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Uniform Practical Output-Feedback Stabilization of Spacecraft Relative Rotation

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Abstract— **In this paper we present a solution to the problem of tracking relative rotation in a leader-follower spacecraft formation using feedback from relative attitude only. The controller incorporates a linear approximation filter to achieve knowledge of angular velocity, and the controller structure renders the equilibrium points of the closed-loop system uniformly practically asymptotically stable (UPAS). That is, the state errors in the closed-loop system are proved to converge from any initial conditions in a region of attraction to a ball in close vicinity of the origin in a stable way, and this ball can be diminished arbitrarily by increasing the gains in the control law. The controller assumes boundedness of angular velocities of spacecraft relative to an inertial frame. Simulation results of a leader-follower spacecraft formation using the proposed controller structure are also presented.**

I. INTRODUCTION

Synchronization, coordination and cooperative control are new and promising trends within mechanical systems technology. Replacing complex single units with several simpler and less expensive agents makes it possible to achieve larger operational areas with greater flexibility and performance. In the space industry, the concept makes the way for new and better applications, such as improved monitoring of the Earth and its surrounding atmosphere, geodesy, deep-space imaging and exploration and even in-orbit spacecraft servicing and maintenance. However, the advantages of using spacecraft formations come at a cost of increased control complexity and technological challenges. Formation flying introduces a control problem with strict and time-varying boundaries on spacecraft reference trajectories, and requires detailed knowledge and tight control of relative distances and velocities for participating spacecraft.

In the last decade research on output-feedback control of relative spacecraft motion has mainly focused on the relative translation case, while relative rotation has received scant focus. Attitude output-feedback control of single spacecraft has received larger interest over the last few years –*cf.* [1], [2], [3]. Furthermore, some results on output-feedback relative translation control have been extended to 6DOF motion directly; in [4], a nonlinear tracking controller for both translation and rotation was presented, including an adaptation law to account for unknown mass and inertia parameters of the spacecraft. The controller ensures asymptotic convergence of position and velocity errors for all initial states; the proof relies on a standard signal-chasing analysis and Barbalat's lemma. Based on the latter reference, semi-global asymptotic convergence of relative translation and rotation errors was claimed in [5] for an adaptive output-feedback controller using relative position and attitude only. As stated above, output-feedback control of relative rotation in spacecraft formation has received scant detail, and a usual approach in this matter is to introduce large model-based nonlinear observers to estimate the relative angular velocity –*cf.* [6], [7]. For the relative translation cases, different velocity filters have been suggested that, if not supplying the controller with the correct velocity, at least provides enough velocity information to solve the control problem. In addition, for the relative rotation and 6DOF motion cases, most results aim for global or almost global results, which involves controlling the relative attitude to one isolated equilibrium point in the closed-loop system.

It must be stressed that the qualifier *global* is used, for different reasons, with an abuse of notation for systems evolving on the rotational sphere. Firstly, global (asymptotic) stability *cannot* be achieved in general (*cf.* [8]), since there are often more than one equilibrium point in the system's state space due to topological constraints in the mapping from the rotational sphere to $\mathbb{R}^n$. For instance, in quaternion coordinates as we use in this paper, there exist two equilibria but which correspond to the *same* physical configuration. Secondly, even if we consider the two equilibria as the same point the term global refers to the whole state space $\mathbb{R}^n$ –*cf.* [9].

In this paper, we solve the problem of output-feedback relative rotation *tracking* control for a leader-follower spacecraft formation. The spacecraft model that we use is expressed in quaternion coordinates hence, enters in the case of study described above. However, a fact that is often neglected and that we use in this paper is that multiple equilibrium cases can be exploited to achieve shorter rotation paths on attitude manoeuvres, by working on different equilibrium points (in the quaternion-space) for different manoeuvres. Our main result extends previous work on attitude control –*cf.* [10], [11], [2]. We assume that only relative positions are measured; an approximate-differentiation filter as in [12] is used to compensate for the lack of velocity measurements. In a strict sense we show that each of the resulting closed-loop system is uniformly practically asymptotically stable (UPAS), meaning that the state errors in the closed-loop system converge from any initial condition within a domain

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of attraction to a ball in close vicinity of the origin in a stable way; moreover, the radius of this ball can be arbitrarily diminished by increasing the gains in the control law.

The rest of the paper is organized as follows: Section II presents the notation and stability definitions. Section III defines the different reference frames used and presents the mathematical models of relative attitude dynamics and kinematics in a leader-follower spacecraft formation. The control solution is presented in Section IV, and simulation results of a system with the derived controller are presented in Section V. Concluding remarks are given in Section VI.

II. Mathematical Preliminaries

In the following, we denote by $\dot{\mathbf{x}}$ the time derivative of a vector $\mathbf{x}$, *i.e.* $\dot{\mathbf{x}} = d\mathbf{x}/dt$. Moreover, $\ddot{\mathbf{x}} = d^2\mathbf{x}/dt^2$. We denote by $\mathbf{x}(t, t_0, \mathbf{x}_0)$ the solution to the nonlinear differential equation $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ with initial conditions $(t_0, \mathbf{x}_0)$. We denote by $\|\cdot\|$ the Euclidian norm of a vector and the induced $\mathcal{L}_2$ norm of a matrix. We denote by $\mathcal{B}_\delta$ the closed ball in $\mathbb{R}^n$ of radius δ, *i.e.* $\mathcal{B}_\delta := \{\mathbf{x} \in \mathbb{R}^n | \ \|\mathbf{x}\| \leq \delta\}$. For such a ball we denote

$$\|\mathbf{z}\|_\delta = \inf_{\mathbf{x} \in \mathcal{B}_\delta} \|\mathbf{z} - \mathbf{x}\| . \tag{1}$$

A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is strictly increasing and $\alpha(0) = 0$. Moreover, α is of class $\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if, in addition, $\alpha(s) \to \infty$ as $s \to \infty$. A continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{L}$ ($\sigma \in \mathcal{L}$), if it is strictly decreasing and $\sigma(s) \to 0$ as $s \to \infty$. When the context is sufficiently explicit, we may omit to write arguments of functions, vectors and matrices.

Our main results rely on the following stability definitions for parametrised[1] nonlinear systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \theta) \tag{2}$$

where $\mathbf{f}(t, \mathbf{x}, \theta) : \mathbb{R}_{\geq 0} \times \mathbb{D} \times \mathbb{R}^m \to \mathbb{D}$ is locally Lipschitz in $\mathbf{x}$ and piecewise continuous in t for each θ.

Definition 1 (UPAS): Let $\Theta \subset \mathbb{R}^m$ be a set of parameters. The system in (2) is said to be uniformly practically asymptotically stable (UPAS) on Θ if, there exists $\Delta > 0$ such that for all $\delta > 0$, there exist $\theta^\star \in \Theta$ and $\beta \in \mathcal{KL}$ such that the solutions of $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \theta^\star)$ satisfy

$$\|\mathbf{x}_{\theta^\star}(t, t_0, \mathbf{x}_0)\|_\delta \leq \beta(\|\mathbf{x}_0\|, t - t_0) \quad \forall t \geq t_0$$

for all $t_0 \geq 0$, $\mathbf{x}_0 \in \mathcal{B}_\Delta$.

The following statement, which is reminiscent of [13, Proposition 2], gives conditions for UPAS. While the assumptions of Corollary 2 below are more conservative than those imposed in the latter reference, Corollary 2 is simpler and fits a number of concrete applications, including the one treated here. An example to illustrate its use can be found in [11].

Corollary 2: Let $\sigma_i : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$, $i \in \{1, ..., N\}$, be continuous functions, positive over Θ, and let $\underline{a}$, $\overline{a}$, q and Δ be positive constants with $\mathcal{B}_\Delta \subset \mathbb{D}$. Assume that, for any $\theta \in \Theta$, there exists a continuously differentiable Lyapunov function $V : \mathbb{R}_{\geq 0} \times \mathbb{D} \times \Theta \to \mathbb{R}_{\geq 0}$ satisfying, for all $\mathbf{x} \in \mathbb{D}$ and all $t \geq 0$,

$$\underline{a} \min\{\sigma_i(\theta)\} \|\mathbf{x}\|^q \leq V(t, \mathbf{x}, \theta) \leq \overline{a} \max\{\sigma_i(\theta)\} \|\mathbf{x}\|^q . \tag{3}$$

Assume also that for any $\delta \in (0, \Delta)$, there exists a parameter $\theta^\star(\delta) \in \Theta$ and a class $\mathcal{K}$ function α_δ such that, for all $\|\mathbf{x}\| \in [\delta, \Delta]$ and all $t \geq 0$,

$$\frac{\partial V}{\partial t}(t, \mathbf{x}, \theta^\star) + \frac{\partial V}{\partial \mathbf{x}}(t, \mathbf{x}, \theta^\star)\mathbf{f}(t, \mathbf{x}, \theta^\star) \leq -\alpha_\delta(\|\mathbf{x}\|) . \tag{4}$$

Assume also that for all $i \in \{1, ..., N\}$,

$$\lim_{\delta \to 0} \sigma_i(\theta^\star(\delta))\delta^q = 0 \quad \text{and} \quad \lim_{\delta \to 0} \sigma_i(\theta^\star(\delta)) \neq 0 . \tag{5}$$

Then, the system (2) is UPAS on the parameter set Θ. Moreover, when $\delta = 0$ and the parameter $\theta^\star$ is independent of δ, the conditions in (5) are no longer required, and the system (2) is UAS[2].

As it is apparent, conditions (3) and (4) are reminiscent of conditions imposed for ultimate boundedness –*cf.* [14] however, we stress that UPAS is a stronger property since it implies that the ball $\mathcal{B}_\delta$ can be arbitrarily diminished. In that respect it is convenient to stress the condition (5) which ensures that the latter is possible in spite of the fact that θ depends on δ. More general results on practical stability are stated in [13].

III. Model of relative attitude

In the following, coordinate reference frames are denoted by $\mathcal{F}_\cdot$, and we denote by $\omega^c_{b,a}$ the angular velocity of $\mathcal{F}_a$ relative to $\mathcal{F}_b$, referenced in $\mathcal{F}_c$. Matrices representing rotation or coordinate transformation between $\mathcal{F}_a$ and $\mathcal{F}_b$ are denoted $\mathbf{R}^b_a$.

A. Cartesian coordinate frames

To form the basis of our relative attitude model, we use the standard definition of the Earth-Centered Inertial (ECI) frame $\mathcal{F}_i$, with z axis towards celestial north. In addition, we employ a standard LVLH-definition of the leader orbit reference frame $\mathcal{F}_l$, with unit vectors defined as

$$\mathbf{e}_r = \frac{\mathbf{r}_l}{r_l}, \qquad \mathbf{e}_\theta = \mathbf{e}_h \times \mathbf{e}_r \qquad \text{and} \qquad \mathbf{e}_h = \frac{\mathbf{h}}{h}, \tag{6}$$

where $\mathbf{r}_l$ is the vector from the center of the Earth to the leader spacecraft, $\mathbf{h} = \mathbf{r}_l \times \dot{\mathbf{r}}_l$ is the angular momentum vector of the leader orbit, and $h = |\mathbf{h}|$. Moreover, we define a follower orbit reference frame $\mathcal{F}_f$ with origin specified by the relative orbit position vector

$$\mathbf{p} = \mathbf{r}_f - \mathbf{r}_l = x\mathbf{e}_r + y\mathbf{e}_\theta + z\mathbf{e}_h \tag{7}$$

and with unit vectors aligned with the unit vectors in $\mathcal{F}_l$ at all times. We also define leader and follower body frames $\mathcal{F}_{lb}$ and $\mathcal{F}_{fb}$ respectively, with origin in the corresponding centers of mass and axes fixed to the spacecraft body.

[1] In a number of concrete control problems, including the one solved here, the parameter θ corresponds to the values of the control gains.

[2] Note however that the resulting domain of attraction may be smaller than $\mathcal{B}_\Delta$.

B. Body frame rotation

Frame and relative orientations are described in terms of the four-parameter representation known as unit quaternions. A unit quaternion (*cf.* [15])

$$\mathbf{q} = \begin{bmatrix} \eta & \varepsilon^\top \end{bmatrix}^\top \quad (8)$$

is composed of a scalar parameter $\eta \in \mathbb{R}$ and a vector $\varepsilon \in \mathbb{R}^3$, satisfying the quaternion constraint

$$\eta^2 + \varepsilon^\top \varepsilon = 1\,. \quad (9)$$

The rotation matrix describing rotations from an orbit frame to a body frame is related to the corresponding unit quaternion through the Rodriguez formula

$$\mathbf{R}_o^b = \mathbf{I} + 2\eta \mathbf{S}(\varepsilon) + 2\mathbf{S}^2(\varepsilon) \quad (10)$$

where $\mathbf{S}(\varepsilon) = \varepsilon \times$ is the cross product operator. Moreover, the inverse rotation is given by the inverse unit quaternion $\bar{\mathbf{q}} = \begin{bmatrix} \eta, & -\varepsilon^\top \end{bmatrix}^\top$. The set of unit quaternions is a vector space over $\mathbb{R}^4$ known as S^3, which is a covering manifold of the SO(3) group of rotation matrices, and provides a globally nonsingular parametrization of the latter. The set forms a group with quaternion multiplication, and the quaternion product of two quaternions $\mathbf{q}_1$ and $\mathbf{q}_2$ is defined as

$$\mathbf{q}_1 \otimes \mathbf{q}_2 := \begin{bmatrix} \eta_1\eta_2 - \varepsilon_1^\top \varepsilon_2 \\ \eta_1\varepsilon_2 + \eta_2\varepsilon_1 + \mathbf{S}(\varepsilon_1)\varepsilon_2 \end{bmatrix} \quad (11)$$

with the identity element $\begin{bmatrix} 1, \mathbf{0}^\top \end{bmatrix}^\top \in \mathbb{R}^4$.

C. Relative rotational motion

The relative attitude kinematics can be expressed as

$$\dot{\mathbf{q}} = \mathbf{T}(\mathbf{q})\,\omega, \quad \mathbf{T}(\mathbf{q}) = \frac{1}{2}\begin{bmatrix} -\varepsilon^\top \\ \eta \mathbf{I} + \mathbf{S}(\varepsilon) \end{bmatrix} \quad (12)$$

where

$$\omega = \omega_{i,fb}^{fb} - \mathbf{R}_{lb}^{fb}\omega_{i,lb}^{lb} \quad (13)$$

is the relative angular velocity between the *leader-body* and the *follower-body* frames $\mathcal{F}_{lb}$ and $\mathcal{F}_{fb}$. Moreover, the relative attitude dynamics can be expressed as

$$\mathbf{J}_f\dot{\omega} + \mathbf{C}\left(\omega_{i,fb}^{fb}\right)\omega + \mathbf{n}\left(\omega_{i,fb}^{fb}, \omega_{i,lb}^{lb}\right) = \Upsilon_d + \Upsilon_a \quad (14)$$

where $\mathbf{J}_f, \mathbf{J}_l$ are spacecraft inertia matrices,

$$\Upsilon_d = \tau_{df}^{fb} - \mathbf{J}_f \mathbf{R}_{lb}^{fb}\mathbf{J}_l^{-1}\tau_{dl}^{lb}, \quad \Upsilon_a = \tau_{af}^{fb} - \mathbf{J}_f \mathbf{R}_{lb}^{fb}\mathbf{J}_l^{-1}\tau_{al}^{lb} \quad (15)$$

are the relative disturbance torques and relative actuator torques, respectively,

$$\mathbf{C}\left(\omega_{i,fb}^{fb}\right) = \mathbf{J}_f\mathbf{S}\left(\omega_{i,fb}^{fb}\right) \quad (16)$$

and

$$\mathbf{n}\left(\omega_{i,fb}^{fb}, \omega_{i,lb}^{lb}\right) = \mathbf{S}\left(\omega_{i,fb}^{fb}\right)\mathbf{J}_f\omega_{i,fb}^{fb} - \mathbf{J}_f\mathbf{R}_{lb}^{fb}\mathbf{J}_l^{-1}\mathbf{S}\left(\omega_{i,lb}^{lb}\right)\mathbf{J}_l\omega_{i,lb}^{lb}$$

IV. Relative attitude control

A. Problem statement

The control problem is to design a control law that makes the state $\mathbf{q}$ converge to a time-varying smooth trajectory $\mathbf{q}_d(t)$, satisfying the kinematic relation $\dot{\mathbf{q}}_d = \mathbf{T}(\mathbf{q}_d)\,\omega_d$ and under the assumption that only $\tilde{\mathbf{q}}$ is measurable.

The relative attitude error quaternion $\tilde{\mathbf{q}}$ is defined by the quaternion product such that $\tilde{\mathbf{q}} := \mathbf{q} \otimes \bar{\mathbf{q}}_d$ and, according with (12), the error kinematics is

$$\dot{\tilde{\mathbf{q}}} = \mathbf{T}(\tilde{\mathbf{q}})\,\tilde{\omega}\,. \quad (17)$$

Note that due to the inherent redundancy of the quaternion representation $\tilde{\mathbf{q}}$ and $-\tilde{\mathbf{q}}$ represent the same *physical* orientation however, one is rotated 2π relative to the other about an arbitrary axis. Accordingly, the closed-loop system has *two* equilibrium points in the $(\tilde{\mathbf{q}}, \tilde{\omega})$ space which correspond to the *same* physical orientation, namely $\tilde{\mathbf{q}}_\pm = \begin{bmatrix} \pm 1, \mathbf{0}^\top \end{bmatrix}^\top$. Therefore, strictly speaking it is not possible to claim any *global* property for the closed-loop system using quaternion coordinates. From a physical viewpoint, since both equilibria correspond to the same orientation it is important to make a *choice* of the equilibrium point to be stabilized, depending on the given initial condition; logically, one aims at minimizing the path length for the desired rotation which can be ensured by choosing the equilibrium point corresponding to the sign of $\tilde{\eta}(t_0)$. In other words, we choose $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}_+$ if $\tilde{\eta}(t_0) \geq 0$, and $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}_-$ otherwise.

For the equilibrium points $\tilde{\mathbf{q}}_\pm$, we define the relative attitude error as $\mathbf{e}_{1\pm} := [1 \mp \tilde{\eta},\ \tilde{\varepsilon}]$, together with the relative angular velocity error $\mathbf{e}_2 := \omega - \omega_d$ hence, according with general kinematic relations, we have

$$\dot{\mathbf{e}}_{1\pm} = \Lambda_e(\mathbf{e}_{1\pm})\,\mathbf{e}_2 \quad (18)$$

where

$$\Lambda_e(\mathbf{e}_{1\pm}) = \frac{1}{2}\begin{bmatrix} \pm\tilde{\varepsilon}^\top \\ \tilde{\eta}\mathbf{I} + \mathbf{S}(\tilde{\varepsilon}) \end{bmatrix} \quad (19)$$

for $\mathbf{e}_{1+}$ and $\mathbf{e}_{1-}$ respectively. Note that for both cases we have $4\Lambda_e^\top\Lambda_e = \mathbf{I}$, which can be shown by direct calculation using $\mathbf{S}(\tilde{\varepsilon})^\top = -\mathbf{S}(\tilde{\varepsilon})$, $\mathbf{S}(\tilde{\varepsilon})^\top\mathbf{S}(\tilde{\varepsilon}) = \tilde{\varepsilon}^\top\tilde{\varepsilon}\mathbf{I} - \tilde{\varepsilon}\tilde{\varepsilon}^\top$ and (9). Note also that $\Lambda_e^\top(\mathbf{e}_{1\pm})\,\mathbf{e}_{1\pm} = \pm\tilde{\varepsilon}$ hence, from (17),

$$\dot{\Lambda}_e^\top(\mathbf{e}_{1\pm})\,\mathbf{e}_{1\pm} = \mathbf{G}_\pm\mathbf{e}_2 \quad (20)$$

where

$$\mathbf{G}_\pm = \pm\frac{1}{2}\left[\tilde{\eta}\mathbf{I} + \mathbf{S}(\tilde{\varepsilon})\right] - \frac{1}{4}\mathbf{I}\,. \quad (21)$$

For simplicity, we shall assume that the scalar parameter of the quaternion does not change sign *i.e.*,

$$\operatorname{sgn}(\tilde{\eta}(t_0)) = \operatorname{sgn}(\tilde{\eta}(t)) \quad (22)$$

for all $t > t_0$.

Remark 3: The assumption of sign-definiteness of the scalar quaternion parameter $\tilde{\eta}$ is imposed for technical reasons to obtain negative definite bounds on Lyapunov function derivatives. An analytical result along the same lines as ours, where a similar assumption is shown to hold, may be found in [16, Lemma 1].

B. Main result

Under the assumptions that the leader spacecraft motion and the orbital perturbations working on the follower are bounded, such that, respectively $\|\mathbf{J}_f \mathbf{R}_{lb}^{fb} \mathbf{J}_l^{-1}(\tau_{al}^{lb} + \tau_{dl}^{lb})\| \le \beta_{\tau l}$ and $\|\tau_{df}^{fb}\| \le \beta_{\tau f}$, we have the following proposition.

Proposition 4: Let $\mathbf{e}_1$ be defined either by $\mathbf{e}_1 := \mathbf{e}_{1+}$ or $\mathbf{e}_1 := \mathbf{e}_{1-}$ and respectively, let $\tilde{\eta}(t_0) > 0$ or $\tilde{\eta}(t_0) < 0$; assume that (9) and (22) hold for all $t \ge t_0 \ge 0$; that $\omega_{i,fb}^{fb}$, $\omega_{i,lb}^{lb}$, ω_d are bounded functions. Consider the control law

$$\tau_{af}^{fb} = -k_q \Lambda_e^\top \mathbf{e}_1 - k_\omega \left[\mathbf{I} + 4\mathbf{G}^\top\right] \vartheta + \mathbf{J}_f \dot{\omega}_d \tag{23a}$$

$$\dot{\mathbf{q}}_c = -a\vartheta \tag{23b}$$

$$\vartheta = \mathbf{q}_c + b\Lambda_e^\top \mathbf{e}_1 \,. \tag{23c}$$

Then, the system (12), (14), (15) in closed loop with (23) is uniformly practically asymptotically stable (UPAS) with k_q, k_ω, a and b as tuning parameters.

Remark 5: As mentioned earlier, to ensure that the shortest rotation path is chosen on attitude manoeuvres the equilibrium point must be chosen to include $\mathbf{e}_{1+}$ when $\tilde{\eta} > 0$, and $\mathbf{e}_{1-}$ otherwise. From the relation $\Lambda_e^\top(\mathbf{e}_{1\pm})\mathbf{e}_{1\pm} = \pm\tilde{\varepsilon}$ and the control law in (23a) we see that the first term in the controller changes sign when we switch between the two equilibrium points. An alternative is to include a switching law such that whenever $\tilde{\eta} < 0$, we change the sign of the of the entire quaternion error $\tilde{\mathbf{q}}$. This corresponds to rotating the quaternion error 2π about an arbitrary axis, which leaves the closed-loop system in the same orientation, but makes $\tilde{\eta}$ positive. The switching law can be implemented by defining $\mathbf{e}_1 = [1 - |\tilde{\eta}|,\ \tilde{\varepsilon}^\top]^\top$, so that

$$\Lambda_e(\mathbf{e}_1) = \frac{1}{2}\begin{bmatrix} \operatorname{sgn}(\tilde{\eta})\tilde{\varepsilon}^\top \\ \tilde{\eta}\mathbf{I} + \mathbf{S}(\tilde{\varepsilon}) \end{bmatrix} \tag{24}$$

and $\Lambda_e \mathbf{e}_1 = \operatorname{sgn}(\tilde{\eta})\tilde{\varepsilon}$, where $\operatorname{sgn}(\cdot)$ denotes the usual sign function. Moreover, by defining this sign function nonzero, so that

$$\operatorname{sgn}(\tilde{\eta}) := \begin{cases} -1, & \tilde{\eta} < 0 \\ \ \ 1, & \tilde{\eta} \ge 0 \end{cases} \tag{25}$$

we avoid the singularity when $\tilde{\eta} = 0$. There is, however, a technical obstacle when using (24): the Lyapunov function –*cf.* (30), is not continuous. An alternative may be to analyze the system as a switched system but we shall not pursue this here.

1) Proof of Proposition 4: We first consider the positive equilibrium point, that is we assume that $\tilde{\eta} > 0$ and we denote $\mathbf{e}_1 = \mathbf{e}_{1+}$ and $\Lambda_e = \Lambda_e(\mathbf{e}_{1+})$. Inserting the control law (23a) into the model (14), results in the closed-loop system

$$\mathbf{J}_f \dot{\mathbf{e}}_2 = -k_q \Lambda_e^\top \mathbf{e}_1 - k_\omega \left[\mathbf{I} + 4\mathbf{G}^\top\right]\vartheta + \Delta_\omega + \Delta_\tau \tag{26}$$

where

$$\Delta_\omega = -\mathbf{C}\left(\omega_{i,fb}^{fb}\right)\omega - \mathbf{n}\left(\omega_{i,fb}^{fb}, \omega_{i,lb}^{lb}\right) \tag{27}$$

$$\Delta_\tau = \tau_{df}^{fb} - \mathbf{J}_f \mathbf{R}_{lb}^{fb} \mathbf{J}_l^{-1}\left(\tau_{al}^{lb} + \tau_{dl}^{lb}\right). \tag{28}$$

Differentiating (23c) and inserting (23b) and (18) results in

$$\dot{\vartheta} = -a\vartheta + b\left[\frac{1}{4}\mathbf{I} + \mathbf{G}\right]\mathbf{e}_2 \tag{29}$$

since $4\Lambda_e^\top \Lambda_e = \mathbf{I}$. Now, let us define the state vector $\mathbf{x} := \left[\mathbf{e}_1^\top \Lambda_e, \mathbf{e}_2^\top, \vartheta^\top\right]^\top$ and introduce the Lyapunov function candidate

$$V(\mathbf{x}) = \frac{1}{8}\mathbf{e}_1^\top k_q \mathbf{e}_1 + \frac{1}{8}\mathbf{e}_2^\top \mathbf{J}_f \mathbf{e}_2 + \frac{1}{2}\vartheta^\top \frac{k_\omega}{b}\vartheta + \lambda_1 \mathbf{e}_2^\top \mathbf{J}_f\left(\Lambda_e^\top \mathbf{e}_1 - \vartheta\right) + \frac{1}{2}\left(\vartheta^\top - b\mathbf{e}_1^\top \Lambda_e\right)\lambda_2\left(\vartheta - b\Lambda_e^\top \mathbf{e}_1\right) \tag{30}$$

where

$$\lambda_2 = \frac{\lambda_1(k_\omega - k_q)}{ba} \tag{31}$$

and $\lambda_1 > 0$ are design parameters. If $\tilde{\eta} > 0$ we have

$$0 \le (1-\tilde{\eta})^2 \le (1-\tilde{\eta})(1+\tilde{\eta}) = 1 - \tilde{\eta}^2 = \tilde{\varepsilon}^\top\tilde{\varepsilon} \tag{32}$$

so for the first term in (30) we find

$$\frac{1}{8}\mathbf{e}_1^\top k_q \mathbf{e}_1 = \frac{1}{8}k_q\left((1-\tilde{\eta})^2 + \tilde{\varepsilon}^\top\tilde{\varepsilon}\right) \le \frac{1}{4}\mathbf{e}_1^\top \Lambda_e k_q \Lambda_e^\top \mathbf{e}_1 \,. \tag{33}$$

Accordingly, we may write

$$V(\mathbf{x}) \le \frac{1}{2}\mathbf{x}^\top \mathbf{P}\mathbf{x} \tag{34}$$

where

$$\mathbf{P} := \begin{bmatrix} \left(\frac{1}{2}k_q + \lambda_2 b^2\right)\mathbf{I} & \lambda_1 \mathbf{J}_f & -b\lambda_2 \mathbf{I} \\ \lambda_1 \mathbf{J}_f & \frac{1}{4}\mathbf{J}_f & -\lambda_1 \mathbf{J}_f \\ -b\lambda_2 \mathbf{I} & -\lambda_1 \mathbf{J}_f & \left(\frac{k_\omega}{b} + \lambda_2\right)\mathbf{I} \end{bmatrix}. \tag{35}$$

On the other hand, we have

$$\frac{1}{8}\mathbf{e}_1^\top k_q \mathbf{e}_1 \ge \frac{1}{8}\mathbf{e}_1^\top \Lambda_e k_q \Lambda_e^\top \mathbf{e}_1 \tag{36}$$

hence, it follows that $V(\mathbf{x})$ satisfies

$$\frac{1}{4}P_m\|\mathbf{x}\|^2 \le V(\mathbf{x}) \le \frac{1}{2}P_M\|\mathbf{x}\|^2 \tag{37}$$

where P_m and P_M are the smallest and largest eigenvalues of $\mathbf{P}$, respectively. Imposing the bound $j_m \le \|\mathbf{J}_f\| \le j_M$, we find that $\mathbf{P}$ is positive definite if

$$\lambda_1^2 \le \min\left\{\frac{k_q}{16 j_M},\ \frac{k_q k_\omega}{16\left(b j_M + k_\omega j_M^2\right)}\right\} \tag{38}$$

with the additional condition that

$$k_\omega \ge k_q \,. \tag{39}$$

Evaluating the total derivative of V along the closed-loop trajectories of (26), (29); using (31), $\mathbf{G} = \mathbf{G}_+$ and $4\Lambda_e^\top \Lambda_e = \mathbf{I}$, we obtain

$$\begin{aligned} \dot{V}(\mathbf{x}) = & -\lambda_1 \mathbf{e}_1^\top \Lambda_e k_q \Lambda_e^\top \mathbf{e}_1 - \lambda_1(b-1)\mathbf{e}_2^\top \mathbf{J}_f\left[\frac{1}{4}\mathbf{I} + \mathbf{G}\right]\mathbf{e}_2 \\ & - \vartheta^\top\left[\left(\frac{a}{b} - \lambda_1\right)k_\omega - \lambda_1 4\mathbf{G}^\top k_\omega + \lambda_2 a\right]\vartheta \\ & - \lambda_1 4 k_\omega \mathbf{e}_1^\top \Lambda_e \mathbf{G}^\top \vartheta + \lambda_1 \mathbf{e}_2^\top \mathbf{J}_f a\vartheta \\ & + \left[\frac{1}{4}\mathbf{e}_2^\top + \lambda_1 \mathbf{e}_1^\top \Lambda_e - \lambda_1 \vartheta^\top\right](\Delta_\omega + \Delta_\tau)\,. \end{aligned} \tag{40}$$

Hence, we may write

$$\dot{V}(\mathbf{x}) \leq -\mathbf{x}^\top \mathbf{Q}\mathbf{x} + \left[\frac{1}{4}\mathbf{e}_2^\top + \lambda_1 \mathbf{e}_1^\top \Lambda_e - \lambda_1 \vartheta^\top\right](\Delta_\omega + \Delta_\tau) \quad (41)$$

where

$$\mathbf{Q} = [\mathbf{q}_{ij}], \quad i,j = 1,2,3, \quad (42)$$

$$\mathbf{q}_{11} = \lambda_1 k_q, \qquad \mathbf{q}_{22} = \lambda_1 (b-1)\mathbf{J}_f \left[\frac{1}{4}\mathbf{I} + \mathbf{G}\right], \quad (43)$$

$$\mathbf{q}_{33} = \left(\frac{a}{b} - \lambda_1\right) k_\omega + \lambda_2 a - \lambda_1 4\mathbf{G}^\top k_\omega, \quad (44)$$

$$\mathbf{q}_{12} = \mathbf{0}, \qquad \mathbf{q}_{13} = 2\lambda_1 k_\omega \mathbf{G}^\top, \qquad \mathbf{q}_{23} = -\frac{1}{2}\lambda_1 \mathbf{J}_f a \quad (45)$$

and $\mathbf{q}_{ij} = \mathbf{q}_{ji}^\top$ for $i \neq j$. Note that the eigenvalues of $(1/4)\mathbf{I} + \mathbf{G}$ equal $\tilde{\eta}/2$.

To verify the conditions of Corollary 2 we exhibit a quadratic upper-bound on $-\mathbf{x}^\top\mathbf{Q}\mathbf{x}$. To that end, we use the formula $2|ab| \leq a^2 + b^2$ for any $a, b \in \mathbb{R}$, to obtain

$$\begin{aligned}\mathbf{x}^\top\mathbf{Q}\mathbf{x} \geq & (q_{11,m} - q_{12,M} - q_{13,M})\|x_1\|^2 \\ & + (q_{22,m} - q_{12,M} - q_{23,M})\|x_2\|^2 \\ & + (q_{33,m} - q_{13,M} - q_{23,M})\|x_3\|^2,\end{aligned}$$

where $q_{ij,m}$ and $q_{ij,M}$ denote, respectively, lower and upper bounds on the induced norm of the submatrices $\mathbf{q}_{ij}$. Choosing the gains k_q, k_ω, a and b large enough so that

$$q_{11,m} \geq 2(q_{12,M} + q_{13,M}) \quad (46)$$

$$q_{22,m} \geq 2(q_{12,M} + q_{23,M}) \quad (47)$$

$$q_{33,m} \geq 2(q_{13,M} + q_{23,M}), \quad (48)$$

which is always possible due to the structure of the submatrices $\mathbf{q}_{ij}$, we obtain

$$\mathbf{x}^\top\mathbf{Q}\mathbf{x} \geq \frac{1}{2}(q_{11,m}\|x_1\|^2 + q_{22,m}\|x_2\|^2 + q_{33,m}\|x_3\|^2).$$

In particular, using $g_M \geq \|\mathbf{G}\|$, we have from (43)-(45) and (46)-(48) the requirements that

$$k_q \geq 2 g_M k_\omega \quad (49)$$

$$b \geq 1 + \frac{j_M a}{j_m\sqrt{2}} \quad (50)$$

$$\left(\frac{a}{b} - \lambda_1\right) k_\omega + \lambda_2 a - \lambda_1 4 g_M k_\omega \geq \lambda_1 4 g_M k_\omega + \lambda_1 j_M a \quad (51)$$

where the last requirement is satisfied for

$$\lambda_1 \leq \frac{a k_\omega}{k_q + (b-1)k_\omega + b j_M a + b 8 g_M k_\omega}. \quad (52)$$

If conditions (49) and (50) hold, we can choose $q_m \leq \frac{1}{2}\min\{q_{11,m}, q_{22,m}, q_{33,m}\}$ such that $0 < q_m < \|\mathbf{Q}\|$. Note that each of these terms can be arbitrarily enlarged by an appropriate choice of k_q, k_ω, a and b, and thus, q_m can be enlarged accordingly.

Using the relation $\mathbf{e}_2 = \omega - \omega_d$, we obtain from (27)

$$\begin{aligned}\Delta_\omega = & -\mathbf{J}_f\left(\mathbf{S}\left(\omega_{i,fb}^{fb}\right)\mathbf{e}_2 + \mathbf{S}\left(\omega_{i,fb}^{fb}\right)\omega_d\right) - \mathbf{S}\left(\omega_{i,fb}^{fb}\right)\mathbf{J}_f\omega_{i,fb}^{fb} \\ & + \mathbf{J}_f\mathbf{R}_{lb}^{fb}\mathbf{J}_l^{-1}\mathbf{S}\left(\omega_{i,lb}^{lb}\right)\mathbf{J}_l\omega_{i,lb}^{lb}. \quad (53)\end{aligned}$$

If we further impose the bounds $\|\mathbf{J}_f\mathbf{R}_{lb}^{fb}\mathbf{J}_l^{-1}\left(\tau_{al}^{lb} + \tau_{dl}^{lb}\right)\| \leq \beta_{\tau l}$, $\|\tau_{df}^{fb}\| \leq \beta_{\tau f}$, $\|\mathbf{R}_{lb}^{fb}\| \leq 3\sqrt{3}$, $\|\omega_{i,fb}^{fb}\| \leq \beta_{\omega f}$, $\|\omega_{i,lb}^{lb}\| \leq \beta_{\omega l}$ and $\|\omega_d\| \leq \beta_{\omega d}$, we find from (28) and (53) that

$$\|\Delta_\omega\| \leq 2 j_M \beta_{\omega f}\|\mathbf{x}\| + 2 j_M\left(\beta_{\omega f}\beta_{\omega d} + \beta_{\omega f}^2 + 3\sqrt{3}\beta_{\omega l}^2\right) \quad (54)$$

$$\|\Delta_\tau\| \leq \beta_{\tau l} + \beta_{\tau f}. \quad (55)$$

Using (41) and (54), we find that

$$\dot{V}(\mathbf{x}) \leq -q_m\|\mathbf{x}\|^2 + \left(\frac{1}{4} + 2\lambda_1\right)\|\mathbf{x}\|\|\Delta_\omega + \Delta_\tau\| \quad (56)$$

$$\leq -q_m\|\mathbf{x}\|^2 + c_1\|\mathbf{x}\| + c_2\|\mathbf{x}\|^2 \quad (57)$$

with

$$c_1 = 2 j_M\left(\frac{1}{4} + 2\lambda_1\right)\left(\beta_{\omega f}\beta_{\omega d} + \beta_{\omega f}^2 + 3\sqrt{3}\beta_{\omega l}^2 + \beta_{\tau l} + \beta_{\tau f}\right)$$

$$c_2 = 2 j_M\left(\frac{1}{4} + 2\lambda_1\right)\beta_{\omega f} + \lambda_1 j_M a.$$

Restricting the norm of the system states such that $\|\mathbf{x}\| \geq \delta$, we have $\|\mathbf{x}\|/\delta \geq 1$, and hence

$$\dot{V}(\mathbf{x}) \leq -\frac{1}{2}q_m\|\mathbf{x}\|^2 - \left(\frac{1}{2}q_m - \frac{c_1}{\delta} - c_2\right)\|\mathbf{x}\|^2. \quad (58)$$

By choosing the gains sufficiently large, such that

$$\frac{1}{2}q_m \geq \frac{c_1}{\delta} + c_2 \quad (59)$$

we obtain

$$\|x\| \in \mathbb{D} \cap \{\|x\| \geq \delta\} \quad \Longrightarrow \quad \dot{V}(\mathbf{x}) \leq -\frac{1}{2}q_m\|\mathbf{x}\|^2. \quad (60)$$

To verify Corollary 2, we define $\theta = (k_q,\ k_\omega,\ a,\ b)$ and obtain from (37) that (3) is satisfied with $\underline{a} = 1/4$, $\overline{a} = 1/2$, $\sigma_1(\theta) = k_q/2 + \lambda_2 b^2$, $\sigma_2(\theta) = j_M/4$ and $\sigma_3(\theta) = k_\omega/b + \lambda_2$. We may now choose $\theta = (k_q,\ k_\omega,\ a,\ b) = (2,4,6,20)$, to satisfy (39), (49) and (50) with $g_M = 1$, and from (38), (52) we may obtain the appropriate condition for $\lambda_1 > 0$ based on follower spacecraft moments of inertia. Hence, for this choice of gain parameters, (60) holds and so does (4). Moreover, from (59) we see that (5) is satisfied. Accordingly, all the conditions in Corollary 2 are satisfied.

The arguments and computations above hold for the trajectories $\mathbf{x} = \mathbf{x}(t)$ generated by any pair of initial conditions t_0, $\mathbf{x}_0$ compatible with the constraints, for all $t \geq t_0 \geq 0$, and under the assumption that (22) holds. Thus, we conclude that the (equilibrium $(\mathbf{e}_{1+}, \mathbf{e}_2) = (0,0)$ of the) closed-loop system is uniformly practically asymptotically stable (UPAS).

The proof for the negative equilibrium point $\mathbf{e}_{1-}$ follows along the same lines as the proof for $\mathbf{e}_{1+}$ however, with (32) replaced by

$$0 \leq (1+\tilde{\eta})^2 \leq (1-\tilde{\eta})(1+\tilde{\eta}) = 1 - \tilde{\eta}^2 = \tilde{\varepsilon}^\top\tilde{\varepsilon} \quad (61)$$

which holds for $\tilde{\eta} < 0$.

■

V. A NUMERICAL EXAMPLE

In this section, simulation results are presented to illustrate the performance of the presented control law. In this case-study, both the leader and the follower have moments of inertia given as $\mathbf{J} = \mathrm{diag}\left\{ 4.350 \quad 4.3370 \quad 3.6640 \right\}$ kgm^2. The leader is assumed to follow an equatorial orbit with a perigee altitude of 250 km and eccentricity $e = 0.3$, and the leader body and orbit coordinate frames are perfectly aligned at all times. The follower is assumed to have available continuous torque about all body axes, with a maximum torque of 0.05 Nm. In the simulations, the controller gains $k_q = 1.2$, $k_\omega = 4$, $a = 5$ and $b = 20$ have been used. The initial relative attitude is standstill at $[-75°,\ -175°,\ 70°]$, corresponding to the quaternion values $[0.3772,\ 0.4329,\ -0.6645,\ -0.4783]$, and the follower spacecraft is commanded to follow a smooth sinusoidal trajectory around the origin.

The simulation results of a spacecraft formation where the follower uses the controller structure in (23a)-(23c) are presented in Figure 1. The figure shows (from top to bottom) relative attitude error, relative angular velocity error, angular velocity filter output and power consumption. The asymptotic convergence from the initial states to the reference trajectory is seen in the left figures, while the right figures shows the practical stability property as oscillations around the origin. Note that $\tilde{\eta}$ is always separated from zero, in accordance with the assumption in (22). The total power consumption for the entire orbit is 1.875 W, with a peak power of is 8.8 mW, and the state error accuracy (*i.e.* the radius of the $\mathcal{B}_\delta$ ball) with the chosen gains amounts to approximately $\delta = 3.5 \cdot 10^{-6}$.

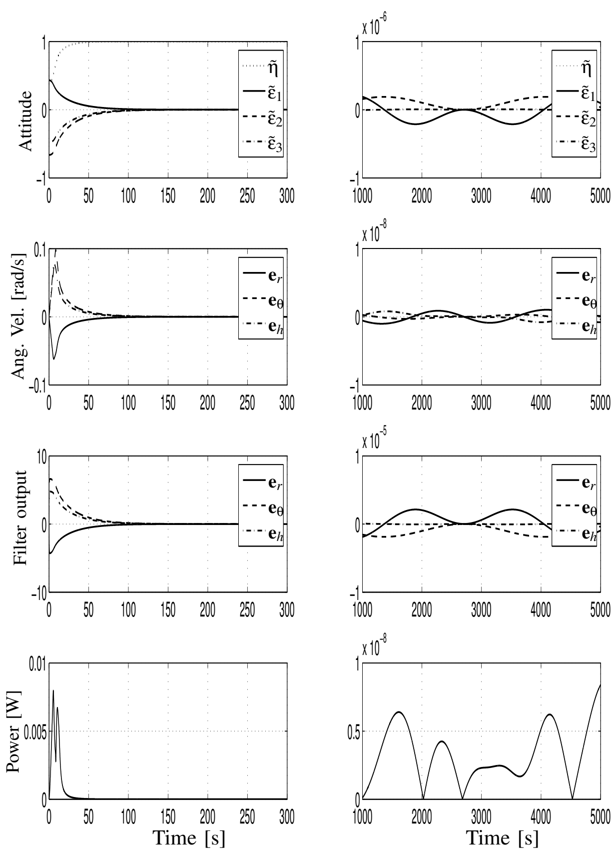

Fig. 1. Simulation result of a spacecraft formation using attitude output feedback.

VI. CONCLUSION

We have presented a solution to the problem of tracking relative rotation in a leader-follower spacecraft formation using feedback from relative attitude only, employing an approximate differentiation filter of the attitude error to provide sufficient knowledge about the angular velocity error. The resulting stability properties of the closed-loop systems left by the controller configuration have been derived, and proved to result in a uniformly practically asymptotically stable (UPAS) closed-loop system.

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