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Stability and performance analysis for linear systems with actuator and sensor saturations subject to unmodeled dynamics

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Abstract— **This paper considers linear systems subject to sensor and actuator saturations, for which a dynamic output feedback controller has been a priori designed. The effects of unmodeled dynamics appearing as additive uncertainties are studied with respect to the regional (local) stability of the closed-loop system. Constructive conditions based on matrix inequalities, or at least linear matrix inequalities (LMI) are proposed in order to minimize the influence of these unmodeled dynamics on both the estimate of the closed-loop system basin of attraction and on the performance.**

I. INTRODUCTION

In the context of high performance specification, the process of control design can result in control laws producing high magnitude signals. However, due to physical, technological or safety reasons, every system has constraints in its operation, resulting in amplitude or rate saturations in both actuators and sensors [4]. When such limitations are not taken into account during the design process or during the controller implementation, unexpected phenomena can occur. Besides degradation of the closed-loop performance, the system can even become unstable (see, for example, [9], [11], and references therein). For these reasons, systems subject to saturating signals presents numerous challenges for stability and performance control laws design. Recent advances on the domain are presented in [13].

Moreover, in order to simplify the design model, some dynamics are intentionally neglected. For example, the high frequency behavior of actuators and sensors is often removed, without much affecting the design phase. In many cases, especially in aerospace applications (satellite, launcher vehicles, large aircrafts, missiles), further simplifications are necessarily done to reduce the order of the plant to be controlled. This reduction phase usually consists in removing the structural modes whose natural frequency lies outside the bandwidth of the control system. However this elimination technique is a bit tricky since it is very hard to predict whether the removed flexible modes will not finally interact with the control system.

It is then of interest to study the influence of such neglected dynamics. In a linear context many methods exist among which μ-analysis is one of the most popular [7]. In a nonlinear context however, the subject has received less attention. Let us cite [1] where the effect of neglected actuator dynamics is studied, [2] where the backstepping method is modified to be robust face to neglected dynamics and [3] a redesign method is proposed to account for some preliminarily neglected nonlinear dynamics.

This paper focuses on the stability analysis and performance of linear systems subject to input and output saturations, controlled by dynamic output feedback. Unmodeled linear dynamics issued from flexible modes, neglected during control design, are then considered as additive uncertainties. The objective is to analyze their influence on both the estimate of the closed-loop system basin of attraction and on the performance. It is important to underline that the closed-loop system presents a nested saturation term. Therefore, based on the use of some modified sector conditions and appropriate change of variables, stability and performance analysis conditions are stated in regional (local) context. The conditions are expressed as linear matrix inequalities. In particular, unmodeled dynamics issued from flexible modes being considered, the uncertainty on the natural frequency of those flexible modes is also taken into account. Admissible bound on this uncertainty is then characterized.

The paper is organized as follows. The addressed problem is formally stated in section II. Some preliminary results are then given in section III. Section IV is dedicated to the main results of the paper, concerning conditions in regional context. Computational issues are discussed in section V. A numerical example illustrating the application of the approach is also presented. The paper ends by a conclusion giving some perspectives.

Notation. Notation used in the paper is standard. For any vector $x \in \Re^n$, $x \succeq 0$ means that all components of x denoted $x_{(i)}$ are nonnegative. For two vectors $x, y \in \Re^n$, the notation $x \succeq y$ means that $x_{(i)} - y_{(i)} \geq 0$, for all $i = 1, ..., n$. The elements of a matrix $A \in \Re^{m\times n}$ are denoted by $A_{(i,j)}$, $i = 1,...,m$, $j = 1,...,n$. $A_{(i)}$ denotes the ith row of matrix A. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. A' denotes the transpose of A. $Diag(x_1;\ldots;x_n)$ denotes the block-diagonal matrix obtained from vectors or matrices $x_1, ..., x_n$. When no confusion is possible, identity and null matrices will be denoted respectively by I and 0. Furthermore, in the case of partitioned symmetric matrices, the symbol $\star$ denotes generically each of its symmetric blocks. For $v \in \Re^m$, $sat_{v_0}(v) : \Re^m \to \Re^m$ denotes the classical saturation function defined as $(sat_{v_0}(v))_{(i)} = sat_{v_0}(v_{(i)}) = sign(v_{(i)}) \min(v_{0(i)}, |v_{(i)}|)$, $\forall i = 1, ..., m$, where $v_{0(i)} > 0$ denotes the ith magnitude bound.

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II. PROBLEM STATEMENT

Consider the following continuous-time system consisting of a plant with input and output saturations controlled by a dynamic output feedback compensator (for example obtained as in [8]):

$$\left\{\begin{array}{lcl} \dot{x}(t) & = & Ax(t)+Bsat_{u_0}(y_c(t)) \\ y(t) & = & Cx(t) \\ \dot{x}_c(t) & = & A_cx_c(t)+B_csat_{y_0}(y(t)) \\ & & +E_c(sat_{u_0}(y_c(t))-y_c(t)) \\ y_c(t) & = & C_cx_c(t)+D_csat_{y_0}(y(t)) \\ z(t) & = & C_zx(t) \end{array}\right. \quad (1)$$

where $x \in \Re^n$ is the state, $y_c \in \Re^m$ is the input of the actuator and the controller output, $x_c \in \Re^{n_c}$ is the controller state, $sat_{y_0}(y) \in \Re^p$ is the controller input, $z \in \Re^l$ is the regulated output for performance purpose. Matrices A, B, C, A_c, B_c, C_c, D_c and E_c are constant matrices of appropriate dimensions. Note that E_c corresponds to a static anti-windup gain [10]. The levels of the saturation on both the actuator and sensor outputs are given respectively by the componentwise positive vectors $u_0 \in \Re^m$ and $y_0 \in \Re^p$.

The design of the dynamic output feedback compensator has been done for the linear case (without saturation) or for the nonlinear case (with saturation) but without taking into account any unmodeled dynamics especially issued from neglected flexible modes described as follows:

$$\left\{\begin{array}{lcl} \dot{x}_f(t) & = & A_fx_f(t)+B_fsat_{u_0}(y_c(t)) \\ y_f(t) & = & C_fx_f(t) \end{array}\right. \quad (2)$$

where $x_f \in \Re^{n_f}$ is the state, $y_f \in \Re^p$ is the output and $sat_{u_0}(y_c(t))$ is the input of the neglected dynamics. In the case of $r = n_f/2$ flexible modes, the matrix A_f is defined as:

$$A_f = Diag(\omega_1 A_{f1},...,\omega_r A_{fr})$$

with

$$A_{fi} = \left[\begin{array}{cc} 0 & 1 \\ -1 & -2\delta_i \end{array}\right]$$

and $0 < \delta_i < 1$. While B_f and C_f, which in an appropriate basis will not depend on the natural frequencies ω_i are computed such that the static gain of the system (2) is null. Such matrices are not detailed here, since they highly depend on the application to be treated. An example is given in section V.

By considering the connection of systems (1) and (2), the complete closed-loop system reads:

$$\left\{\begin{array}{lcl} \dot{x}(t) & = & Ax(t)+Bsat_{u_0}(y_c(t)) \\ y(t) & = & Cx(t)+C_fx_f(t) \\ \dot{x}_c(t) & = & A_cx_c(t)+B_csat_{y_0}(y(t)) \\ & & +E_c(sat_{u_0}(y_c(t))-y_c(t)) \\ y_c(t) & = & C_cx_c(t)+D_csat_{y_0}(y(t)) \\ \dot{x}_f(t) & = & A_fx_f(t)+B_fsat_{u_0}(y_c(t)) \\ z(t) & = & C_zx(t) \end{array}\right. \quad (3)$$

and is depicted in figure 1.

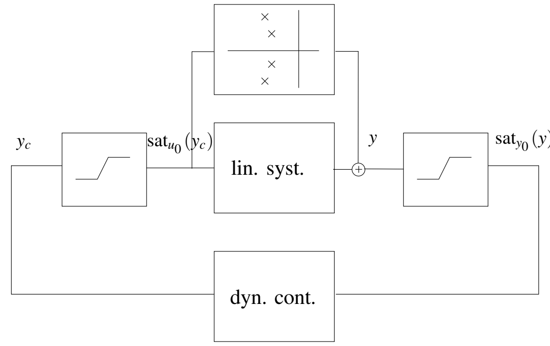

Fig. 1. Closed-loop plant with additive flexible dynamics and input/output saturations.

The basin of attraction of system (3), denoted $\mathcal{B}_a$, is defined as the set of all $(x, x_c, x_f) \in \Re^n \times \Re^{n_c} \times \Re^{n_f}$ such that for $(x(0), x_c(0), x_f(0)) \in \mathcal{B}_a$ the corresponding trajectory converges asymptotically to the origin. Note, however, that the exact characterization of the basin of attraction is in general not possible. In this case, it is important to obtain estimates of the basin of attraction. Regions of asymptotic stability can represent such estimates. On the other hand, in some practical applications one can be interested in ensuring the stability for a given set of admissible initial conditions. This set can be seen as a practical operation region for the system, or a region where the states of the system can be brought by the action of temporary disturbances.

In this work, we are particularly interested in evaluating the basin of attraction of the complete closed-loop system. More precisely, we want to guarantee that taking into account *a posteriori* the unmodeled dynamics does not degrade too much the size of the region of stability beforehand in the directions $(x, x_c) \in \Re^n \times \Re^{n_c}$, obtained neglecting it. Similarly, in presence of the unmodeled dynamics, we want to minimize the potential degradation on the performance, this last one being measured from the upper bound on the energy of the regulated output z.

The problem we intend to solve can be summarized as follows.

Problem 1: Given the unmodeled dynamics (2):

1) **Stability.** Minimize the degradation of the region of stability of the complete closed-loop system (3) in the directions $(x, x_c) \in \Re^n \times \Re^{n_c}$ and characterize the associate admissible uncertainty on the natural frequencies ω_i, $i = 1, \ldots, r = n_f/2$.
2) **Performance.** Regarding the energy of the regulated output z, minimize its degradation and characterize the associate admissible uncertainty on the natural frequencies ω_i, $i = 1, \ldots, r = n_f/2$.

III. PRELIMINARIES

Let us define the augmented state vector

$$\xi = \left[\begin{array}{ccc} x' & x_c' & x_f' \end{array}\right]' = \left[\begin{array}{cc} \bar{\xi}' & x_f' \end{array}\right]' \in \Re^{n+n_c+n_f} \quad (4)$$

and the two nonlinearities $\phi_{y_0} = sat_{y_0}(y(t)) - y(t)$ and $\phi_{u_0} = sat_{u_0}(y_c(t)) - y_c(t)$:

$$\begin{array}{rcl} \phi_{y_0} & = & sat_{y_0}(C_1\xi) - C_1\xi \\ \phi_{u_0} & = & sat_{u_0}(C_2\xi + D_1\phi_{y_0}) - (C_2\xi + D_1\phi_{y_0}) \end{array}$$

with

$$\begin{array}{c} C_1 = \left[\begin{array}{ccc} C & 0 & C_f \end{array} \right] \\ C_2 = \left[\begin{array}{ccc} D_cC & C_c & D_cC_f \end{array} \right]; D_1 = D_c \end{array} \tag{5}$$

These nonlinearities are nested decentralized dead-zone functions since ϕ_{u_0} depends on ϕ_{y_0}. Hence, Lemma 1 in [14] applies.

By using the augmented state ξ defined in (4), the closed-loop system (3) can be written as:

$$\left\{ \begin{array}{rcl} \dot{\xi}(t) & = & A_1\xi(t) + B_1\phi_{y_0} + B_2\phi_{u_0} \\ y(t) & = & C_1\xi(t) \\ y_c(t) & = & C_2\xi(t) + D_1\phi_{y_0} \\ z(t) & = & C_3\xi(t) \end{array} \right. \tag{6}$$

with matrices C_1, C_2 and D_1 defined in (5) and

$$A_1 = \left[\begin{array}{ccc} A + BD_cC & BC_c & BD_cC_f \\ B_cC & A_c & B_cC_f \\ B_fD_cC & B_fC_c & A_f + B_fD_cC_f \end{array} \right]$$

$$B_1 = \left[\begin{array}{c} BD_c \\ B_c \\ B_fD_c \end{array} \right]; B_2 = \left[\begin{array}{c} B \\ E_c \\ B_f \end{array} \right]; C_3 = \left[\begin{array}{ccc} C_z & 0 & 0 \end{array} \right]$$

Let us give a preliminary result regarding the system before taking into account the unmodeled dynamics, i.e. system (1). For this purpose, consider the following sub-matrices A_0, B_{01}, B_{02}, C_{01}, C_{02} and C_{03}:

$$\begin{array}{c} A_0 = \left[\begin{array}{cc} A + BD_cC & BC_c \\ B_cC & A_c \end{array} \right] \\ B_{01} = \left[\begin{array}{c} BD_c \\ B_c \end{array} \right]; B_{02} = \left[\begin{array}{c} B \\ E_c \end{array} \right]; C_{01} = \left[\begin{array}{cc} C & 0 \end{array} \right] \\ C_{02} = \left[\begin{array}{cc} D_cC & C_c \end{array} \right]; C_{03} = \left[\begin{array}{cc} C_z & 0 \end{array} \right] \end{array} \tag{7}$$

Proposition 1: If there exist a symmetric positive definite matrix $W \in \Re^{(n+n_c)\times(n+n_c)}$, two diagonal positive definite matrices $S_1 \in \Re^{p\times p}$, $S_2 \in \Re^{m\times m}$, two matrices $Y_1 \in \Re^{p\times(n+n_c)}$, $Y_2 \in \Re^{m\times(n+n_c)}$, and a positive scalar γ satisfying the following LMIs:

$$\left[\begin{array}{cccc} WA_0' + A_0W & B_{01}S_1 - WC_{01}' + Y_1' & B_{02}S_2 - WC_{02}' + Y_2' & WC_{03}' \\ \star & -2S_1 & -S_1D_1' & 0 \\ \star & \star & -2S_2 & 0 \\ \star & \star & \star & -\gamma I \end{array} \right] < 0 \tag{8}$$

$$\left[\begin{array}{cc} W & Y_{1(i)}' \\ \star & y_{0(i)}^2 \end{array} \right] \geq 0, i = 1, ..., p \tag{9}$$

$$\left[\begin{array}{cc} W & Y_{2(i)}' \\ \star & u_{0(i)}^2 \end{array} \right] \geq 0, i = 1, ..., m \tag{10}$$

then system (1) is asymptotically stable for all initial conditions in the set

$$\mathcal{E}(W) = \{\bar{\xi} \in \Re^{n+n_c}; \bar{\xi}'W^{-1}\bar{\xi} \leq 1\} \tag{11}$$

Moreover, the energy of the regulated output satisfies:

$$\int_0^\infty z(t)'z(t)dt \leq \gamma \tag{12}$$

Proof: The satisfaction of relations (8), (9) and (10) guarantees that one gets [14]: $\dot{V}(\bar{\xi}) + \frac{1}{\gamma}z'z \leq \dot{V}(\bar{\xi}) + \frac{1}{\gamma}z'z - 2\phi_{y_0}S_1^{-1}(\phi_{y_0} - (Y_1W^{-1} - C_{01})\bar{\xi}) - 2\phi_{u_0}S_2^{-1}(\phi_{u_0} - (Y_2W^{-1} - C_{02})\bar{\xi} + D_1\phi_{y_0}) < 0$, for any $\bar{\xi} \in \mathcal{E}(W)$. ■

IV. Main results

Consider now the complete closed-loop system (3) or (6). First, suppose that the matrices A_f, B_f and C_f of the flexible modes are given without uncertainty.

Proposition 2: Given $W \in \Re^{(n+n_c)\times(n+n_c)}$ and $\gamma > 0$ solutions to Proposition 1. If there exist a symmetric positive definite matrix $X_2 \in \Re^{n_f\times n_f}$, two diagonal positive definite matrices $S_1 \in \Re^{p\times p}$, $S_2 \in \Re^{m\times m}$, three matrices $X_1 \in \Re^{(n+n_c)\times n_f}$, $Y_1 = \left[\begin{array}{cc} Y_{11} & Y_{12} \end{array} \right] \in \Re^{p\times(n+n_c+n_f)}$, $Y_2 = \left[\begin{array}{cc} Y_{21} & Y_{22} \end{array} \right] \in \Re^{m\times(n+n_c+n_f)}$, and three positive scalars α, β and σ satisfying:

$$M_0 = \left[\begin{array}{ccccc} M_1 & M_2 & M_3 & M_4 & M_5 \\ \star & M_6 & M_7 & M_8 & M_9 \\ \star & \star & -2S_1 & -S_1D_1' & 0 \\ \star & \star & \star & -2S_2 & 0 \\ \star & \star & \star & \star & -\gamma\beta I \end{array} \right] < 0 \tag{13}$$

$$\left[\begin{array}{ccc} \alpha W & X_1 & Y_{11(i)}' \\ \star & X_2 & Y_{12(i)}' \\ \star & \star & y_{0(i)}^2 \end{array} \right] \geq 0, i = 1, ..., p \tag{14}$$

$$\left[\begin{array}{ccc} \alpha W & X_1 & Y_{21(i)}' \\ \star & X_2 & Y_{22(i)}' \\ \star & \star & u_{0(i)}^2 \end{array} \right] \geq 0, i = 1, ..., m \tag{15}$$

$$\left[\begin{array}{cc} (\alpha - \sigma)W & X_1 \\ \star & X_2 \end{array} \right] \geq 0 \tag{16}$$

with

$$\begin{array}{l} M_1 = \alpha(WA_0' + A_0W) + B_{01}C_fX_1' + X_1C_f'B_{01}' \\ M_2 = A_0X_1 + B_{01}C_fX_2 + \alpha WC_{02}'B_f' \\ \quad + X_1(A_f + B_fD_cC_f)' \\ M_3 = B_{01}S_1 - \alpha WC_{01}' - X_1C_f' + Y_{11}' \\ M_4 = B_{02}S_2 - \alpha WC_{02}' - X_1C_f'D_c' + Y_{21}' \\ M_5 = \alpha WC_{03}' \\ M_6 = B_fC_{02}X_1 + X_1'C_{02}'B_f' + X_2(A_f + B_fD_cC_f)' \\ \quad + (A_f + B_fD_cC_f)X_2 \\ M_7 = B_fD_cS_1 - X_1'C_{01}' - X_2C_f' + Y_{12}' \\ M_8 = B_fS_2 - X_1'C_{02}' - X_2C_f'D_c' + Y_{22}' \\ M_9 = X_1'C_{03}' \end{array} \tag{17}$$

then:

1) System (6) is asymptotically stable for all initial conditions in the set

$$\mathcal{E}(Q) = \{\xi \in \Re^{n+n_c+n_f}; \xi' Q^{-1} \xi \leq 1\} \quad (18)$$

with

$$Q = \begin{bmatrix} \alpha W & X_1 \\ \star & X_2 \end{bmatrix} > 0 \quad (19)$$

2) The output energy of the regulated output satisfies:

$$\int_0^{\infty} z(t)' z(t) dt \leq \beta\gamma \quad (20)$$

3) Moreover, the degradation relative to the size of the region of stability in the directions $(x, x_c) \in \Re^n \times \Re^{n_c}$ and the degradation of the performance can be measured via the scalars σ and β, respectively.

Proof: Let us first note that the matrix A_1 of system (6) reads:

$$A_1 = \begin{bmatrix} A_0 & B_{01} C_f \\ B_f C_{02} & A_f + B_f D_c C_f \end{bmatrix}$$

with matrices A_0, B_{01} and C_{02} defined in (7). Relations (13), (14) and (15) are obtained by considering the quadratic Lyapunov function $V(\xi) = \xi' Q^{-1} \xi$, with matrix Q defined in (19), and by invoking similar arguments like in Proposition 1 with respect to the complete closed-loop system (6): $\dot{V}(\xi) + \frac{1}{\beta\gamma} z'z \leq \dot{V}(\xi) + \frac{1}{\beta\gamma} z'z - 2\phi_{y_0} S_1^{-1}(\phi_{y_0} - (Y_1 Q^{-1} - C_1)\xi) - 2\phi_{u_0} S_2^{-1}(\phi_{u_0} - (Y_2 Q^{-1} - C_2)\xi + D_1 \phi_{y_0}) < 0$, for any $\xi \in \mathcal{E}(Q)$. Hence points 1) and 2) readily follow.

Moreover, it is important to note that for $x_f = 0$, the region of stability is defined from (18) by

$$\begin{bmatrix} x' & x_c' & 0 \end{bmatrix} \begin{bmatrix} \alpha W & X_1 \\ \star & X_2 \end{bmatrix}^{-1} \begin{bmatrix} x \\ x_c \\ 0 \end{bmatrix} = \bar{\xi}' Z \bar{\xi} \leq 1$$

with $\bar{\xi}' = \begin{bmatrix} x' & x_c' \end{bmatrix}$, $Z = (\alpha W)^{-1} + (\alpha W)^{-1} X_1 (X_2 - X_1'(\alpha W)^{-1} X_1)^{-1} X_1' (\alpha W)^{-1} = (\alpha W - X_1 X_2^{-1} X_1')^{-1}$. Thus, from (16) one gets:

$$\sigma W \leq (\alpha W - X_1 X_2^{-1} X_1') \leq \alpha W$$

or equivalently

$$(\alpha W)^{-1} \leq Z \leq (\sigma W)^{-1}$$

This last inequality is equivalent to:

$$\bar{\xi}' Z \bar{\xi} \leq \bar{\xi}' (\sigma W)^{-1} \bar{\xi} \leq 1$$

Therefore, the ellipsoid $\mathcal{E}_1 = \{\bar{\xi} \in \Re^{n+n_c}; \bar{\xi}'(\sigma W)^{-1}\bar{\xi} \leq 1\}$ is included in the ellipsoid $\mathcal{E}_2 = \{\bar{\xi} \in \Re^{n+n_c}; \bar{\xi}' Z \bar{\xi} \leq 1\}$, which corresponds to the region of stability defined from (20) for $x_f = 0$. Then, a way to measure the degradation of the region of stability in the direction of $\bar{\xi}$ can be done through the value of the positive scalar σ.

Similarly, a way to measure the degradation of the upper bound on the energy of the regulated output with respect to (12) can be done through the value of the positive scalar β in (20). ■

Remark 1: Note that since A_f, B_f, C_f, W and γ are given, relations (13), (14), (15) and (16) are LMIs in the decision variables.

Let us consider the presence of uncertainty on the frequencies ω_i. At this aim, two ways to represent the uncertainty are considered: norm-bounded and polytopic representation. A detailed description of such kinds of uncertainties can be found in [5], [6], [12].

When the frequencies ω_i are supposed uncertain as $\omega_i = \omega_{0i} + \Delta\omega_i$, we can consider that the matrix A_f of system (2) can be written as follows

$$A_f = A_{f0} + FG \quad (21)$$

with

$$\begin{array}{rcl} A_{f0} & = & Diag(\omega_{01} A_{f1}; \ldots; \omega_{0r} A_{fr}) \\ F & = & Diag(\Delta\omega_1 I_2; \ldots; \Delta\omega_r I_2) \\ G & = & Diag(A_{f1}; \ldots; A_{fr}) \end{array}$$

matrices B_f and C_f are kept unchanged. Then we have

$$\left\{ \begin{array}{rcl} \dot{\xi}(t) & = & \mathcal{A}_1 \xi(t) + B_1 \phi_{y_0} + B_2 \phi_{u_0} \\ y(t) & = & C_1 \xi(t) \\ y_c(t) & = & C_2 \xi(t) + D_1 \phi_{y_0} \\ z(t) & = & C_3 \xi(t) \end{array} \right.$$

where

$$\mathcal{A}_1 = A_1 + \tilde{F} \tilde{A}$$

with

$$\begin{array}{rcl} \tilde{F} & = & \begin{bmatrix} 0 & 0 & F \end{bmatrix}' \\ \tilde{A} & = & \begin{bmatrix} 0 & 0 & G \end{bmatrix}. \end{array}$$

The following result gives a solution to Problem 1.

Proposition 3: Given $W \in \Re^{(n+n_c)\times(n+n_c)}$ and $\gamma > 0$ solutions to Proposition 1. If there exist a symmetric positive definite matrix $X_2 \in \Re^{n_f \times n_f}$, two diagonal positive definite matrices $S_1 \in \Re^{p\times p}$, $S_1 \in \Re^{m\times m}$, three matrices $X_1 \in \Re^{(n+n_c)\times n_f}$, $Y_1 = \begin{bmatrix} Y_{11} & Y_{12} \end{bmatrix} \in \Re^{p\times(n+n_c+n_f)}$, $Y_2 = \begin{bmatrix} Y_{21} & Y_{22} \end{bmatrix} \in \Re^{m\times(n+n_c+n_f)}$, three positive scalars α, β, σ, and positive scalars θ_i, $i = 1, \ldots, n_f/2$ satisfying (14), (15), (16) and

$$\begin{bmatrix} M_0 & \mathcal{D} & \mathcal{E}' \\ \star & -I & 0 \\ \star & \star & -\Theta \end{bmatrix} < 0 \quad (22)$$

with

$$\begin{array}{rcl} \mathcal{E} & = & \begin{bmatrix} \tilde{A}Q & 0_{n_f \times p+m+l} \end{bmatrix} \\ \mathcal{D}' & = & \begin{bmatrix} \begin{bmatrix} 0_{n_f \times n+n_c} & I \end{bmatrix} & 0_{n_f \times p+m+l} \end{bmatrix} \\ \Theta & = & (Diag(\theta_1 I_2; \ldots; \theta_r I_2)) \end{array}$$

then:

1) System (6) is asymptotically stable in the set (18) with Q as in (19) for any value of $\omega_i \in [\omega_{0i} - \Delta\omega_i, \omega_{0i} + \Delta\omega_i]$ with $\Delta\omega_i \leq \sqrt{\frac{1}{\theta_i}}$, $i = 1, \ldots, n_f/2$.
2) The output energy of the regulated output satisfies (20).
3) The degradation relative to the size of the region of stability in the directions $(x, x_c) \in \Re^n \times \Re^{n_c}$ and the

degradation of the performance can be measured via the scalars σ and β, respectively.

Proof: If we consider A_f given by (21), the terms M_2 and M_6 in (13) are modified and to verify the stability of the uncertain system, the following condition must be verified:

$$M_0 + \mathcal{D}F\mathcal{E} + \mathcal{E}'F'\mathcal{D}' < 0 \quad (23)$$

with

$$\begin{array}{rcl} \mathcal{E} & = & \begin{bmatrix} \tilde{A}Q & 0_{n_f \times p+m+l} \end{bmatrix} \\ \mathcal{D}' & = & \begin{bmatrix} \begin{bmatrix} 0_{n_f \times n+n_c} & I \end{bmatrix} & 0_{n_f \times p+m+l} \end{bmatrix} \end{array}$$

Using the fact that [12]

$$\mathcal{D}F\mathcal{E} + \mathcal{E}'F'\mathcal{D}' \leq \mathcal{D}\mathcal{D}' + \mathcal{E}'F'F\mathcal{E}$$

and imposing $F'F \leq \Theta^{-1}$, we have

$$M_0 + \mathcal{D}F\mathcal{E} + \mathcal{E}'F'\mathcal{D}' \leq M_0 + \mathcal{D}\mathcal{D}' + \mathcal{E}'\Theta^{-1}\mathcal{E}$$

Then the satisfaction of (22) guarantees that (23) is verified.

Then if (22), (14), (15) and (16) are satisfied, points 1) and 2) readily follow. As in the case without uncertainties, a measure on the degradation of the upper bound on the energy of the regulated output with respect to (12) is given by the value of β in (20). As in Proposition 2 the satisfaction of (16) implies 3). ■

Considering the bounds of uncertainties $\Delta\omega_i$, the system can also be represented under polytopic form, that is

$$A_{fi}(\eta) = \sum_{j=1}^{n_v} \eta_j A_{fij} \quad (24)$$

where n_v is the number of vertices of the uncertainty and parameters η belong to the set

$$\mathcal{U} = \left\{ \eta \in \Re^{n_v} : \eta_j \geq 0, \sum_{j=1}^{n_v} \eta_j = 1, j = 1 \ldots n_v \right\}$$

An immediate result can be obtained from Proposition 1.

Proposition 4: Given $W \in \Re^{(n+n_c)\times(n+n_c)}$ and $\gamma > 0$ solutions to Proposition 1. If there exist a symmetric positive definite matrix $X_2 \in \Re^{n_f \times n_f}$, two diagonal positive definite matrices $S_1 \in \Re^{p\times p}$, $S_1 \in \Re^{m\times m}$, three matrices $X_1 \in \Re^{(n+n_c)\times(n_f)}$, $Y_1 = \begin{bmatrix} Y_{11} & Y_{12} \end{bmatrix} \in \Re^{p\times(n+n_c+n_f)}$, $Y_2 = \begin{bmatrix} Y_{21} & Y_{22} \end{bmatrix} \in \Re^{m\times(n+n_c+n_f)}$, and three positive scalars α, β and σ satisfying (14), (15), (16) and

$$\begin{bmatrix} M_1 & M_{2j} & M_3 & M_4 & M_5 \\ \star & M_{6j} & M_7 & M_8 & M_9 \\ \star & \star & -2S_1 & -S_1D_1' & 0 \\ \star & \star & \star & -2S_2 & 0 \\ \star & \star & \star & \star & -\gamma\beta I \end{bmatrix} < 0,$$

$$j = 1 \ldots n_v$$

with

$$\begin{array}{l} M_{2j} = A_0X_1 + B_{01}C_fX_2 + \alpha W C_{02}'B_f' \\ +X_1(A_{fj} + B_fD_cC_f)' \\ M_{6j} = B_fC_{02}X_1 + X_1'C_{02}'B_f' + X_2(A_{fj} + B_fD_cC_f)' \\ +(A_{fj} + B_fD_cC_f)X_2 \end{array}$$

and M_1, M_3, M_4, M_5, M_7, M_8 and M_9 given by (17). Then points 1), 2) and 3) given in Proposition 2 are satisfied for the uncertain system defined from (24).

V. Computational and Numerical Issues

Based on Proposition 2, a way to minimize the degradation of the region of stability in the direction of $\bar{\xi}$ consists in maximizing σ. Indeed, we would like to obtain σ very close to 1. Similarly a way to minimize the degradation of the upper bound on the energy of the regulated output with respect to (12) can be done by minimizing β in (20). Another approach, derived from Proposition 4, is the maximization of uncertainties $\Delta\omega_i$ which can be done by imposing bounds on the degradations β and σ.

Consider the controlled system (1) defined by matrices

$$\begin{array}{c} A = \begin{bmatrix} -1 & 0.3 \\ 0 & 0.5 \end{bmatrix}; B = \begin{bmatrix} 0.5 \\ -10 \end{bmatrix}; C = \begin{bmatrix} 1 & 2 \end{bmatrix}; \\ A_c = \begin{bmatrix} -4 & 1 \\ 0 & -8 \end{bmatrix}; B_c = \begin{bmatrix} -1 \\ -0.5 \end{bmatrix}; C_c = \begin{bmatrix} 0.3 & -2 \end{bmatrix}; \\ D_c = 1; E_c = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}; C_z = \begin{bmatrix} 3 & 0 \end{bmatrix} \end{array} \quad (25)$$

and the saturation limits $u_0 = 0.3$ and $y_0 = 0.5$.

Applying conditions from Proposition 1 and maximizing the size of the set defined by (11) we obtained the following matrix W and performance index γ:

$$W = \begin{bmatrix} 0.7056 & -0.3263 & -0.7862 & -0.1033 \\ -0.3263 & 0.8061 & -0.0699 & -0.0755 \\ -0.7862 & -0.0699 & 63.9651 & 12.4161 \\ -0.1033 & -0.0755 & 12.4161 & 2.8070 \end{bmatrix}$$

$$\gamma = 2.9287$$

Considering an unmodeled flexible dynamics with $\omega_1 = 20$ and $\delta_1 = 0.1$, which gives

$$A_f = \begin{bmatrix} 0 & 20 \\ -20 & -4 \end{bmatrix} \quad (26)$$

and

$$B_f = \begin{bmatrix} 0 \\ 3 \end{bmatrix}; C_f = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad (27)$$

we test conditions from Proposition 2 keeping the performance index γ (imposing $\beta = 1$) and trying to maximize σ. The values for the degradation on the estimate of the region of attraction is given by

$$\sigma = 0.8634$$

Applying conditions from Proposition 3, imposing bounds on β and σ given respectively as $\beta < 10$ and $\sigma > 0.5$ the maximum value obtained for $\Delta\omega_1$ was

$$\Delta\omega_1 = 0.7692$$

and the corresponding degradations β on the performance and σ on the region of stability are given by

$$\beta = 7.5205 \ ; \ \sigma = 0.5$$

Consider an uncertain system in the form of (2) where the uncertainty is given by $\Delta\omega_1 \in [1,9]$ and $\delta_1 = 0.5$ and therefore described under polytopic form (24) , that is

$$A_{f11} = \begin{bmatrix} 0 & 25 \\ -25 & -5 \end{bmatrix}; A_{f12} = \begin{bmatrix} 0 & 15 \\ -15 & -3 \end{bmatrix}; \quad (28)$$

with B_f and C_f defined in (27).

We test conditions given in Proposition 4 with respect to system (25), (27), (28) , imposing $\beta < 5$ and $\sigma > 0.2$. The system remains stable for any value of the uncertainty. The degradations on the performance and on the region of stability are given by

$$\beta = 4.1744 \ ; \ \sigma = 0.7427.$$

Figure 3 shows the time response of the saturated input $\text{sat}_{u_0}(u)$ and of the output $\text{sat}_{y_0}(y)$ to the initial state $\xi(0)' = \begin{bmatrix} 0.35 & 0.87 & 0 & 0 \end{bmatrix}$ for the system (1) considering three cases: without neglected dynamics (dark solid line), the dynamics described by (26)-(27) (grey dashed line) and the vertex of the polytope corresponding to $\omega_1 = 25$ from (28) (grey solid line). A stable behavior is identified, as expected, when the flexible modes are influencing the system. Note that the responses of the input and of the output remain saturated for $t < 0.3$ for the three cases.

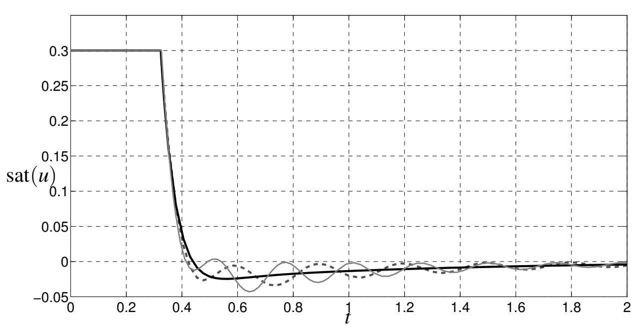

Fig. 2. Control input for nominal model and different flexible modes.

VI. CONCLUSION

This paper considered linear systems with sensor and actuator saturations, subject to unmodeled dynamics appearing as additive uncertainties. The influence of neglected flexible modes on both the estimate of the closed-loop stability region and the performance was studied. Conditions based on

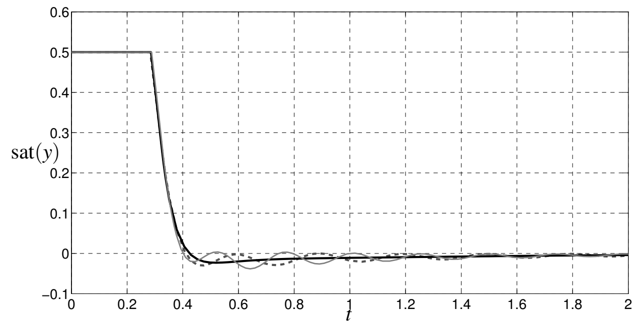

Fig. 3. Output considering nominal model and admissible neglected dynamics.

quadratic Lyapunov function and modified sector conditions are proposed in order to minimize the degradation of the size of the region of stability and of the performance. Admissible uncertainties on the natural frequencies of flexible modes were also characterized.

When dealing with such a problem, there are still several open issues. In particular, a problem of interest is to take into account not only additive flexible modes but any kind of neglected dynamics or parametric uncertainties. For this purpose, a possible direction which is currently under investigation is to adapt some results to fit in with the LFT framework.

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