2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Stability Conditions for Discrete Time Fault Tolerant Control Systems With State Delays

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Abstract— **A DTFTCS in noisy environments subject to delays and driven by a state feedback controller is developed. Second moment stability for the proposed delayed DTFTCS is investigated. A delay-independent sufficient condition that guarantee the H_∞ second moment stability and achieve δ-level of disturbance rejection is derived and proved. Results are obtained using Lyapunov function approach and formulated as feasibility solution for a set of linear matrix inequalities (LMI). The theory is demonstrated by a numerical example.**

I. Introduction

Fault Tolerant Control Systems (FTCS) are a class of modern control systems designed to maintain high levels of system survivability and performance. Two classes of FTCS are defined: passive and active designs. A passive FTCS can tolerate faulty operation while maintaining satisfactory performance without any control reconfiguration. An active FTCS (AFTCS) is composed of two major blocks a fault detection and isolation (FDI) scheme and a control reconfiguration mechanism. The FDI scheme continuously monitor system behavior to detect and locate failed component(s). The decisions made by the FDI scheme command a reconfiguration mechanism to restructure the control law in real time basis accordingly. It was documented that the dynamical behavior of AFTCS is governed by stochastic differential equations and modeled as a general hybrid system combining the Euclidean space for system dynamics and the discrete space for fault-induced changes [11], [13], [14].

The research of hybrid systems evolved into two major classes: Jump Linear Systems (JLS) and Fault Tolerant Control Systems with Markovian Parameters (FTCSMP). In JLS, the random jump process of the coefficients is represented by a finite state Markov chain called plant regime mode [3], [7]. The restrictive assumption of perfect regime knowledge in JLS was the motivation to introduce the model of FTCSMP [13]. In FTCSMP two separate random processes with different state spaces are defined: one represents system component failures and the second represents decisions of the FDI process used to reconfigure the control law. This unique modeling allows the consideration of practical implementation issues and physical limitations of a particular FTCSMP. Stability properties of FTCSMP in the presence of noise, detection errors, detection delays, parameter uncertainties and actuator saturation were studied in [8],[9] and [10]. A comprehensive review of the stochastic stability and stabilization of continuous AFTCSMP using Lyapunov function approach can be found in [11]. Just lately, the analysis of stochastic stability and H_∞ stabilization of continuous FTCSMP was revisited in [1] and [2] using convex programming framework. The results provided an LMI characterization of output feedback controllers that stochastically stabilize FTCSMP and ensures H_∞ constraints. Integral Quadratic Constraints were defined for FTCSMP and a stabilizing controller was synthesized in [15] and [16], optimal H_2 performance was investigated in [17] and [18]. In [19] FTCSMP were modeled and analyzed using randomized algorithms. The vital issue of detection delays has been revisited in a more rigorous form in [20].

Very limited results in the literature dealt with discrete time fault tolerant control systems (DTFTCS). The difficulty to characterize the stochastic behavior of DTFTCS was due to the complexity of the model and tools needed to complete the studies. [6] studied the stochastic stability and controller design for the nominal DTFTCS, [12] extended the results to include norm bounded parameter uncertainties. The work in both citations synthesized a control law by solving a set of Riccati-Like matrix inequalities and ended up with a conclusion that this model yield results that are more complex than the case of continuous FTCSMP.

On the other hand, it is well documented that time delays is a major cause for performance degradation of dynamical systems, ultimately, it may lead to loss of stability. The issue of time delay has been extensively researched. In JLS significant results were obtained and several reports can be cited [3]. The effect of time delays was also considered for continuous-time FTCSMP in [11]. It is the objective of this article to characterize the stochastic behavior of DTFTCS subject to state delays and driven by a constant gains state feedback controller. The analysis is to produce delay-independent stability conditions in terms of feasibility solution for some LMIs. The stochastic behavior for proposed DTFTCS in noise-free and noisy environments is considered. The results also test for the existence of a stabilizing H_∞ state feedback controller that guarantee the second moment stability and achieve δ level of noise attenuation. The findings are validated by a numerical example.

This paper is organized as follows: Section II describes the dynamical model of DTFTCS, the failure processes and the FDI process. Stochastic stability properties for DTFTCS with state delays is studied in Section III. Section IV defines and derives conditions for the stability of delayed DTFTCS for a given state feedback controller with constant gains. A numerical example is given in Section V and a concluding summary is briefed in Section VI.

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II. The Model of State Delayed DTFTCS

A discrete-time fault tolerant control system (DTFTCS) under normal operating conditions is described by

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + \varphi_x w_k \\ y_k &= Cx_k + \varphi_y w_k \\ z_k &= Dx_k + Eu_k \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the system state, $u_k \in \mathbb{R}^m$ is the system input, $y_k \in \mathbb{R}^p$ is the system measured output, and $z_k \in \mathbb{R}^q$ is the system controlled output. $w_k \in \mathbb{R}^t$, $\{w_k\} \in l_2[0, \infty)$ is an exogenous disturbance input which belongs to the space of square summable infinite vector sequences on $[0, \infty)$, that is;

$$\|w\|_2^2 = \mathcal{E}\{\sum_{k\in\mathbb{N}} |w_k|^2\} < \infty \tag{2}$$

$\mathbb{N}$ is the set of natural numbers. The occurrence of of failures in plant components changes the dynamics of the DTFTCS (1). In FTCSMP components failures have Markovian behavior represented by a Markov chain $\{\eta(k)\}$ and the failure process is not directly measurable instead it is detectable by an FDI process [11], [13] represented by another Markovian chain, $\{\Psi(k)\}$. Moreover, the control law is only a function of the measurable FDI process with constant feedback controller gains. Given a probability space $(\Omega,\ \mathcal{F},\ \mathbb{P})$ where Ω is the sample space, $\mathcal{F}$ is the algebra events, and $\mathbb{P}$ is the probability measure defined on $\mathcal{F}$. A DTFTCS subject to component failures and time delays in the state belongs to a class of hybrid stochastic discrete-time linear systems described by

$$\begin{aligned} x_{k+1} &= A(\eta_k)x_k + A_\tau(\eta_k)x_{k-d} + B(\eta_k)u(\Psi_k, k) + \varphi_x(\eta_k)w_k \\ y_k &= C(\eta_k)x_k + \varphi_y(\eta_k)w_k \\ z_k &= D(\eta_k)x_k + D_\tau(\eta_k)x_{k-d} + E(\eta_k)\ u(\Psi_k, k) \\ x_l &= \zeta_l, \quad l \in \{-d, ..., 0\}, \quad \eta(0) = \eta_0,\ \Psi(0) = \Psi_0 \end{aligned} \tag{3}$$

A state feedback controller has the form

$$u(\Psi_k, k) = -K(\Psi_k)x_k \tag{4}$$

where $A(\eta_k)$, $A_\tau(\eta_k)$, $B(\eta_k)$, $D(\eta_k)$, $D_\tau(\eta_k)$, $E(\eta_k)$, $\varphi_x(\eta_k)$, and $\varphi_y(\eta_k)$ are properly dimensioned real-valued system matrices, and are random in nature with Markovian transition characteristics. It is assumed that for all $l \in \{-d, ..., 0\}$, there exists a scalar $1 \le \epsilon < \infty$ such that $\|x_{k+1}\| \le \epsilon\|x_k\|$. η_k and Ψ_k are homogeneous discrete-time discrete-state Markov chains [5] with finite state spaces $S = \{1, 2, ..., s\}$ and $R = \{1, 2, ..., r\}$, respectively. The one-step transition probabilities from state (m) at time instant (k) to state (n) at time instant $(k+1)$ for the plant component failure process, $\{\eta_k\}$, is

$$\begin{aligned} \alpha_{mn} &= Pr\ \{\ \eta_{k+1} = n,\ \eta_k = m\ \} \\ \sum_{n=1}^{s} \alpha_{mn} &= 1.0 \quad \forall\, m \in S, \quad \alpha_{mn} \ \ge\ 0 \end{aligned} \tag{5}$$

The conditional transition probability for $\{\Psi_k\}$, is

$$\begin{aligned} q^i_{mn} &= Pr\ \{\ \Psi_{k+1} = m,\ \Psi_k = n,\ \eta_k = j\} \\ \sum_{n=1}^{r} q^i_{mn} &= 1.0\ \forall\, m \in R \text{ and } i \in S, \quad q^i_{mn} \ \ge\ 0 \end{aligned} \tag{6}$$

α_{mn} and q^i_{mn} are directly related to the component failure rates, and the FDI transition rates, respectively. These rates play a key-role in modelling different behaviors for the general class of AFTCSMP [11], [13].

Notations The following notations are used in the paper, the notation $M > N(\ge, <, \le\ ) \ 0$ is used to denote that $M - N$ is positive definite (positive semi-definite, negative definite, negative semi-definite) matrix. $\lambda_{min}(\cdot)$, $\lambda_{max}(\cdot)$ denote the minimum and the maximum eigenvalue, respectively. $\|.\|$ represents the Euclidean norm of [.]. $\mathcal{E}[\cdot]$ stands for the mathematical expectation. Also, for simplicity, $A(\eta_k) = A_i$, $A_\tau(\eta_k) = A_{\tau i}$, $B(\eta_k) = B_i$, $D(\eta_k) = D_i$, $D_\tau(\eta_k) = D_{\tau i}$ and $E(\eta_k) = E_i$ when $\eta_k = i \in S$ and $K(\Psi_k, k) = K_j$ when $\Psi_k = j \in R$. A symmetric matrix is equivalently written as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{12}^T & a_{22} \end{bmatrix} \triangleq \begin{bmatrix} a_{11} & a_{12} \\ * & a_{22} \end{bmatrix}$$

III. Stochastic Stability for State Delayed DTFTCS

The stochastic stability properties will be defined and studied first for for the DTFTCS (3) without exogenous noise ($w_k \equiv 0$) and without any input ($u_k \equiv 0$), in such a case the DTFTCS will be called autonomous system. The results will be extended to the case of noisy environment and new set of conditions will be derived and proved.

A. Stability Properties for Autonomous Delayed DTFTCS

An autonomous DTFTCS with delays can be described by

$$x_{k+1} = A_i x_k, \quad y_k = C_i x_k, \quad z_k = D_i x_k \tag{7}$$

There exist several stability definitions for the general class of DTFTCS. It can be shown, omitted for space limitation, that these definitions are equivalent for DTFTCS, that is, the satisfaction of one form implies the satisfaction of the other forms. In this article, second moment stochastic stability is used and it is defined as

Definition 1: The equilibrium point, $x = 0$, for the noise-free input-free DTFTCS (7) is said to be stochastically second moment stable, if for all finite $\zeta_i \in \mathbb{R}^n$ defined on $\{-d, ..., 0\}$, and any initial state $\{x_0, \eta_0, \Psi_0\}$ that lead to a sample solution $\{x_k, \eta_k, \Psi_k\}$, there exists a finite positive number $\tilde{\mathbb{N}}(\zeta_0, \eta_0, \Psi_0)$, such that

$$\sum_{k=0}^{\infty} \mathcal{E}\{\|x_k, \eta_k, \Psi_k|x_0, \eta_0, \Psi_0\|^2\} \le \tilde{\mathbb{N}}(\zeta_0, \eta_0, \Psi_0) < \infty$$

Theorem 1: The autonomous DTFTCS (7) is second moment stable, if there exist a positive definite matrix $Q > 0$ and $P_{ij} > 0\, \forall\, i \in S, j \in R$ satisfying

$$\begin{bmatrix} A_i^T \tilde{P}_{ij} A_i - P_{ij} + Q & A_i^T \tilde{P}_{ij} A_{\tau i} \\ * & A_{\tau i}^T \tilde{P}_{ij} A_{\tau i} - Q \end{bmatrix} < 0$$

where

$$\tilde{P}_{ij} = \sum_{n=1}^{s} \alpha_{nj} \sum_{m=1}^{r} q^j_{im} P_{nm} \tag{8}$$

Proof
Selecting a Lyapunov function as

$$V(x_k,\eta_k,\Psi_k)=x_k^TP_{ij}x_k+\sum_{l=k-d}^{k-1}x_l^TQx_l \quad \forall\, i\in S,\ j\in R \tag{9}$$

If the failure process represented by the Markov chain η_k is in state i at time k, then the DTFTCS will be described as

$$\begin{array}{l} x_{k+1}=A_ix_k, \quad y_k=C_ix_k, \quad z_k=D_ix_k \\ x_l=\zeta_l, \quad l\in\{-d,...,0\}, \quad \eta(0)=\eta_0,\ \Psi(0)=\Psi_0 \end{array} \tag{10}$$

Define

$$\mathfrak{X}=\{x_{k-d},x_{k-d+1},...,x_k\}$$

The one-step forward difference equation of $V(\mathfrak{X}_k,\eta_k,\Psi_k)$ is

$$\begin{array}{l} \mathcal{E}\{V(\mathfrak{X}_{k+1},\eta_{k+1},\Psi_{k+1})|\mathfrak{X}_k,\eta_k,\Psi_k-V(\mathfrak{X}_k,\eta_k,\Psi_k)\}= \\ \mathcal{E}\{x_{k+1}^TP_{ij}x_{k+1}+\sum\limits_{l=k+1-d}^{k}x_l^TQx_l|x_k,\eta_k,\Psi_k\} \\ -x_k^TP_{ij}x_k-\sum\limits_{l=k-d}^{k-1}x_l^TQx_l \end{array} \tag{11}$$

The one-step forward increment for the noise-free input-free DTFTCS with delays (7) gives

$$\begin{array}{l} \mathcal{E}\{x_{k+1}^TP_{ij}x_{k+1}|x_k,\eta_k,\Psi_k\}-x_k^TP_{ij}x_k \\ =\mathcal{E}\{[x_k^TA_i^T+x_{k-d}^TA_{\tau i}^T]P_{ij}[A_ix_k+A_{\tau i}x_{k-d}]\} \\ +\sum\limits_{l=k+1-d}^{k}x_l^TQx_l-x_k^TP_{ij}x_k-\sum\limits_{l=k-d}^{k-1}x_l^TQx_l<0 \\ \\ =\mathcal{E}\{x_k^TA_i^TP_{ij}A_ix_k+x_k^TA_i^TP_{ij}A_{\tau i}x_{k-d} \\ +x_{k-d}^TA_{\tau i}^TP_{ij}A_ix_k+x_{k-d}^TA_{\tau i}^TP_{ij}A_{\tau i}x_{k-d}\} \\ -x_k^TP_{ij}x_k+x_k^TQx_k-x_{k-d}^TQx_{k-d}<0 \\ \\ =x_k^TA_i^T\tilde{P}_{ij}A_ix_k-x_k^TP_{ij}x_k+x_k^TQx_k \\ +x_k^TA_i^T\tilde{P}_{ij}A_{\tau i}x_{k-d}+x_{k-d}^TA_{\tau i}^T\tilde{P}_{ij}A_ix_k \\ +x_{k-d}^TA_{\tau i}^T\tilde{P}_{ij}A_{\tau i}x_{k-d}-x_{k-d}^TQx_{k-d}<0 \end{array} \tag{12}$$

Define

$$y_k^T=[x_k \quad x_{k-d}]$$

and

$$\Theta_{ij}=\begin{bmatrix} A_i^T\tilde{P}_{ij}A_i-P_{ij}+Q & A_i^T\tilde{P}_{ij}A_{\tau i} \\ * & A_{\tau i}^T\tilde{P}_{ij}A_{\tau i}-Q \end{bmatrix} \tag{13}$$

then we have $y_k^T\Theta y_k<0$
For $x_k\neq 0$,

$$\begin{array}{l} \dfrac{\mathcal{E}\{V(x_{k+1},\eta_{k+1},\Psi_{k+1})|x_k,\eta_k,\Psi_k\}-V(x_k,\eta_k,\Psi_k)}{V(x_k,\eta_k,\Psi_k)} \\ <\dfrac{y_k^T\Theta_{ij}y_k}{x_k^TP_{ij}x_k}<-\min\limits_{i\in S,j\in R}\{\dfrac{\lambda_{min}(-\Theta_{ij})}{\lambda_{max}(P_{ij})}\} \end{array} \tag{14}$$

Define

$$\gamma=1-\min_{i\in S,j\in R}\{\frac{\lambda_{min}(-\Theta_{ij})}{\lambda_{max}(P_{ij})}\} \tag{15}$$

The upper bound of γ can be calculated as

$$\begin{array}{l} \dfrac{\mathcal{E}\{V(x_{k+1},\eta_{k+1},\Psi_{k+1})|x_k,\eta_k,\Psi_k\}-V(x_k,\eta_k,\Psi_k)}{V(x_k,\eta_k,\Psi_k)} \\ =\gamma-1<0 \end{array}$$

and the lower bound can be found as

$$0<\frac{\mathcal{E}\{V(x_{k+1},\eta_{k+1},\Psi_{k+1})|x_k,\eta_k,\Psi_k\}}{V(x_k,\eta_k,\Psi_k)}<\gamma$$

This gives

$$0.0<\gamma<1.0 \tag{16}$$

As a result, we have

$$\mathcal{E}\{V(x_k,\eta_k,\Psi_k)|x_k,\eta_k,\Psi_k\}<\gamma^kV(x_o,\eta_o,\Psi_o) \tag{17}$$

Substituting $V(x_o,\eta_o,\Psi_o)=x_o^TP_{i_oj_o}x_o$ and taking the summation for both hand sides, we get

$$\begin{array}{l} \sum\limits_{k=0}^{\infty}\mathcal{E}\{\|x_k,\eta_k,\Psi_k|x_0,\eta_0,\Psi_0\|^2\}< \\ \sum\limits_{k=0}^{\infty}\gamma^k\|x_o\|^2\lambda_{max}(P_{i_oj_o})\leq\|x_o\|^2\lambda_{max}(P_{i_oj_o})\sum\limits_{k=0}^{\infty}\gamma^k\leq \\ \left\{\|x_o\|^2\lambda_{max}(P_{i_oj_o})\lim\limits_{N\to\infty}\dfrac{1-\gamma^{N+1}}{1-\gamma}=\|x_o\|^2\lambda_{max}(P_{i_oj_o})\dfrac{1}{1-\gamma}\right\} \\ <\infty \end{array} \tag{18}$$

This result satisfies Definition 1 and hence the autonomous DTFTCS (7) is second moment stable and the proof is complete.

B. H_∞ *Performance For Input-Free Delayed DTFTCS*

H_∞ performance of delayed DTFTCS will be studied for the case of input-free system. The work starts by defining the stability of DTFTCS with δ-level of noise rejection. Necessary and sufficient conditions for the second moment stability will be derived and proved. An equivalent set of sufficient conditions will be constructed. The results are stated in terms of feasibility solution for LMI. A noisy input-free DTFTCS with state delays can be described by

$$\begin{array}{r} x_{k+1}=A_ix_k+A_{\tau_i}x_{k-d}+\varphi_{x_i}w_k \\ y_k=C_ix_k+\varphi_{y_i}w_k \\ z_k=D_ix_k+D_{\tau_i}x_{k-d} \end{array} \tag{19}$$

Definition 2: The equilibrium point, $x=0$, for the noisy input-free DTFTCS with delays (19) is said to be second moment stable with δ-disturbance attenuation, if the autonomous DTFTCS (7) is second moment stable, and for any noise disturbance $w_k\in l_2$ with δ prescribed level of disturbance attenuation, the output response $z_k\in\mathbb{R}^P$ satisfies

$$\sum_{k=0}^{\infty}\mathcal{E}[z_k^Tz_k|x_0,\eta_0,\Psi_0]<\delta^2\sum_{k=0}^{\infty}w_k^Tw_k$$

The following theorem states necessary and sufficient conditions for the stochastic stability with δ-disturbance rejection level.

Theorem 2: The noisy input-free DTFTCS (19) is second moment stable with δ-disturbance attenuation for any noise disturbance $w_k \in l_2$, if there exist two set of matrices $Q > 0,\ and\ P_{ij} > 0\ \forall i \in S\ and\ j \in R$ satisfying the following matrix inequalities

$$\begin{bmatrix} A_i^T \tilde{P}_{ij} A_i - P_{ij} + Q + D_i^T D_i & A_i^T \tilde{P}_{ij} A_{\tau i} + D_i^T D_{\tau i} & A_i^T \tilde{P}_{ij} \varphi_{x_i} \\ * & A_{\tau i}^T \tilde{P}_{ij} A_{\tau i} - Q + D_{\tau i}^T D_{\tau i} & A_{\tau i}^T \tilde{P}_{ij} \varphi_{x_i} \\ * & * & \varphi_{x_i}^T \tilde{P}_{ij} \varphi_{x_i} - \delta^2 \mathcal{I} \end{bmatrix} < 0$$

where $\tilde{P}_{ij}$ is defined in (8).

Proof: Selecting a Lyapunov function as given in (9), the one-step forward difference equation for the noisy input-free state-delayed DTFTCS is

$$\mathcal{E}\{x_{k+1}^T P_{ij} x_{k+1} + \sum_{l=k+1-d}^{k} x_l^T Q x_l | x_k, \eta_k, \Psi_k\} - x_k^T P_{ij} x_k - \sum_{l=k-d}^{k-1} x_l^T Q x_l =$$

$$\mathcal{E}\{[x_k^T A_i^T + x_{k-d}^T A_{\tau i}^T + w_k^T \varphi_{x_i}^T] P_{ij} [A_i x_k + A_{\tau i} x_{k-d} + \varphi_{x_i} w_k]\} + \sum_{l=k+1-d}^{k} x_l^T Q x_l - x_k^T P_{ij} x_k - \sum_{l=k-d}^{k-1} x_l^T Q x_l < 0$$

$$\begin{aligned} = \mathcal{E}\{ & x_k^T A_i^T P_{ij} A_i x_k + x_k^T A_i^T P_{ij} A_{\tau i} x_{k-d} + x_k^T A_i^T P_{ij} \varphi_{x_i} w_k \\ & + x_{k-d}^T A_{\tau i}^T P_{ij} A_i x_k + x_{k-d}^T A_{\tau i}^T P_{ij} A_{\tau i} x_{k-d} + x_{k-d}^T A_{\tau i}^T P_{ij} \varphi_{x_i} w_k \\ & + w_k^T \varphi_{x_i}^T P_{ij} A_i x_k + w_k^T \varphi_{x_i}^T P_{ij} A_{\tau i} x_{k-d} + w_k^T \varphi_{x_i}^T P_{ij} \varphi_{x_i} w_k\} \\ & - x_k^T P_{ij} x_k + x_k^T Q x_k - x_{k-d}^T Q x_{k-d} < 0 \end{aligned}$$

$$\begin{aligned} = & x_k^T A_i^T \tilde{P}_{ij} A_i x_k + x_k^T A_i^T \tilde{P}_{ij} A_{\tau i} x_{k-d} + x_k^T A_i^T \tilde{P}_{ij} \varphi_{x_i} w_k \\ & + x_{k-d}^T A_{\tau i}^T \tilde{P}_{ij} A_i x_k + x_{k-d}^T A_{\tau i}^T \tilde{P}_{ij} A_{\tau i} x_{k-d} + x_{k-d}^T A_{\tau i}^T \tilde{P}_{ij} \varphi_{x_i} w_k \\ & + w_k^T \varphi_{x_i}^T \tilde{P}_{ij} A_i x_k + w_k^T \varphi_{x_i}^T \tilde{P}_{ij} A_{\tau i} x_{k-d} + w_k^T \varphi_{x_i}^T \tilde{P}_{ij} \varphi_{x_i} w_k \\ & - x_k^T P_{ij} x_k + x_k^T Q x_k - x_{k-d}^T Q x_{k-d} \leq 0 \end{aligned} \tag{20}$$

The input-free delayed DTFTCS (19) possess a disturbance rejection property with attenuation level δ, if the condition in Definition 2 is satisfied, that is

$$z_k^T z_k - \delta^2 w_k^T w_k \leq 0$$

The one-step forward increment for the system controlled output (23) gives

$$\begin{aligned} & [x_k^T D_i^T + x_{k-d}^T D_{\tau i}^T][D_i x_k + D_{\tau i} x_{k-d}] - \delta^2 w_k^T \mathcal{I} w_k \leq 0 \\ & x_k^T D_i^T D_i x_k + x_k^T D_i^T D_{\tau i} x_{k-d} + x_{k-d}^T D_{\tau i}^T D_i x_k \\ & + x_{k-d}^T D_{\tau i}^T D_{\tau i} x_{k-d} - \delta^2 w_k^T \mathcal{I} w_k \leq 0 \end{aligned}$$

Define

$$y_k^T = [x_k^T \ \ x_{k-d}^T \ \ w_k^T]$$

we have $y_k^T \Theta y_k \leq 0$, where Θ is the matrix inequality in Theorem 2. Following similar arguments used to prove Theorem 1, the input-free delayed DTFTCS (19) is second moment stable and achieve δ level of noise attenuation. The proof is complete.

IV. Stochastic Stabilization for State Delayed DTFTCS

In this section, stochastic stability properties will be characterized for the state delayed DTFTCS (3) when driven by state feedback in ideal noise-free and more practical noisy environment. The results provide a test criteria for the second moment stability given a state feedback control law with constant gains, K_j. Several equivalent sufficient conditions will be constructed in terms of feasibility solutions for some LMIs.

A. Stabilization of Delayed DTFTCS in Noisy-Free Environments

A noise-free DTFTCS with delays is described as

$$\begin{aligned} x_{k+1} &= \hat{A}_{ij} x_k + A_{\tau_i} x_{k-d} \\ y_k &= C_i x_k \\ z_k &= \hat{D}_{ij} x_k + D_{\tau_i} x_{k-d} \end{aligned} \tag{21}$$

where

$$\begin{aligned} \hat{A}_{ij} &= A_i - B_i K_j \\ \hat{D}_{ij} &= D_i - E_i K_j \end{aligned} \tag{22}$$

Theorem 3: The noise-free DTFTCS state-delayed DTFTCS (21) is second moment stable, if there exist $Q > 0$ and $P_{ij} > 0, \forall\, i \in S, j \in R$ satisfying

$$\begin{bmatrix} \hat{A}_i^T \tilde{P}_{ij} \hat{A}_i - P_{ij} + Q & \hat{A}_i^T \tilde{P}_{ij} A_{\tau i} \\ * & A_{\tau i}^T \tilde{P}_{ij} A_{\tau i} - Q \end{bmatrix} < 0$$

where $\tilde{P}_{ij}$ is defined in (8).

Proof: Arguments similar to those in Theorem 1 can be followed with the augmented closed-loop system matrix, $\hat{A}$, is used instead of the open loop system matrix, A. The proof is omitted.

B. Stabilization of Delayed DTFTCS in Noisy Environments

The closed-loop noisy DTFTCS with state delays can be written as

$$\begin{aligned} x_{k+1} &= \hat{A}_{ij} x_k + A_{\tau_i} x_{k-d} + \varphi_{x_i} w_k \\ y_k &= C_i x_k + \varphi_{y_i} w_k \\ z_k &= \hat{D}_{ij} x_k + D_{\tau_i} x_{k-d} \end{aligned} \tag{23}$$

where $\hat{A}_{ij}$ and $\hat{D}_{ij}$ are defined in (22).

Theorem 4: A necessary and sufficient condition for a linear state feedback control law with constant gains K_j to stabilize the noisy state delayed DTFTCS (23) with δ-disturbance attenuation for any noise disturbance $w_k \in l_2$, is the existence of a preselected $Q > 0$ and $P_{ij} > 0\ \forall i \in S,\ j \in R$ that satisfy the following matrix inequality

$$\begin{bmatrix} \hat{A}_i^T \tilde{P}_{ij} \hat{A}_i - P_{ij} + Q + \hat{D}_i^T \hat{D}_i & A_i^T \tilde{P}_{ij} A_{\tau i} + \hat{D}_i^T D_{\tau i} & \hat{A}_i^T \tilde{P}_{ij} \varphi_{x_i} \\ * & A_{\tau i}^T \tilde{P}_{ij} A_{\tau i} - Q + D_{\tau i}^T D_{\tau i} & A_{\tau i}^T \tilde{P}_{ij} \varphi_{x_i} \\ * & * & \varphi_{x_i}^T \tilde{P}_{ij} \varphi_{x_i} - \delta^2 \mathcal{I} \end{bmatrix} < 0$$

$\tilde{P}_{ij}$ is defined in (8), and $\hat{A}_{ij}$ and $\hat{D}_{ij}$ are defined in (22).

Proof: Similar to Theorem 3 with the augmented closed-loop matrix, $\hat{A}$, replaces the open loop system matrix A.

Theorem 5: A sufficient condition for a linear state feedback control law with constant gains K_j to stabilize the noisy state delayed DTFTCS (23) with δ-disturbance attenuation for any noise disturbance $w_k \in l_2$, is the existence of a preselected $Q > 0$ and $\chi_{ij} > 0\ \forall i \in S,\ j \in R$ that satisfy the following matrix inequality

$$\begin{bmatrix} -\chi_{ij} + \chi_{ij} Q \chi_{ij} & 0 & 0 & \mathbb{A}_{ij}^T & \chi_{ij}\hat{D}_i^T \\ * & -Q + D_{\tau i}^T D_{\tau i} & 0 & \mathbb{A}_{\tau ij}^T & 0 \\ * & * & -\delta^2 \mathcal{I} & \mathbb{W}_{ij}^T & 0 \\ * & * & * & -\mathbb{Z} & 0 \\ * & * & * & * & -\mathcal{I} \end{bmatrix} < 0$$

where

$$\begin{aligned} \nabla_{ij}[.] = [&\sqrt{\alpha_{1i} q_{j1}^i}[.] ... \sqrt{\alpha_{1i} q_{jr}^i}[.]\ \sqrt{\alpha_{2i} q_{j1}^i}[.] ... \\ &\sqrt{\alpha_{2i} q_{jr}^i}[.] ... \sqrt{\alpha_{si} q_{j1}^i}[.]\ ...\ \sqrt{\alpha_{si} q_{jr}^i}[.]\ ] \\ \mathbb{Z} = diag\{&[\chi_{i1},\ \chi_{i2},\ ..., \chi_{ir}]_{i=1,2,...,s}\} \end{aligned} \tag{24}$$

and

$$\begin{aligned} \mathbb{A}_{ij}^T &= \nabla_{ij}[\chi_{ij}(A_i - B_i K_j)^T] \\ \mathbb{A}_{\tau ij}^T &= \nabla_{ij}[A_{\tau i}^T] \\ \mathbb{W}_{ij}^T &= \nabla_{ij}[\varphi_{x_i}^T] \end{aligned} \tag{25}$$

Proof: Using the results of Theorem 4 an equivalent set of matrix inequalities can be obtained using Shur complement.

Lemma 1: (Shur Complement [4]) For appropriately dimensioned constant matrices $\phi = \phi^T$, $\omega = \omega^T$, and θ, the linear matrix inequality

$$\begin{bmatrix} \phi & \theta^T \\ \theta & -\omega \end{bmatrix} < 0$$

is equivalent to $\phi + \theta^T \omega^{-1} \theta < 0$ and $\omega > 0$.

Define $\chi_{ij} = P_{ij}^{-1}$, pre- and post- multiply the matrix inequality in Theorem 4 by $diag(\chi_{ij}, \mathcal{I}, \mathcal{I})$, we get

$$\begin{bmatrix} \chi_{ij}\hat{A}_i^T \tilde{\chi}_{ij}^{-1} \hat{A}_i \chi_{ij} - \chi_{ij} + \chi_{ij} Q \chi_{ij} + \chi_{ij}\hat{D}_i^T \hat{D}_i \chi_{ij} & \chi_{ij}\hat{A}_i^T \tilde{\chi}_{ij}^{-1} A_{\tau i} + \chi_{ij}\hat{D}_i^T D_{\tau i} & \chi_{ij}\hat{A}_i^T \tilde{\chi}_{ij}^{-1} \varphi_{x_i} \\ * & A_{\tau i}^T \tilde{\chi}_{ij}^{-1} A_{\tau i} - Q + D_{\tau i}^T D_{\tau i} & A_{\tau i}^T \tilde{\chi}_{ij}^{-1} \varphi_{x_i} \\ * & * & \varphi_{x_i}^T \tilde{\chi}_{ij}^{-1} \varphi_{x_i} - \delta^2 \mathcal{I} \end{bmatrix} < 0 \tag{26}$$

The identifications in (24) and (25) give

$$\begin{aligned} \chi_{ij}\hat{A}_{ij}^T \tilde{\chi}_{ij}^{-1} \hat{A}_{ij} \chi_{ij} &= \mathbb{A}_{ij}^T \mathbb{Z}^{-1} \mathbb{A}_{ij} \\ A_{\tau i}^T \tilde{\chi}_{ij}^{-1} A_{\tau i} &= \mathbb{A}_{\tau ij}^T \mathbb{Z}^{-1} \mathbb{A}_{\tau ij} \\ \varphi_{x_i}^T \tilde{\chi}_{ij}^{-1} \varphi_{x_i} &= \mathbb{W}_{ij}^T \mathbb{Z}^{-1} \mathbb{W}_{ij} \end{aligned} \tag{27}$$

Using Shur complement, Lemma 1, the coupled matrix inequality (26) is equivalent to

$$\begin{bmatrix} \Omega_{11} & 0 & 0 & \mathbb{A}_{ij}^T \\ * & -Q + D_{\tau i}^T D_{\tau i} & 0 & \mathbb{A}_{\tau ij}^T \\ * & * & -\delta^2 \mathcal{I} & \mathbb{W}_{ij}^T \\ * & * & * & -\mathbb{Z} \end{bmatrix} < 0 \tag{28}$$

where $\Omega_{11} = -\chi_{ij} + \chi_{ij} Q \chi_{ij} + \chi_{ij}\hat{D}_i^T \hat{D}_i \chi_{ij}$. Applying Shur complement to the term $\chi_{ij}\hat{D}_i^T \hat{D}_i \chi_{ij}$, we get

$$\begin{bmatrix} -\chi_{ij} + \chi_{ij} Q \chi_{ij} & 0 & 0 & \mathbb{A}_{ij}^T & \chi_{ij}\hat{D}_i^T \\ * & -Q + D_{\tau i}^T D_{\tau i} & 0 & \mathbb{A}_{\tau ij}^T & 0 \\ * & * & -\delta^2 \mathcal{I} & \mathbb{W}_{ij}^T & 0 \\ * & * & * & -\mathbb{Z} & 0 \\ * & * & * & * & -\mathcal{I} \end{bmatrix} < 0 \tag{29}$$

The proof is complete. The LMI of Theorem 5 are non-linear in the term $\chi_{ij} Q \chi_{ij}$. In the following, a set of less expensive conditions that are easier to solve for χ_{ij} and to test for a given set of K_j will be constructed by proper parametrization for the preselected matrix $Q > 0$.

Case 1: Select

$$Q = \tau \mathbb{I}$$

This selection is consistent with the fact that a positive definite matrix has positive eigenvalues, the LMI in (29) is equivalent to

$$\begin{bmatrix} -\chi_{ij} & 0 & 0 & \mathbb{A}_{ij}^T & \chi_{ij}\hat{D}_i^T & \chi_{ij} \\ * & -\tau\mathbb{I} + D_{\tau i}^T D_{\tau i} & 0 & \mathbb{A}_{\tau ij}^T & 0 & 0 \\ * & * & -\delta^2 \mathcal{I} & \mathbb{W}_{ij}^T & 0 & 0 \\ * & * & * & -\mathbb{Z} & 0 & 0 \\ * & * & * & * & -\mathcal{I} & 0 \\ * & * & * & * & * & -\tau^{-1}\mathcal{I} \end{bmatrix} < 0 \tag{30}$$

Case 2: Select

$$Q = N^T N$$

Since Q is real symmetric positive definite, then the existence of a real nonsingular matrix N is guaranteed. The LMI in (29) is equivalent to

$$\begin{bmatrix} -\chi_{ij} & 0 & 0 & \mathbb{A}_{ij}^T & \chi_{ij}\hat{D}_i^T & \chi_{ij} N^T \\ * & -N^T N + D_{\tau i}^T D_{\tau i} & 0 & \mathbb{A}_{\tau ij}^T & 0 & 0 \\ * & * & -\delta^2 \mathcal{I} & \mathbb{W}_{ij}^T & 0 & 0 \\ * & * & * & -\mathbb{Z} & 0 & 0 \\ * & * & * & * & -\mathcal{I} & 0 \\ * & * & * & * & * & -\mathcal{I} \end{bmatrix} < 0 \tag{31}$$

Other selections for Q can be made to construct other sets of equivalent LMI to test for for the second moment stability for the noisy state delayed DTFTCS (23).

V. NUMERICAL EXAMPLE

A second-order DTFTCS subject to single component failure has the following system parameters

$$A_1 = \begin{bmatrix} -2.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.0 & 1.0 \\ -1.0 & 0.0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix},$$

$$A_{\tau_1} = \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.5 \end{bmatrix}, \quad A_{\tau_2} = \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & -0.5 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.5 & 0.0 \end{bmatrix}, \quad D_{\tau_1} = \begin{bmatrix} 0.0 & 0.5 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.1 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.5 & 0.0 \end{bmatrix}, \quad D_{\tau_2} = \begin{bmatrix} 0.0 & 0.5 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.1 \end{bmatrix},$$

$$\varphi_{y_1} = \varphi_{y_2} = \begin{bmatrix} 0.1 \end{bmatrix}, \quad \varphi_{x_1} = \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix}, \quad \varphi_{x_2} = \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix},$$

For such system, η_k the component failure is a finite Markov chain with $S = \{1, 2\}$ and the FDI process Ψ_k also has a finite state space $R = \{1, 2\}$. The two modes of operation are: fault-free system represented by state 1, and impaired system operation represented by state 2. The failure rates representing these two modes of operation are

$$\alpha_{mn} = \begin{bmatrix} 0.70 & 0.30 \\ 0.40 & 0.60 \end{bmatrix},$$

$$q^1_{mn} = \begin{bmatrix} 0.20 & 0.80 \\ 0.2 & 0.80 \end{bmatrix}, \quad q^2_{mn} = \begin{bmatrix} 0.15 & 0.85 \\ 0.25 & 0.75 \end{bmatrix}.$$

As per Theorem 5, the second moment stability was examined and verified by the existence of positive definite symmetric matrices χ_{ij}. The following design parameters were selected and used.

- Level of attenuation $\delta = 1.25$
- Weighting matrices $R_{ij} = \mathcal{I}$
- Constant state feedback controller

$$K_1 = \begin{bmatrix} 2.15 & -1.50 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 2.35 & -1.25 \end{bmatrix}$$

A sample path simulation for the controlled output are shown in Figure 1.

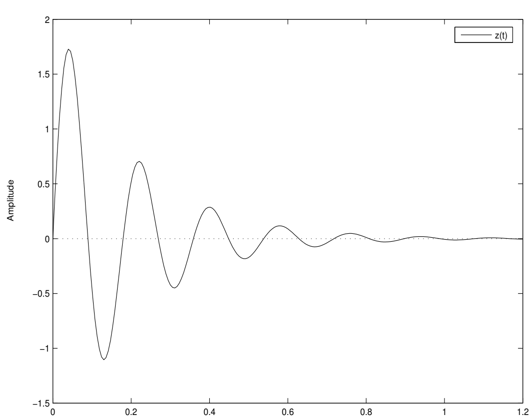

Fig. 1: Single sample path simulation: Controlled Output

VI. Conclusion

A DTFTCS in noisy environments subject to state delays has been developed. A delay-independent sufficient condition that guarantee the H_∞ second moment stability and achieve δ-level of disturbance rejection was derived. A test condition for a given state feedback controller was formulated as a feasibility solution for a set of LMIs. The obtained result was validated by a numerical example. In future work, a design methodology to synthesize a stabilizing fault tolerant state feedback controller is to be developed.

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