mathrm{T}} - P_3 + h^2 P_{2i} & P_3^{\mathrm{T}} A_{di} \\ * & * & -\bar{d} P_{1i} - P_{2i} \end{bmatrix} < 0, \ i = 1, \cdots, N, \quad (23)$$

where $\Pi_{11} = P_3^{\mathrm{T}} A_{cli} + A_{cli}^{\mathrm{T}} P_3 + P_{1i} - P_{2i}$, $A_{cli} = A_i + B_i K$. For the closed-loop system (21), application of Theorem 1 completes the proof.

Remark 3. A relevant result was also given in Theorem 2 of [9]. With the introduction of extra scalar parameters α and ε, a stabilizability criterion is derived in terms of LMIs. This criterion depends upon the α and ε that must be positive. Thus, this treatment, which estimates the α and ε in advance to secure feasible solutions, causes conservativeness.

IV. Simulation Results

For comparison purposes, the performance of the proposed method is compared with those of other methods.

A. Stability

Consider system (1) with the following matrices borrowed from [6]

$$A_1 = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.09 \end{bmatrix}, \ A_2 = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}, \ A_2 = \begin{bmatrix} -1.9 & 0 \\ 0 & -1 \end{bmatrix},$$

$$A_{d1} = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \ A_{d2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ A_{d3} = \begin{bmatrix} -0.9 & 0 \\ -1 & -1.1 \end{bmatrix}.$$

Several previous stability conditions have been applied to this system. For constant delay, the upper bound h of $\tau(t)$ has been found to be 0.0853 by [5], 0.4149 by [6], 0.6142 by [8], 4.2423 by [9]. According to Theorem 1, it is found that the system is robustly stable for $h = 4.0301$, which shows that Theorem 1 yields less conservative stability criterion than [5], [6] and [8]. According to Corollary 1, the upper bound h is 4.2423, which means that Corollary 1 yields better results than those obtained in [5], [6] and [8] and the same with that of [9]. To provide relatively complete information, we calculate the upper bound h for different time-varying cases, listed in Table 1.

TABLE I
DELAY BOUNDS BY DIFFERENT APPROACHES

d	0	0.1	0.5	0.9	Any d
By [5]	0.0853	-	-	-	-
By [6]	0.4149	-	-	-	-
By [8]	0.6142	-	-	-	-
By [9]	4.2423	3.3555	1.8088	0.9670	0.7963
By ours	4.2423	3.3555	1.8088	0.9670	0.7963

Besides, the average CPU time and numbers of iterations by [9] and Corollary 1 are presented in Table 2 in order to compare computational burden involved in solving LMIs. We carry out all LMI computations by Matlab LMI Control Toolbox on a PC (PentiumIII 866 MHz). It can be seen that our method achieves exactly the same upper bound of delay with less computational effort.

TABLE II
COMPUTATIONAL BURDEN BY DIFFERENT APPROACHES

	d	0	0.05	0.1	0.5	0.9
No. of	By [9]	46	48	46	35	25
iteration	By ours	29	29	28	24	17
Average	By [9]	49.6 s	51.1 s	51.8 s	41.6 s	34.2 s
CPU time	By ours	4.82 s	5s	4.83 s	4.66 s	4.61 s

B. Application to flight control

An example of the robust control for a linearized *F-18* aircraft is given to illustrate the effectiveness of our approach The state equations of longitudinal motion of the aircraft are described by

$$\begin{bmatrix}\dot\alpha(t)\\ \dot q(t)\end{bmatrix}=\begin{bmatrix}Z_\alpha^{'}(\rho) & Z_q^{'}(\rho)\\ M_\alpha^{'}(\rho) & M_q^{'}(\rho)\end{bmatrix}\begin{bmatrix}\alpha(t-\tau(t))\\ q(t-\tau(t))\end{bmatrix} +\begin{bmatrix}Z_{\delta E}(\rho) & Z_{\delta PTV}(\rho)\\ M_{\delta E}(\rho) & M_{\delta PTV}(\rho)\end{bmatrix}\begin{bmatrix}\delta_E(t)\\ \delta_{PTV}(t)\end{bmatrix} +\begin{bmatrix}Z_\alpha(\rho) & Z_q(\rho)\\ M_\alpha(\rho) & M_q(\rho)\end{bmatrix}\begin{bmatrix}\alpha(t)\\ q(t)\end{bmatrix}, \tag{24}$$

where $\alpha(t)$ and $q(t)$ represent angle-of-attack (AOA) and pitch rate, respectively; δ_E and δ_{PTV} represent symmetric elevator position and symmetric pitch thrust velocity nozzle position, respectively; $\rho=(M,h)$ denotes Mach and altitude; $\tau(t)$ represents a time-varying flight delay and satisfies $\tau(t)=2+0.3\sin t$. From this information, it is clearly known that h and d are 2.3 and 0.3, respectively.

Denoting $x(t)=\begin{bmatrix}\alpha(t) & q(t)\end{bmatrix}^{\mathrm{T}}$, $u(t)=\begin{bmatrix}\delta_E(t) & \delta_{PTV}(t)\end{bmatrix}^{\mathrm{T}}$, we can rewrite the aircraft system as

$$\dot x(t)=A(\lambda)x(t)+A_d(\lambda)x(t-\tau(t))+B(\lambda)u(t), \tag{25}$$

where $A(\lambda)$, $A_d(\lambda)$ and $B(\lambda)$ are the system matrices defined in (19) and $N=3$. For each operating point, the values of the system matrix parameters, which are obtained by trimming and linearizing a high-fidelity, nonlinear simulation model at level flight for different flight conditions, are given in the Appendix.

By solving (22), the state feedback gain is given in the Appendix. The closed-loop dynamic responses are depicted in Fig. 1–2 with the initial flight state $x(0)=\begin{bmatrix}2 & 1\end{bmatrix}^{\mathrm{T}}$.

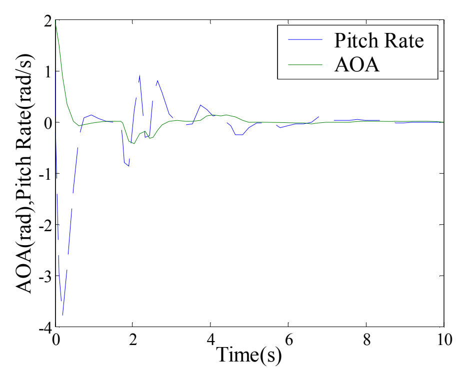

Fig. 1. The state trajectories by [9].

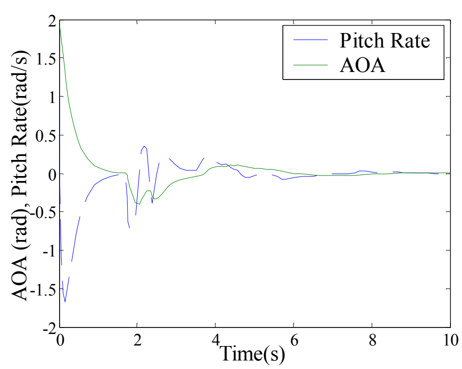

Fig. 2. The state trajectories by our method

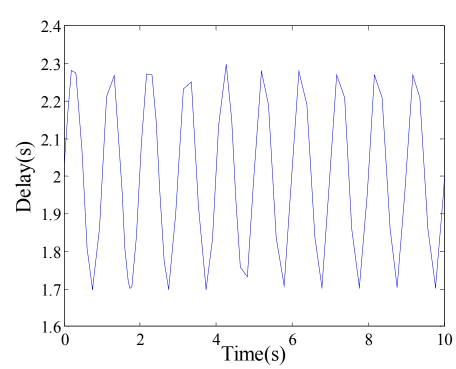

Fig. 3. The time-varying flight delay

The aircraft state trajectories by virtue of the methods proposed in [9] and this paper are shown in Fig. 1 and Fig. 2, respectively. From these figures, it is obvious that by our controller the pitch rate and AOA converge to equilibrium with less overshoots compared with the one designed in [9]. So the controller in this paper can overcome the adverse effect caused by time-varying flight delay which is given in Fig. 3.

V. CONCLUSION

This paper has presented a control strategy using a descriptor system approach to deal with robust stability analysis and control synthesis for a time delay system. The parameters of the system are not exactly given but known to reside in a given polytope. Based on parameter-dependent Lyapunov functions, an efficient delay-dependent stability criterion is derived via LMI approach. Furthermore, the simulation analysis demonstrates that with less computational effort this criterion achieves exactly the same upper bound of delay with that of [9]. By applying the stability criterion to the system with state feedback, a stabilization condition follows immediately. A robust flight control problem is solved which demonstrates the applicability of the stabilization condition.

APPENDIX

For each operating point, system matrices of the aircraft are

$$A_1 = \begin{bmatrix} -1.1750 & 0.9871 \\ -8.4580 & 0.8776 \end{bmatrix}, A_{d1} = \begin{bmatrix} -0.3525 & 0.2961 \\ -2.5374 & -0.2633 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -2.3280 & 0.9831 \\ -30.440 & -1.493 \end{bmatrix}, A_{d2} = \begin{bmatrix} -0.6984 & 0.2949 \\ -9.1320 & -0.4479 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -2.4520 & 0.9856 \\ -38.610 & -1.340 \end{bmatrix}, A_{d3} = \begin{bmatrix} -0.7356 & 0.29570 \\ -11.583 & -0.4020 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.194 & -0.0359 \\ -19.29 & -3.803 \end{bmatrix}, B_2 = \begin{bmatrix} -0.3012 & -0.0587 \\ -38.430 & -7.8150 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} -0.2757 & -0.0523 \\ -37.360 & -7.2470 \end{bmatrix}.$$

The state feedback gain and corresponding matrices are

$$X_{01} = \begin{bmatrix} 15.5900 & -37.5352 \\ -37.5352 & 299.4617 \end{bmatrix}, X_{02} = \begin{bmatrix} 17.6657 & -23.7534 \\ -23.7534 & 430.2555 \end{bmatrix},$$

$$X_{03} = \begin{bmatrix} 17.55030 & -8.39650 \\ -8.39650 & 655.5171 \end{bmatrix}, X_{11} = \begin{bmatrix} 20.7566 & -41.4734 \\ -41.4734 & 242.3862 \end{bmatrix}$$

$$X_{12} = \begin{bmatrix} 27.1775 & -13.4730 \\ -13.4730 & 440.0925 \end{bmatrix}, X_{13} = \begin{bmatrix} 29.3382 & 11.4506 \\ 11.4506 & 569.1790 \end{bmatrix},$$

$$X_{21} = \begin{bmatrix} 0.30070 & -0.38460 \\ -0.38460 & 7.90770 \end{bmatrix}, X_{22} = \begin{bmatrix} 0.28140 & -0.56970 \\ -0.5697 & 3.479200 \end{bmatrix},$$

$$X_{23} = \begin{bmatrix} 0.23430 & -0.50360 \\ -0.50360 & 2.70910 \end{bmatrix}, X_3 = \begin{bmatrix} 2.55870 & 9.54630 \\ -13.2489 & 49.8373 \end{bmatrix}$$

$$Y = \begin{bmatrix} -9.1289 & 121.8597 \\ 33.4714 & -585.7444 \end{bmatrix}, K = \begin{bmatrix} 4.5652 & 1.5707 \\ -23.9859 & -7.1586 \end{bmatrix}.$$

REFERENCES

[1] P. Gahinet, P. Apkarian and M. Chilali, "Affine parameter-dependent Lyapunov functions and real parametric uncertainty," *IEEE Trans. on Automatic Control*, vol. 41, no. 3, pp. 436-442, 1996.

[2] Domingos C. W. Ramos and Pedro L. D. Peres, "An LMI Condition for the Robust Stability of Uncertain Continuous-Time Linear Systems," *IEEE Trans. on Automatic Control*, vol. 47, no. 4, pp. 675-678, 2002.

[3] V. Vesely, "Robust output feedback control synthesis: LMI approach," in *2nd IFAC Conf. Control System Design*, Bratislava, Slovakia, 2003, CD-ROM.

[4] Yong-Yan Cao and Zongli Lin, "A Descriptor System Approach to Robust Stability Analysis and Controller Synthesis," *IEEE Trans. on Automatic Control*, vol. 49, no. 11, pp. 2081-2084, 2004.

[5] Carlos E. de Souza and Xi Li, "Delay-dependent robust H_∞ control of uncertain linear state-delayed systems," *Automatica*, vol. 35, no. 7, pp. 1313-1321, 1999.

[6] Y. Xia and Y. Jia, "Robust stability functionals of state delayed systems with polytopic type uncertainties via parameter-dependent Lyapunov functions," *Internat. J. Control*, vol. 75, no. 10, pp. 1427-1334, 2002.

[7] Li Yu, "Comments andimprovement on Robust control of state delayed systems with polytopic type uncertainties via parameter-dependent Lyapunov functionals," *Systems & Control Letters*, vol. 53, no. 3, pp. 321-323, 2004.

[8] Yuanqing Xia and Yingmin Jia, "Robust control of state delayed systems with polytopic type uncertainties via parameter-dependent Lyapunov functionals," *Systems & Control Letters,* vol. 50, no. 3, pp. 183-193, 2003.

[9] E. Fridman and U. Shaked, "Parameter Dependent Stability and Stabilization of Uncertain Time-Delay Systems," *IEEE Trans. on Automatic Control*, vol. 48, no. 5, pp. 861-866, 2003.

[10] E. Fridman and U. Shaked, "A descriptor system approach to *H* control of linear time-delay systems," *IEEE Trans. on Automatic Control*, vol. 47, no. 2, pp. 253-279, 2002.

[11] H. J. Gao and C. H. Wang, "Comments and Further Results on A Descriptor System Approach to H_∞ Control of Linear Time-Delay Systems," *IEEE Trans. on Automatic Control*, vol. 48, no. 3, pp. 520-525, 2003.

[12] K. Gu, V. L. Kharitonov and J. Chen, *Stability of Time-Delay Systems*. Boston, MA: Birkhauser, 2003.