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Stabilization of discrete time-varying delay systems: a convex parameter dependent approach

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***Abstract*— A parameter dependent Lyapunov-Krasovskii based approach is developed to deal with robust stability analysis as well as with the robust stabilization of discrete time-varying systems with time-varying delay. It uses a polytopic representation of uncertainties which can affect dynamic and control matrices. Both robust analysis conditions and synthesis of robust state feedback gains conditions are presented as simple convex feasibility tests. Some numerical examples are presented to illustrate the efficacy of the proposed LMI conditions.**

I. INTRODUCTION

In this paper it is considered the existence of linear parameter dependent Lyapunov-Krasovskii functionals assuring the robust stability of uncertain discrete time-varying systems with time-varying state delay, or simply uncertain discrete time-varying delay system (DTVDS). These functionals are also used to obtain convex synthesis conditions that allow the design of robust state feedback gains.

The study of DTVDS is an important issue and, recently, lots of attention have been paid to this subject mainly because digital control systems have unavoidable delays, which can degenerate performance and even lead systems to instability, see for example [5], [6], [2], [16], [8]. In other cases, control actions can introduce delayed states into the closed-loop system [10, pp. 3]. However, the problems of robust stability and robust stabilization of DTVDS remain open, since most of the solutions found in the literature are based on quadratic stability (QS) approach, i.e., constant and parameter independent Lyapunov-Krasovskii matrices are employed.

Notice that, stability of time-varying systems has been focused on delay-free systems during the last decades. In this context, several results for linear time-varying systems can be found in the literature. For instance, see [1] for QS approach; [4] for a study on parameterized Lyapunov functions on stability problems of time-invariant systems and their application to the stability analysis for a class of time-varying systems; [24] and [13] for switched Lyapunov functions applied to time-varying systems; [3] for affine parameter dependent Lyapunov functions applied to time-varying systems; [23] for the proposition of the biquadratic stability analysis with a Lyapunov function depending quadratically on the uncertain parameters and on the states; [11] and references therein for the stability of piecewise linear systems. Although the existence of all these different approaches to deal with discrete time-varying delay-free systems, in case of DTVDS, QS approach has been the most used [17]. QS approach has been widely used to deal with norm-bounded and polytopic type uncertainties, yielding convex optimization problems that can be efficiently solved by means of commercially available LMI solvers [22], [7]. For polytopic systems with time-varying uncertainties subject to arbitrary variation rates, a parameter dependent Lyapunov function is proposed in [3], encompassing the quadratic stability case.

However, to the best author's knowledge, convex formulations to deal with DTVDS are limited to QS based approach and, usually, consider delayed systems affected by norm-bounded uncertainties. Note that, in this case, parameter dependent Lyapunov-Krasovskii functionals are useless [9]. The conditions presented in [3] have motivated the development of the conditions formulated in this paper, for uncertain DTVDS. These conditions are convex for both robust stability analysis and robust state feedback gains design.

In the next section, some definitions and the problem statement are presented. Then, in section III the main results are given. Some examples are given in section IV. Finally, some conclusions are given in the section V.

Notation: The notation used here is quite standard. $\mathbb{R}$ is the set of real numbers and $\mathbb{N}^*$ is the set of natural numbers excluded the zero. $\mathbf{I}_n$ and $\mathbf{0}$ denotes, respectively, the $n \times n$ identity matrix and the null matrix of appropriate dimensions. $M > \mathbf{0}$ ($M < \mathbf{0}$) means that matrix M is positive (negative) definite. M' is the transpose of M. The symbol $\star$ stands for symmetric blocks in the LMIs. α_k (d_k) means $\alpha(k)$ ($d(k)$) and α_{ki} is the i-th entry of α_k.

II. PRELIMINARIES AND PROBLEM STATEMENT

Consider the following uncertain time-varying discrete-time system with delayed states given by

$$x_{k+1} = A(\alpha_k)x_k + A_d(\alpha_k)x_{k-d_k} + B(\alpha_k)u_k,$$
$$x_k = \phi(k),\ k \in [-\bar{d},\ 0] \quad (1)$$

where k is the sampling time, $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^p$ is the input control signal, d_k is the time-varying state delay which is limited by

$$\underline{d} \le d_k \le \bar{d} \quad (2)$$

with $(\underline{d}, \bar{d}) \in \mathbb{N}^* \times \mathbb{N}^*$ representing the possible variation band of the delay value, d_k. $B \in \mathbb{R}^{n\times p}$ is a fixed input control matrix and $[A(\alpha_k)|A_d(\alpha_k)|B(\alpha_k)] \equiv [A|A_d|B](\alpha_k) \in$

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$\mathbb{R}^{n\times 2n+p}$ are unknown time-varying matrices belonging to polytope $\mathcal{P}$

$$\mathcal{P} \equiv \Big\{[A|A_d|B](\alpha_k) \;:\; [A|A_d|B](\alpha_k) = \sum_{i=1}^{N}[A|A_d|B]_i\alpha_{ki}, \quad \alpha_k \in \Omega\Big\} \quad (3)$$

$$\Omega \equiv \Big\{\alpha_k : \alpha_k \in \mathbb{R}^N, \sum_{i=1}^{N}\alpha_{ki} = 1, \alpha_{ki} \geq 0\Big\} \quad (4)$$

where the vertices $\Upsilon_i = [A_i|A_{di}|B_i] \equiv [A|A_d|B]_i$ are precisely known and α_k is the bounded time-varying parameter. In particular, observe that if $\alpha_k = \alpha, \ \forall k$, then the description given in (1) recovers the case where the dynamic matrices of the system are time-invariant, but the delay is still time-varying.

In this paper, it is considered the following control law

$$u(k) = Kx_k + K_d x_{k-d_k} \quad (5)$$

where $[K|K_d] \in \mathbb{R}^{p\times 2n}$ are the robust state feedback gains that assure the robust stability of the closed-loop system (1)-(4), i.e., the stability of (1)-(4) with (5) is assured $\forall\ \alpha_k \in \Omega$. Therefore, the uncertain time-varying closed-loop system (1)-(4) with (5) is given by

$$x_{k+1} = \tilde{A}(\alpha_k)x_k + \tilde{A}_d(\alpha_k)x_{k-d_k} \quad (6)$$

with

$$\left.\begin{array}{c} \tilde{A}(\alpha_k) \equiv A(\alpha_k) + B(\alpha_k)K \\ \tilde{A}_d(\alpha_k) \equiv A_d(\alpha_k) + B(\alpha_k)K_d \end{array}\right\} \quad (7)$$

where $[\tilde{A}|\tilde{A}_d|B](\alpha_k) \in \tilde{\mathcal{P}}$ with

$$\tilde{\mathcal{P}} \equiv \Big\{[\tilde{A}|\tilde{A}_d](\alpha_k) \;:\; [\tilde{A},\tilde{A}_d](\alpha_k) = \sum_{i=1}^{N}[\tilde{A}|\tilde{A}_d]_i\alpha_{ki}, \quad \alpha_k \in \Omega\Big\} \quad (8)$$

It is worth to mention that, if the delay d_k is not known, then it is enough to make $K_d = \mathbf{0}$ in equation (5). If d_k is known, then the possibility of using K and K_d may improve the performance of the closed-loop system (6). Notice that this can occur in systems where some kind of time-stamped measurements or state estimative is used [21].

The objective of this paper is to give convex conditions solving the following problems:

Problem 1 *Given $d(k)$ subject to (2), determine if the time-varying uncertain DTVDS given in (6) is robustly stable.*

Problem 2 *Find a pair of gains $[K|K_d]$ such that the system (1)-(4) controlled by (5) be robustly stable.*

III. Main Results

In this section, it is considered the following Lyapunov-Krasovskii candidate functional

$$V(\alpha_k, k) = \sum_{v=1}^{3} V_v(\alpha_k, k) \quad (9)$$

with

$$V_1(\alpha_k, k) = x_k' P(\alpha_k) x_k, \quad (10)$$

$$V_2(\alpha_k, k) = \sum_{j=k-d_k}^{k-1} x_j' Q(\alpha_j) x_j, \quad (11)$$

$$V_3(\alpha_k, k) = \sum_{\ell=2-\bar{d}}^{1-\underline{d}} \ \sum_{j=k+\ell-1}^{k-1} x_j' Q(\alpha_j) x_j, \quad (12)$$

where matrices $P(\alpha_k)$ and $Q(\alpha_k)$ can assume different values at each instant k.

A. Robust Stability Analysis

Theorem 1 *The time-varying system (6) subject to (2), (4) and (8) is robustly stable if there exist symmetric matrices $\mathbf{0} < P(\alpha_k) \in \mathbb{R}^{n\times n}$, $\mathbf{0} < Q(\alpha_k) \in \mathbb{R}^{n\times n}$ and a scalar $\beta = \bar{d} - \underline{d} + 1$, with $\underline{d}$ and $\bar{d}$ known, such that one of the following equivalent conditions is verified*

a)

$$\Gamma(\alpha_k) \equiv \begin{bmatrix} \Gamma_{11}(k) & \tilde{A}(\alpha_k)' P(\alpha_{k+1}) \tilde{A}_d(\alpha_k) \\ \star & \Gamma_{22}(k) \end{bmatrix} < \mathbf{0} \quad (13)$$

with $\Gamma_{11}(k) \equiv \tilde{A}(\alpha_k)' P(\alpha_{k+1}) \tilde{A}(\alpha_k) + \beta Q(\alpha_k) - P(\alpha_k)$ and $\Gamma_{22}(k) \equiv \tilde{A}_d(\alpha_k)' P(\alpha_{k+1}) \tilde{A}_d(\alpha_k) - Q(\alpha_{k-d_k})$.

b) *There exist parameter dependent matrices $F(\alpha_k) \in \mathbb{R}^{n\times n}$, $G(\alpha_k) \in \mathbb{R}^{n\times n}$ and $H(\alpha_k) \in \mathbb{R}^{n\times n}$, such that*

$$\mathcal{M}(\alpha_k) \equiv \begin{bmatrix} \mathcal{M}_{11}(\alpha_k) & G(\alpha_k)' - F(\alpha_k)\tilde{A}(\alpha_k) & H(\alpha_k)' - F(\alpha_k)\tilde{A}_d(\alpha_k) \\ \star & \mathcal{M}_{22}(\alpha_k) & -\tilde{A}(\alpha_k)' H(\alpha_k)' - G(\alpha_k)\tilde{A}_d(\alpha_k) \\ \star & \star & \mathcal{M}_{33}(\alpha_k) \end{bmatrix} < \mathbf{0} \quad (14)$$

with $\mathcal{M}_{11}(\alpha_k) = P(\alpha_{k+1}) + F(\alpha_k)' + F(\alpha_k)$, $\mathcal{M}_{22}(\alpha_k) = \beta Q(\alpha_k) - P(\alpha_k) - \tilde{A}(\alpha_k)' G(\alpha_k)' - G(\alpha_k)\tilde{A}(\alpha_k)$ and $\mathcal{M}_{33}(\alpha_k) = -(Q(\alpha_{k-d_k}) + H(\alpha_k)\tilde{A}_d(\alpha_k) + \tilde{A}_d(\alpha_k)' H(\alpha_k)')$.

In both cases, functional (9)-(12) verifies

$$V(\alpha_k, k) > 0, \ \Delta V(\alpha_k, k) < 0, \ \forall\ [x_k' \ \ x_{k-d_k}']' \neq \mathbf{0} \quad (15)$$

and is called a Lyapunov-Krasovskii functional, assuring the robust stability of (6).

Proof: The positivity of the functional (9) is assured with the hypothesis of $P(\alpha_k) = P(\alpha_k)' > \mathbf{0}$, $Q(\alpha_k) = Q(\alpha_k)' > \mathbf{0}$. For (9) being a Lyapunov-Krasovskii functional, besides its positivity, it is necessary to verify (15) $\forall\ \alpha_k \in \Omega$. From hereafter, the α_k dependency is omitted in the expressions $V_v(k)$, $v = 1, \ldots, 3$, for simplicity of the notation. To calculate (15), consider

$$\Delta V_1(k) = x_{k+1}' P(\alpha_{k+1}) x_{k+1} - x_k' P(\alpha_k) x_k \quad (16)$$

$$\Delta V_2(k) = x_k' Q(\alpha_k) x_k - x_{k-d_k}' Q(\alpha_{k-d_k}) x_{k-d_k} + \sum_{i=k+1-d_{k+1}}^{k-1} x_i' Q(\alpha_i) x_i - \sum_{i=k+1-d_k}^{k-1} x_i' Q(\alpha_i) x_i \quad (17)$$

and

$$\Delta V_3(k) = (\bar{d} - \underline{d}) x_k' Q(\alpha_k) x_k - \sum_{i=k+1-\bar{d}}^{k-\underline{d}} x_i' Q(\alpha_i) x_i \quad (18)$$

Observe that the third term in equation (17), $\Xi_k \equiv \sum_{i=k+1-d_{k+1}}^{k-1} x_i' Q(\alpha_i) x_i$, can be rewritten as

$$\begin{aligned} \Xi_k &= \sum_{i=k+1-\underline{d}}^{k-1} x_i' Q(\alpha_i) x_i + \sum_{i=k+1-d_{k+1}}^{k-\underline{d}} x_i' Q(\alpha_i) x_i \\ &\leq \sum_{i=k+1-d_k}^{k-1} x_i' Q(\alpha_i) x_i + \sum_{i=k+1-\bar{d}}^{k-\underline{d}} x_i' Q(\alpha_i) x_i \end{aligned} \quad (19)$$

Using (19) in (17), one gets

$$\Delta V_2(k) \leq x_k' Q(\alpha_k) x_k - x_{k-d_k}' Q(\alpha_{k-d_k}) x_{k-d_k} + \sum_{i=k+1-\bar{d}}^{k-\underline{d}} x_i' Q(\alpha_i) x_i \quad (20)$$

So, taking into account (16), (18) and (20), the following upper bound for (15) can be obtained

$$\begin{aligned} \Delta V(k) \leq\; & x_{k+1}' P(\alpha_{k+1}) x_{k+1} \\ & + x_k' [\beta Q(\alpha_k) - P(\alpha_k)] x_k \\ & - x_{k-d_k}' Q(\alpha_{k-d_k}) x_{k-d_k} < 0 \end{aligned} \quad (21)$$

Replacing x_{k+1} in (21) by the right hand side of (6) one gets (13). The equivalence between (13) and (14) can be established as follows. First, note that (13) can be rewritten as

$$\Gamma(\alpha_k) = \Pi(k)' P(\alpha_{k+1})^{-1} \Pi(k) - \begin{bmatrix} P(\alpha_k) - \beta Q(\alpha_k) & \mathbf{0} \\ \mathbf{0} & Q(\alpha_{k-d_k}) \end{bmatrix} < \mathbf{0} \quad (22)$$

with $\Pi(k) = \left[P(\alpha_{k+1}) \tilde{A}(\alpha_k) \;\; P(\alpha_{k+1}) \tilde{A}_d(\alpha_k) \right]$, which by Schur complement is equivalent to

$$\begin{bmatrix} -P(\alpha_{k+1}) & P(\alpha_{k+1}) \tilde{A}(\alpha_k) & P(\alpha_{k+1}) \tilde{A}_d(\alpha_k) \\ \star & \beta Q(\alpha_k) - P(\alpha_k) & \mathbf{0} \\ \star & \star & -Q(\alpha_{k-d_k}) \end{bmatrix} < \mathbf{0} \quad (23)$$

Therefore, the equivalence between **a)** and **b)** is the same of the equivalence between (14) and (23). So, if (23) is verified, then it is possible to assure condition (14) with $F(\alpha_k) = F(\alpha_k)' = -P(\alpha_{k+1})$, $G(\alpha_k) = H(\alpha_k) = \mathbf{0}$. On the other hand, if (14) is verified, then $\Gamma(\alpha_k) = T(\alpha_k)' \mathcal{M}(\alpha_k) T(\alpha_k)$ with

$$T(\alpha_k) = \left[\begin{array}{cc} \tilde{A}(\alpha_k) & \tilde{A}_d(\alpha_k) \\ \hline \multicolumn{2}{c}{\mathbf{I}_{2n}} \end{array} \right],$$

completing the proof. ■

Observe that, conditions stated in Theorem 1 does not consider a particular structure for matrices $P(\alpha_k)$, $Q(\alpha_k)$, $F(\alpha_k)$, $G(\alpha_k)$ and $H(\alpha_k)$. Although conditions (13) and (14) are not LMIs, nowadays there are some relaxation techniques that can be applied such as those in [12], [20] and [19] to obtain convex conditions as done in [14]. From hereafter, consider that the parameter dependent matrices $P(\alpha_k)$ and $Q(\alpha_k)$ are defined as follows

$$P(\alpha_k) = \sum_{i=1}^{N} \alpha_{ki} P_i \quad (24)$$

$$Q(\alpha_k) = \sum_{i=1}^{N} \alpha_{ki} Q_i \quad (25)$$

In what follows, sufficient convex conditions to verify Theorem 1 are given.

Theorem 2 *If there exist symmetric matrices $\mathbf{0} < P_i \in \mathbb{R}^{n \times n}$, $\mathbf{0} < Q_i \in \mathbb{R}^{n \times n}$, $i = 1, \ldots, N$, and matrices $F \in \mathbb{R}^{n \times n}$, $G \in \mathbb{R}^{n \times n}$ and $H \in \mathbb{R}^{n \times n}$ and a scalar $\beta = \bar{d} - \underline{d} + 1$, with $\underline{d}$ and $\bar{d}$ known, such that the following LMIs are verified*

$$\tilde{\mathcal{M}}(i,j,\ell) \equiv \begin{bmatrix} P_j + F' + F & G' - F\tilde{A}_i & H' - F\tilde{A}_{di} \\ \star & \beta Q_i - P_i - \tilde{A}_i' G' - G\tilde{A}_i & -\tilde{A}_i' H' - G\tilde{A}_{di} \\ \star & \star & -(Q_\ell + H\tilde{A}_{di} + \tilde{A}_{di}' H') \end{bmatrix} < \mathbf{0}, \quad i,j,\ell = 1, \ldots, N \quad (26)$$

then the uncertain and time-varying discrete-time system with state delay described by (6) is robustly stable. Besides this, (9)-(12) with (24)-(25) is a Lyapunov-Krasovskii functional for (6).

Proof: Condition (14) given in Theorem 1 can be obtained from (26) by noting that $\mathcal{M}(\alpha_k) = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{\ell=1}^{N} \tilde{\mathcal{M}}(i,j,\ell) \alpha_{k,i} \alpha_{k+1,j} \alpha_{k-d_k,\ell}$, $\alpha_k \in \Omega$. In this case, $F(\alpha_k) = F$, $G(\alpha_k) = G$, $H(\alpha_k) = H$ and $P(\alpha_k)$ and $Q(\alpha_k)$ are given in (24) and (25), respectively. ■

Both, Theorem 1 and Theorem 2 can deal with system defined by $\tilde{A}(\alpha_k)$ and $\tilde{A}_d(\alpha_k)$ as well as with its dual $\tilde{A}(\alpha_k)'$ and $\tilde{A}_d(\alpha_k)'$

It is worth noting that theorems 1 and 2 encompass the case where the delay is constant, i.e., for $\underline{d} = \bar{d}$. In this case, the conditions presented here are similar to those presented in [15], but in this last one, besides the constant delay, the system is also time-invariant. Also, note that the quadratic stability based conditions can be recovered from the conditions of (13), (14) and (26) as indicated in the next corollary.

Corollary 1 *The time-varying system (6) subject to (2), (4) and (8) is quadratically stable if there exist symmetric matrices $\mathbf{0} < P \in \mathbb{R}^{n \times n}$, $\mathbf{0} < Q \in \mathbb{R}^{n \times n}$ and a scalar $\beta = \bar{d} - \underline{d} + 1$, with $\underline{d}$ and $\bar{d}$ known, such that one of the following equivalent conditions is verified*

a)

$$\begin{bmatrix} \tilde{A}_i' P \tilde{A}_i + \beta Q - P & \tilde{A}_i' P \tilde{A}_{di} \\ \star & \tilde{A}_{di}' P \tilde{A}_{di} - Q \end{bmatrix} < \mathbf{0} \qquad i = 1, \ldots, N \quad (27)$$

b)

$$\begin{bmatrix} -P & P\tilde{A}_i & P\tilde{A}_{di} \\ \star & \beta Q - P & \mathbf{0} \\ \star & \star & -Q \end{bmatrix} < \mathbf{0}, \; i = 1, \ldots, N \quad (28)$$

c) *There exist parameter dependent matrices $F \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times n}$ and $H \in \mathbb{R}^{n\times n}$, such that*

$$\begin{bmatrix} P + F' + F & G' - F\tilde{A}_i & H' - F\tilde{A}_{di} \\ \star & \beta Q - P - \tilde{A}'G' - G\tilde{A}_i & -\tilde{A}_i' H' - G\tilde{A}_{di} \\ \star & \star & -(Q + H\tilde{A}_{di} + \tilde{A}_{di}' H') \end{bmatrix} < \mathbf{0}, \; i = 1, \ldots, N \quad (29)$$

In this case, functional (9)-(12) verifies (15) with $P(\alpha_k) = P$, $Q(\alpha_k) = Q$ and is called a Lyapunov-Krasovskii functional, assuring the robust stability of (6).

Proof: The proof follows steps of the proof of Theorem 1. ■

Observe that, having common variables $F(\alpha_k) = F$, $G(\alpha_k) = G$ and $H(\alpha_k) = H$ may be somewhat restrictive in the robust stability analysis condition, but this seems to be of fundamental importance for developing robust filters and controllers as it can be verified in the case of delay-free systems [18]. This fact is exploited in what follows to achieve a convex design condition for the control law (5).

B. Robust Stabilization

Convex conditions are derived from Theorem 2 to design robust state feedback gains K and K_d for (5) assuring the robust stabilization of (1).

Theorem 3 *If there exist symmetric matrices $\mathbf{0} < P_i \in \mathbb{R}^{n\times n}$, $\mathbf{0} < Q_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, N$, and matrices $F \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times p}$ and $W_d \in \mathbb{R}^{n\times p}$ and a scalar $\beta = \bar{d} - \underline{d} + 1$, with $\underline{d}$ and $\bar{d}$ known, such that the following LMIs are verified*

$$\begin{bmatrix} P_j + F' + F & -WB_i' - FA_i' & -W_d B_i' - FA_{di}' \\ \star & \beta Q_i - P_i & \mathbf{0} \\ \star & \star & -Q_\ell \end{bmatrix} < \mathbf{0}, \qquad i, j, \ell = 1, \ldots, N \quad (30)$$

then the uncertain and time-varying discrete-time system with state delay described by (1) is robustly stabilizable by the control law (5) with

$$K = W'(F')^{-1} \quad \text{and} \quad K_d = W_d'(F')^{-1} \quad (31)$$

Besides this, (9)-(12) with (24)-(25) is a Lyapunov-Krasovskii functional for the resulting uncertain time-varying closed-loop system (6).

Proof: The proof can be obtained by replacing $\tilde{A}_i$ and $\tilde{A}_{di}$ by $(A_i + B_i K)'$ and $(A_{di} + B_i K_d)'$, respectively, choosing $G = \mathbf{0}$ and $H = \mathbf{0}$ and making the changing of variables $W = FK'$, $W_d = FK_d'$ in (26). ■

Note that in case where the delay value is not known, i.e., when $x_{k-d(k)}$ is not available for feedback, then condition (30) can be used with $W_d = \mathbf{0}$. A quadratic stability condition can be recovered from (30) by imposing $P_i = P = P'$. Thus, whenever the time-varying system (1) is quadratically stabilizable, it is also robustly stabilizable. This implies that the proposed conditions represent an improvement on the available tools for dealing with time-varying delay systems. Also note that, conditions **b)** and **c)** of Corollary 1 can be used to obtain convex synthesis conditions. This is presented in Corollary 2. Observe that condition **a)** cannot be directly used, because of the triple product present in all blocks.

Corollary 2 *The uncertain and time-varying discrete-time system with state delay described by (1) is quadratically stabilizable by the control law (5) if there exist symmetric matrices $\mathbf{0} < P \in \mathbb{R}^{n\times n}$, $\mathbf{0} < Q \in \mathbb{R}^{n\times n}$, a scalar $\beta = \bar{d} - \underline{d} + 1$, with $\underline{d}$ and $\bar{d}$ known, such that the following conditions are verified for $i = 1, \ldots, N$:*

a) *There exist matrices $Z \in \mathbb{R}^{n\times p}$ and $Z_d \in \mathbb{R}^{n\times p}$, such that*

$$\begin{bmatrix} -P & P\tilde{A}_i' + ZB_i' & P\tilde{A}_{di}' + Z_d B_i' \\ \star & \beta Q - P & \mathbf{0} \\ \star & \star & -Q \end{bmatrix} < \mathbf{0} \quad (32)$$

and, in this case,

$$K = Z'P^{-1} \quad \text{and} \quad K_d = Z_d' P^{-1} \quad (33)$$

b) *There exist matrices $F \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times p}$ and $W_d \in \mathbb{R}^{n\times p}$, such that*

$$\begin{bmatrix} P + F' + F & -WB_i' - FA_i' & -W_d B_i' - FA_{di}' \\ \star & \beta Q - P & \mathbf{0} \\ \star & \star & -Q \end{bmatrix} < \mathbf{0} \quad (34)$$

and, in this case, K and K_d are determined as in (31).

Besides this, in both cases, (9)-(12) with $P(\alpha_k) = P$, $Q(\alpha_k) = Q$ is a Lyapunov-Krasovskii functional for the resulting uncertain time-varying closed-loop system (6).

An important issue of the presented proposal is that the results can be used to deal with decentralized control by imposing block-diagonal structure to some matrices. In case of Theorem 3 and Corollary 2.**b)**, this can be done by imposing $F = F_D = \text{block-diag}\{F^1, \ldots, F^\kappa\}$, $W = W_D = \text{block-diag}\{W^1, \ldots, W^\kappa\}$, $W_d = W_{dD} = \text{block-diag}\{W_d^1, \ldots, W_d^\kappa\}$ where κ denote the number of subsystems defined. This structure results in block-diagonal state feedback gains $K_D = W_D'(F_D')^{-1}$ and $K_{dD} = W_{dD}'(F_D')^{-1}$ and in matrices $P(\alpha_k)$, $Q(\alpha_k)$ without any restrictions in their structures, which yields in less conservative design conditions. On the other hand, for the conditions presented in Corollary 2.**a)**, a decentralized control can be achieved only by imposing a diagonal structure to the matrices P, Z and Z_d. In this case, one gets $P = P_D = \text{block-diag}\{P^1, \ldots, P^\kappa\}$,

$Z = Z_D = \text{block-diag}\{Z^1, \ldots, Z^\kappa\}$, $Z_d = Z_{dD} = \text{block-diag}\{Z_d^1, \ldots, Z_d^\kappa\}$, achieving the decentralized gains $K = Z_D P_D^{-1}$ and $K_{dD} = Z_{dD} P_D^{-1}$. Therefore, it is expect that results obtained from condition **a)** of Corollary 2 are less conservative than those obtained from its condition **b)**.

The results in this paper can be seen as an extension of the conditions proposed in [3] for discrete time-varying delay-free systems. However, the conditions proposed here can deal with uncertainty in all system matrices and constant gains are used, simplifying the controller implementation w.r.t. [3]. Thus, practical situations like actuator fails can be taken into account by the conditions of theorems 2 and 3 which makes this proposal very interesting for practical applications. Also, it is expected that new filtering and controlling conditions with some performance index can be developed from the conditions presented here, thanks to the common variable F preserved in the convex formulation given in Theorem 3.

Finally, note that the conditions presented in this paper can be straightforward extended to case of multiple delays, by considering new terms like $V_2(\alpha_k, k)$ and $V_3(\alpha_k, k)$, equations (11) and (12), with new matrices $Q(\alpha_k)_\varrho$, $\varrho = 2, \ldots, \overline{\varrho}$, where $\overline{\varrho}$ is the number of delays presented in the system.

C. Numerical complexity

The numerical complexity of the conditions presented in this paper can be determined by the number of scalar variables, $\mathcal{K}$, and the number of rows, $\mathcal{R}$, involved in the optimization problems. In case of using *LMI Control Toolbox* [7], the numerical complexity is $\mathcal{O}(\mathcal{K}^3\mathcal{R})$ and using the solver SeDuMi [22] the numerical complexity is $\mathcal{O}(\mathcal{K}^2\mathcal{R}^{2.5} + \mathcal{R}^{3.5})$. Note that, nowadays efficient algorithms can solve the conditions here, in polynomial time. The number of scalar variables and the number of LMI rows of the feasibility tests proposed in this paper are presented in Table I.

Condition	$\mathcal{K}$	$\mathcal{R}$
Theorem 2	$n[n(3+N)+N]$	$Nn(3N^2+2)$
Corollary 1.**a)**	$n(n+1)$	$2n(N+1)$
Corollary 1.**b)**	$n(n+1)$	$3Nn$
Corollary 1.**c)**	$n(4n+1)$	$n(3N+2)$
Theorem 3	$n[N(n+1)+n+2p]$	$3N^3n$
Corollary 2.**a)**	$n(n+2p+1)$	$3Nn$
Corollary 2.**b)**	$n(2n+2p+1)$	$3Nn$

TABLE I

NUMBER OF SCALAR VARIABLES ($\mathcal{K}$) AND LMI ROWS ($\mathcal{R}$) FOR EACH PROPOSED CONDITION

IV. NUMERICAL EXAMPLES

Example 1 *Consider the uncertain DTVDS described by (6) where where* $\tilde{A}(\alpha_k) = \tilde{A}_n + (2\alpha_k - 1)\rho L' J$ *and* $\tilde{A}_d(\alpha_k) = 0.25(1 - 0.2\alpha_k)\tilde{A}_n$ *with*

$$\tilde{A}_n = \begin{bmatrix} 0.8 & -0.25 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.03 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (35)$$

$L = [0,\ 0,\ 1,\ 0]'$, $J = [0.8,\ -0.5,\ 0,\ 1]$, $\alpha_k \in [0,\ 1]$, $0 \leq |\rho| \leq \delta$. *This defines a polytopic system with 2 vertices, both depending on* ρ *and given by* $\alpha_k = 0$ *(vertex 1) and* $\alpha_k = 1$ *(vertex 2). Considering* $\underline{d} = 1$, *the objective here is to determine stability regions in* $\delta \times \bar{d}$ *plane. Figure 1 shows the regions of stability below each curve provide by Theorem 2 (solide line) and Corollary 1 (dashed line). It is worth of mentioning that all conditions in Corollary 1 achieve the same maximum values of* δ. *Note that conditions of Theorem 2 clearly improve the result obtained by QS approach, enlarging the stability region.*

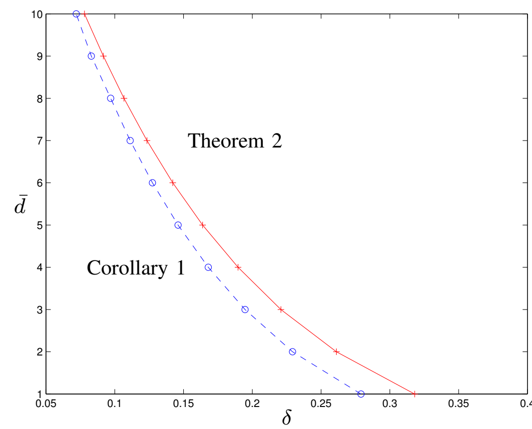

Fig. 1. Stability regions.

Example 2 *Consider a randomly generated discrete time-varying system with time-varying delay given by*

$$x_{k+1} = A(\alpha_k)x_k + A_d(\alpha_k)x_{k-d_k} + B(\alpha_k)u_k + B_d(\alpha_k)u_{k-d_k} \quad (36)$$

where the time-varying system matrices are described by a polytope with two vertices given by $\Upsilon_1 = [A|A_d|B|B_d]_1$ *and* $\Upsilon_2 = [A|A_d|B|B_d]_2$ *as follows*

$$\Upsilon_1 = \left[\begin{array}{cc|cc|c|c} 0.132 & 0.317 & 0.314 & 0.475 & 0 & 0.72 \\ 0.377 & 0.091 & 0.878 & 0.558 & 1 & 0.54 \end{array}\right]$$

$$\Upsilon_2 = \left[\begin{array}{cc|cc|c|c} 0.719 & 0.274 & 0.169 & 0.363 & 0 & 0.10 \\ 0.030 & 0.672 & 0.887 & 0.431 & 2 & 0.39 \end{array}\right]$$

Note that the additional term $B_d(\alpha)u_{k-d(k)}$ *can be taken into account by the conditions presented here replacing* B_i *by* B_{di} *in the entries* $(1,3)$ *and* $(3,1)$ *of the LMIs (30), (32) and (34). In this case, the control law is given by* $u_k = Kx_k$ *and* $u_{k-d_k} = K_d x_{k-d_k}$. *For* $\underline{d} = 1$ *and* $\bar{d} = 5$ *only Theorem 3 yields a solution to the stabilization problem, with* $K = \begin{bmatrix} -0.4236 & -0.4918 \end{bmatrix}$ *and* $K_d = \begin{bmatrix} -1.1691 & -1.0353 \end{bmatrix}$, *thus demonstrating that conditions of this theorem can lead to less conservative results. Simulations for this case are shown in figures 2 and 3. In the top of Figure 2 it is shown the state behaviors of the time varying closed-loop system and in its bottom the control signals. This simulation has been performed with* $\alpha_1(k) = 0.5(1 + \sin k)$ *(*$\alpha_2(k) = 1 - \alpha_1(k)$*) as seen in the top of the Figure 3, with initial conditions* $x(k) = [1\ \ -1]'$, $k = -5, \ldots, 0$.

The time-varying delay $d(k)$ has been randomly generated as indicated in the bottom of Figure 3.

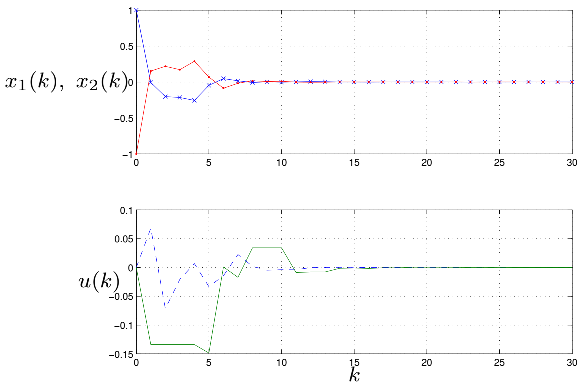

Fig. 2. The behaviors of the states $x_1(k)$, $\times$, and $x_2(k)$, $\cdot$, (top) and the control signals, $u_k = Kx_k$, dashed line, and $u_{k-d_k} = K_d x_{k-d_k}$, solid line (bottom).

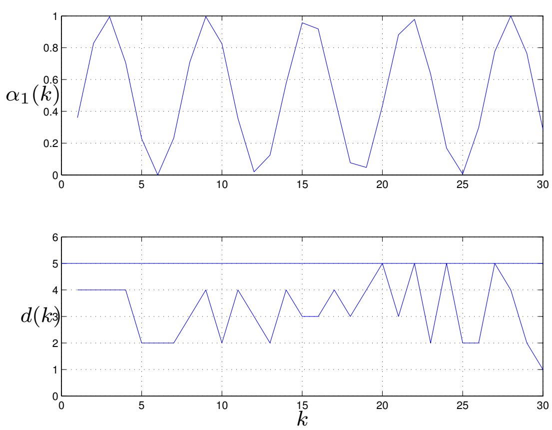

Fig. 3. $\alpha_1(k)$ (top) and time-varying delay, $d(k)$ (bottom).

V. Conclusions

It has been presented some new convex conditions for dealing with robust stability analysis and robust stabilization of discrete time-varying systems with time-varying delays. The novelty of the conditions is due to *i)* the use of time-varying parameter dependent Lyapunov-Krasovskii functionals allowing to treat time-varying discrete-time systems with time-varying delay; *ii)* the use of extra matrices allowing less conservative evaluations of stability domains; and *iii)* the convex design conditions for robust state feedback gains. It has been shown that additional structure restriction on the state feedback gains can easily incorporated without restrictions on the Lyapunov-Krasovskii matrices which allows, for example, to deal with actuator failures. Some numerical examples, including a time-simulation, are given to illustrate the efficacy of the proposed conditions.

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