Multi-Agent Rendezvousing
With A Finite Set Of Candidate Rendezvous Points

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Abstract— The discrete multi-agent rendezvous problem we consider in this paper is concerned with a specified set of points in the plane, called "dwell-points," and a set of mobile autonomous agents with limited sensing range. Each agent is initially positioned at some dwell-point, and is able to determine its distances to dwell-points within sensing range, and also the bearings of those dwell-points with respect to a local reference of the agent. An agent can also determine if each dwell-point within sensing range is occupied by at least one agent. We do not, however, assume that an agent can count the number of agents occupying a dwell-point. We say the agents have "rendezvoused" if all the agents have moved to a set of dwell-points all within sensing radius of each other. The goal is to devise distributed motion strategies for each agent which will cause all the agents to rendezvous in the sense just defined, without any active communication among the agents. We propose a distributed motion rule and use graphs to characterize a class of dwell-points in the plane for which rendezvous is guaranteed under the rule.

I. INTRODUCTION

Growing interest in cooperative control of large multi-agent systems has led to a number of distributed algorithms aimed at causing autonomous agents to perform various tasks in the absence of a centralized control [1], [2], [3], [4], [5], [6], [7], [8]. The aim of this paper is to study a discrete version of the so-called "multi-agent rendezvous problem" considered previously in [9], [10], [11] and elsewhere. The version of the problem addressed in [10], [11] is concerned with the collective behavior of a group of $n > 1$ mobile autonomous agents, labelled 1 through n, which can all move in the plane. Each agent is able to continuously track the positions of all other agents currently within its "sensing region" where by an agent's *sensing region* is meant a closed disk of positive radius r centered at the agent's current position. The *multi-agent rendezvous problem* is to devise "local" control strategies, one for each agent, which without any active communication between agents, cause all members of the group to eventually rendezvous at single unspecified location. In [11], a family of distributed strategies was proposed which solved the multi-agent rendezvous problem by performing a sequence of "stop-and-go" maneuvers. A *stop-and-go maneuver* takes place within a time interval

consisting of two consecutive sub-intervals. The first, called a *sensing period*, is an interval of fixed length during which the agent is stationary. The second, called a *maneuvering period*, is an interval of variable length during which the agent moves from its current position to its next 'way-point' and again come to rest. Successive way-points for each agent are chosen to be within r_M units of each other where r_M is a pre-specified positive distance no larger than r. It is assumed that there has been chosen for each agent i, a positive number τ_{M_i}, called a *maneuver time*, which is large enough so that the required maneuver for agent i from any one way-point to the next can be accomplished in at most τ_{M_i} seconds. The work in [10], [11] deal exclusively with devising *high level* strategies which dictate when and where agents are to move. Accordingly, these works use point models for agents. In this context multiple agents may be positioned at a point in $\mathbb{R}^2$. In addition, issues concerned with with how maneuvers are actually carried out or with how vehicle collisions are to be avoided are not addressed.

In this paper we will continue to use point models for agents; however, in sharp contrast with [10], [11], we will not assume that an agent's next way-point can be just *any* point within r_M units of its prior way-point. Rather we will assume that for each given possible way-point x of each agent there are at most a *finite* number of candidate positions which can serve as the agent's next way-point, and these points are all within r_M units of x. These candidate way-points will be referred to as the *dwell-points* of point x. We assume that only dwell-points can be way-points, and that for each given dwell-point x there is a finite set of dwell-points $\mathcal{N}(x)$ within r_M units of x. Thus if agent i's current way-point is at dwell-point x, the next way-point to which agent i must move must be in the set $\mathcal{N}(x)$. We will assume that $x \in \mathcal{N}(x)$, and that $x \in \mathcal{N}(y)$ for each $y \in \mathcal{N}(x)$. Furthermore, if two dwell-points w and z are within sensing range, then $w \in \mathcal{N}(z)$ and $z \in \mathcal{N}(w)$.

A point in $\mathbb{R}^2$ is said to be *occupied* if there is at least one agent positioned at the point. An agent can determine which dwell-points of its current way-point are occupied; however, we do not assume the agent is able to count the number of agents positioned at an occupied dwell-point. We say the agents have *rendezvoused* if each agent is positioned such that all the agents are within sensing radius of each other. The *discrete multi-agent rendezvous problem* is to devise local

control rules, one for each agent, which without any active communication between agents, will cause all the agents to rendezvous in the sense just defined, at an unspecified set of points.

Our interest in formulating the discrete multi-agent problem is rooted in practical considerations that arise when the environment in which the agents are situated is hostile. In this case, it is not desirable for agents to be stationary at arbitrary points in the plane. Moreover, there may be only a finite and discrete set of points, i.e. where watchtowers and protected areas are located, where the agents may dwell when deciding on the next move. Also, since the agents in our framework are not limited to being actual vehicles, discrete rendezvous can also be regarded as a problem in networking. For example, the agents may represent files or packets moving from computer to computer over a network of computers.

In this paper, we propose a family of stop-and-go strategies, and use graphs to characterize a class of dwell-points for which rendezvous is guaranteed under the proposed strategies. The agent strategies we propose are mutually synchronized in the sense that all depend on a common clock. In Section II, we give the terms and definitions to be used in the exposition that follows, and we propose a family of distributed motion strategies for the agents. In Section III, we give the graphical characterization of dwell-point sets for which rendezvous is guaranteed under the proposed strategies. We conclude with future work in Section IV.

II. Occupied dwell-point induced motion rules

Each maneuver takes place within a time interval that consists of an interval during which the agent is stationary followed by an interval during which the agent moves from its current position to its next "way-point." Successive way-points for each agent are restricted to the set of dwell-points of the agent's current way-point. The real time axis can be partitioned into a sequence of time intervals $[0, t_1]$, $(t_1, t_2]$, $\ldots$, $(t_{k-1}, t_k]$, $\ldots$. For a positive integer k, the kth *time period* denotes the time interval $(t_{k-1}, t_k]$. Each time interval consists of a "sensing period" followed by a "maneuvering period," where each maneuvering period is of fixed length τ_M, and $\tau_M > \tau_{M_i}$ for each agent i. The kth *sensing period* denotes the time interval $(t_{k-1}, t_k - \tau_M)$, and the kth *maneuvering period* denotes the time interval $[t_k - \tau_M, t_k]$.

In the kth sensing period, each agent determines the positions in its local coordinate system of the dwell-points of its current way-point, and which of those dwell-points are occupied. Each agent then computes its next *way-point*, which is the dwell-point the agent is to move to by time t_k. The kth way-point of an agent is restricted to the set of dwell-points of the agent's current way-point. In the kth maneuvering period, each agent moves to its kth way-point, after which it comes to rest until the next maneuvering period. Agent motions are synchronized in the sense that the agents only move during the maneuvering period of each time interval which are the same for all agents.

Let $\mathcal{N}(x)$ denote the set of dwell-points of agent i's $(k-1)$th way-point x, and let $\mathcal{W}(x)$ denote the subset of $\mathcal{N}(x)$ consisting of dwell-points which are within sensing range of all occupied dwell-points in $\mathcal{N}(x)$. Agent i is said to satisfy the *inter-agent motion constraint* during time period k if its next way-point is restricted to the occupied dwell-points of set $\mathcal{W}(x)$. Motivated by [11], we will only be interested in strategies possessing the property whereby if a pair of agents are within sensing range in sensing period k, then the agents are also within sensing range in all subsequent sensing periods. It is easy to see that this property is possessed by any strategy satisfying the following assumption:

Cooperation Assumption: During each maneuvering period k, each agent restricts its motions to satisfy the inter-agent motion constraint.

Let $\mathcal{N}(x)$ and $\mathcal{W}(x)$ be as defined above. Let $\mathcal{O}(x)$ be the set of all dwell-points in $\mathcal{W}(x) - \{x\}$ which are occupied. Consider the following rule for determining the kth way-point of agent i:

1) If $\mathcal{O}(x) \neq \emptyset$, then let the kth way-point of agent i be any point in $\mathcal{O}(x)$

2) If $\mathcal{O}(x) = \emptyset$, then agent i does not move in the kth maneuvering period.

In the following, we will refer to the above as the *occupied dwell-points induced* motion rule. Obviously, the occupied dwell-points induced rule satisfies the cooperation assumption. In the sequel, we will give graphical conditions for when rendezvous will occur under the occupied dwell-points induced rule.

III. Graphical characterizations

In this section, we will use graphs to characterize dwell-point configurations for which agents will rendezvous. We begin with some terms and definitions.

For $k \geq 1$, let $\mathbb{G}(k)$ denote the graph whose vertex set $\mathcal{V}(k)$ corresponds to the set of points in the plane that are occupied by one or more agents in the kth sensing period. The edge set of $\mathbb{G}(k)$, denoted $\mathcal{E}(k)$, is the set of all (i, j) where the point corresponding to j is a dwell-point of the point corresponding to i. We say that $\mathbb{G}(k)$ is the *graph induced by the occupied dwell-points in the kth time period*. For vertex v of $\mathbb{G}(k)$, let $\mathcal{N}(v, k)$ denote the vertices adjacent to v in $\mathbb{G}(k)$.

Suppose $\mathbb{G}(k)$ is connected, and let i and j be any pair of agents. Since $\mathbb{G}(k)$ is connected, we have that in the kth sensing period, there is a sequence of agents $a_1, \ldots, a_n$ such that i is within sensing range of a_1, j is within sensing range of a_n, and each agent a_i, $i < n$, is within sensing range of agent a_{i+1}. Since the occupied dwell-points induced rule satisfies the Cooperation Assumption, we have that if two agents are within sensing range in the kth sensing period,

then the two agents must also be within sensing range in the $(k+1)$th sensing period. Therefore, in the $(k+1)$th sensing period, agent i is within sensing range of a_1, j is within sensing range of a_n, and each agent a_i, $i < n$, is within sensing range of agent a_{i+1}. If $\mathbb{G}(k+1)$ is not connected, then there must exist a pair of agents i and j for which the above does not hold. We have just shown the following:

Property 1: If $\mathbb{G}(k)$ is connected, then $\mathbb{G}(k+1)$ is also connected.

Under the occupied dwell-points induced motion rule, agents restrict their next way-points to be only occupied dwell-points. An obvious consequence of this is that:

Property 2: Graph $\mathbb{G}(k+1)$ is an induced subgraph of $\mathbb{G}(k)$.

In the following sections, we will characterize some graphs $\mathbb{G}(k)$ for which each $\mathbb{G}(k+i)$, $i \geq 1$, is a proper subgraph of $\mathbb{G}(k+i-1)$ assuming the agents have not yet rendezvoused, i.e. $\mathbb{G}(k+i-1)$ is not a complete graph. For if such is the case, then rendezvous will occur in a finite number of steps.

A. Acyclic Graphs

A graph is said to be *acyclic* if the graph does not contain any cycles. It is easy to see that all subgraphs of an acyclic graphs must also be acyclic. Hence, it follows directly from Property 2 that:

Lemma 1: If $\mathbb{G}(k)$ is acyclic, then $\mathbb{G}(k+1)$ is also acyclic.

If the graph induced by the occupied dwell-points in some time period is acyclic and not complete, then it is easy to show that each occupied dwell-point corresponding to a vertex of degree one will no longer be occupied in the next sensing period. It is straightforward to show that:

Theorem 1: If $\mathbb{G}(1)$ is connected and $\mathbb{G}(k)$ is acyclic for some k, then the agents will rendezvous.

Hence, rendezvous will always occur if the graph induced by the occupied dwell-points in some time period is acyclic and connected. In particular, consider the graph whose vertex set corresponds to the set of dwell-points, and where two vertices are adjacent when the corresponding dwell-points are within sensing range. If this graph is acyclic, then rendezvous of the agents is guaranteed regardless of which dwell-points are initially occupied, so long as $\mathbb{G}(1)$ is connected.

B. Generalized Wheel Graphs

A *cycle* of *length m* is a graph whose vertices can be ordered as $c_1, c_2, \ldots, c_m$, $m \geq 3$, so that c_i is adjacent to c_j just in case $|i-j| = 1$ or $|i-j| = m-1$. A *wheel graph* consists of a cycle of length three or more, and an additional *spoke* vertex that is adjacent to each vertex of the cycle. A wheel graph consisting of four vertices is also by definition a complete graph, i.e. a graph whereby each vertex is adjacent to all other vertices. We say a wheel graph is *proper* if it has five or more vertices. It is easy to see that if the occupied dwell-points induced graph of some time period is a proper wheel graph, then all the agents will rendezvous at the dwell-point corresponding to the spoke vertex in the next time period.

A cycle of length three is called a *triangle*, and a graph is said to be *triangle free* if it has no subgraph which is a triangle. The key properties of a proper wheel graph that enable rendezvous in one step are that the graph induced by the non-spoke vertices is triangle free, and each non-spoke vertex is adjacent to two other non-spoke vertices. Hence, we define the *generalized wheel graph* as a graph whose vertex set is the disjoint union of non-empty subsets $\mathcal{K}$ and $\mathcal{F}$ where vertices in $\mathcal{K}$ induce a complete graph, vertices in $\mathcal{F}$ induce a triangle free graph where each vertex has degree at least two when $|\mathcal{F}| \geq 3$, and each vertex of $\mathcal{F}$ is adjacent to each vertex of $\mathcal{K}$.

Lemma 2: If $\mathbb{G}(k)$ is a generalized wheel graph for some k, then the agents will rendezvous in the $(k+1)$th time period.

A *path* of *length m* is a graph whose vertices can be ordered as $p_1, p_2, \ldots, p_m$, $m \geq 2$, so that p_i is adjacent to p_j just in case $|i-j| = 1$. A *broken wheel* graph consists of a path of length two or more, and an additional *spoke* vertex that is adjacent to each vertex of the path. We say a broken wheel graph is *proper* if it has five or more vertices. It is easy to see that if the occupied dwell-points induced graph of some time period is a proper broken wheel graph, then all the agents will rendezvous. In the next section we will use wheel graphs and broken wheel graphs to characterize a class of graphs for which rendezvous is guaranteed if $\mathbb{G}(k)$ belongs to this class for some k.

C. Union of wheel graphs

For subgraphs $\mathbb{H}_1$ and $\mathbb{H}_2$, the *union* of $\mathbb{H}_1$ and $\mathbb{H}_2$, denoted $\mathbb{H}_1 \cup \mathbb{H}_2$, is the graph whose vertex (edge) set is the union of the vertex (edge) sets of $\mathbb{H}_1$ and $\mathbb{H}_2$. For a graph $\mathbb{G}$, let $V(\mathbb{G})$ denote the set of vertices in $\mathbb{G}$. Two proper wheel subgraphs $\mathbb{W}_1$ and $\mathbb{W}_2$ with spoke vertices s_1 and s_2 respectively are said to be *adjacent* if the graph induced in $\mathbb{W}_1 \cup \mathbb{W}_2$ by $V(\mathbb{W}_1) \cap V(\mathbb{W}_2)$ is the union of two non-identical triangles $\mathbb{T}_1$ and $\mathbb{T}_2$, each of which contains (s_1, s_2) as an edge. If $\mathbb{G}$ is the union of proper wheel subgraphs
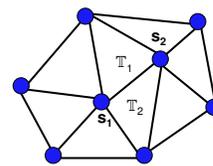
Fig. 1. Adjacent proper wheel graphs

$\mathbb{W}_1, \ldots, \mathbb{W}_m$ of $\mathbb{G}$, then define the *reduced graph* of $\mathbb{G}$

corresponding to $\mathbb{W}_1, \ldots, \mathbb{W}_m$, denoted $\bar{\mathbb{G}}(\mathbb{W}_1, \ldots, \mathbb{W}_m)$, as the graph with m vertices labelled $1, \ldots, m$ so that vertex i corresponds to $\mathbb{W}_i$, and vertices i and j are adjacent just in case the corresponding wheel subgraphs are adjacent.

Let $\mathbb{T}$ be a connected acyclic graph. A vertex of $\mathbb{T}$ adjacent to just one other vertex is called a *leaf* vertex of $\mathbb{T}$. For any two distinct vertices u and v of $\mathbb{T}$, the *path from u to v* is the unique path subgraph of $\mathbb{T}$ containing u and v in which both u and v are leaf vertices. For vertices u, v in $\mathbb{T}$, let $l(u, v)$ denote the number of edges in the path from u to v in in $\mathbb{T}$. For a vertex r of $\mathbb{T}$, we say that $\mathbb{T}$ is *balanced with respected to vertex r* if $l(r, u) = l(r, v)$ for all leaf vertices u and v of $\mathbb{T}$. For vertices r and u of $\mathbb{T}$, let $\mathcal{P}(r, u, \mathbb{T})$ denote the set of vertices v where $l(r, v) \leq l(r, u)$ in $\mathbb{T}$. Let $\mathcal{C}(r, u, \mathbb{T})$ denote the set of vertices v where $l(r, v) > l(r, u)$ and the path from v to r in $\mathbb{T}$ contains u. Let $\bar{\mathcal{C}}(r, u, \mathbb{T})$ denote the vertices in $\mathcal{C}(r, u, \mathbb{T})$ which are adjacent to u.

Lemma 3: The agents will rendezvous at one dwell-point if there is some time period $k \geq 1$ such that the following hold:

1) $\mathbb{G}(k)$ is the union of proper wheel graphs $\mathbb{W}_1, \ldots, \mathbb{W}_m$.
2) $\mathbb{T}(k) = \bar{\mathbb{G}}(k)(\mathbb{W}_1, \ldots, \mathbb{W}_m)$ is acyclic and connected.
3) $\mathbb{T}(k)$ is balanced with respect to some vertex r of $\mathbb{T}(k)$.
4) If vertex i of $\mathbb{T}(k)$ is a leaf and i is adjacent to j in $\mathbb{T}(k)$, then for all vertices u in $V(\mathbb{W}_i) - V(\mathbb{W}_j)$, u is not in any $\mathbb{W}_l$, $l \in \{1, \ldots, m\} - \{i\}$.
5) If vertex i of $\mathbb{T}(k)$ is not a leaf, then for all vertices u in $\mathbb{W}_i$, there is a set $\mathcal{M}(u) \subset (\{u\} \cup \mathcal{N}(u, k)) \cap \bigcup_{h \in \mathcal{P}(r, i, \mathbb{T}(k)) \cup \bar{\mathcal{C}}(r, i, \mathbb{T}(k))} V(\mathbb{W}_h)$ such that the vertices in $\mathcal{M}(u)$ induce a proper (broken) wheel in $\mathbb{G}(k)$ with spoke u, and no vertex in $\mathcal{N}(u, k) - \mathcal{M}(u)$ is adjacent to all vertices of $\mathcal{M}(u)$.
6) If vertex i of $\mathbb{T}(k)$ is not a leaf, and i is adjacent to j in $\mathbb{T}(k)$ where $l(r, j) = l(r, i) - 1$, then each vertex in $V(\mathbb{W}_i) - V(\mathbb{W}_j)$ is not a vertex of any wheel $\mathbb{W}_h$, $h \in \{1, \ldots, m\} - (\{i\} \cup \mathcal{C}(r, i, \mathbb{T}(k)))$.

See Figure 2 for an example of a graph $\mathbb{G}(k)$ satisfying the conditions of Lemma 3.

IV. CONCLUSION

In this paper, we have formulated a discrete version of the multi-agent rendezvous problem, which lends a combinatoric aspect to the problem that was not present in the continuous case. As we noted in the introduction, discrete multi-agent rendezvous have important practical applications, and they are not limited to actual mobile agents moving in a plane. Agents may also represent computer files such as viruses that can transmit themselves from computer to computer on a network. Such viruses may be generated at different computers in a network and seek to congregate at either one or a cluster of computers.

We have presented the occupied dwell-points induced motion rule, and gave graphical conditions for when the agents
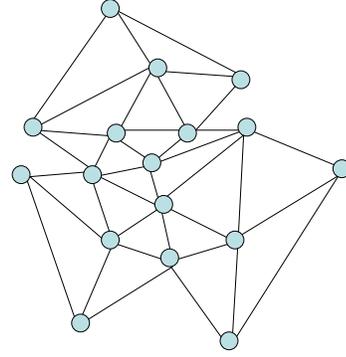
Fig. 2. Graph satisfying conditions of Lemma 3

will rendezvous under the rule. We have conjectures for more general results, and we aim to prove those conjectures as part of future work. Also, the synchronous motion rule we propose in this paper is not truly distributed because it requires that all agents possess a common clock. As a first step towards developing a completely distributed strategy for discrete rendezvous, we will study the occupied dwell-points induced rule in the asynchronous case.

V. APPENDIX: PROOFS

Theorem 1 If $\mathbb{G}(1)$ is connected and $\mathbb{G}(k)$ is acyclic for some k, then the agents will rendezvous.

Proof From property 1, we have that $\mathbb{G}(k)$ must be connected since $\mathbb{G}(1)$ is connected. If $\mathbb{G}(k)$ has less than three vertices, then the agents will have already rendezvoused, so suppose $\mathbb{G}(k)$ has at least three vertices.

If a vertex in $\mathbb{G}(k)$ has degree one, then we call it a *leaf* vertex. Since $\mathbb{G}(k)$ is acyclic, that means there must be at least one leaf vertex. Since $\mathbb{G}(k)$ is acyclic with at least three vertices, it must be that $\mathbb{G}(k)$ has at least one leaf vertex.

We will show in the following that at each occupied dwell-point corresponding to a leaf vertex of $\mathbb{G}(k)$ will no longer be occupied in the $(k+1)$th sensing period.

Let v be a leaf vertex of $\mathbb{G}(k)$, and suppose v is adjacent to u in $\mathbb{G}(k)$. It follows from the occupied dwell-points induced rule that each agent positioned at the dwell-point corresponding to leaf vertex v must have as its next way-point the dwell-point corresponding to vertex u. If vertex u is also a leaf, then u can only be adjacent to v, which implies either $\mathbb{G}(k)$ is not connected or $\mathbb{G}(k)$ has only two vertices. Hence, vertex u cannot be a leaf, and so vertex u must be adjacent to some vertex w in $\mathbb{G}(k)$ where $w \neq v$. No vertex adjacent to u can be adjacent to v for that would imply v is not a leaf. Hence, no agent positioned at the dwell-point corresponding to vertex u can move in the kth time period under the occupied dwell-point induced rule. And since vertex v is only adjacent to vertex u, the previous imply that the dwell point corresponding to vertex v cannot be the kth way-point of any agent, which implies the dwell point corresponding to vertex v will no longer be occupied in the $(k+1)$th sensing period.

We have just shown that if $\mathbb{G}(k)$ is acyclic with three or more vertices, then at least one dwell-point that is occupied in the kth sensing period will no longer be occupied in the $(k+1)$th sensing period. Hence, $\mathbb{G}(k+1)$ will consist of at least one vertex less than $\mathbb{G}$.

From lemma 1 and property 1, we have that $\mathbb{G}(k+1)$ must also be acyclic and connected. By induction, we have that at least one occupied dwell-point in each time period will become unoccupied in the subsequent time period until the graph induced by the occupied dwell-points consists of just one vertex, or is the complete graph on two vertices. In which case the agents will have rendezvoused. $\square$

Lemma 2 If $\mathbb{G}(k)$ is a generalized wheel graph for some k, then the agents will rendezvous.

Proof Let the vertex set of $\mathbb{G}(k)$ be the disjoint union of non-empty subsets $\mathcal{K}$ and $\mathcal{F}$ where vertices in $\mathcal{K}$ induce a complete graph, vertices in $\mathcal{F}$ induce a triangle free graph where each vertex has minimal degree two when $|\mathcal{F}| \geq 3$, and each vertex of $\mathcal{F}$ is adjacent to each vertex of $\mathcal{K}$. Since each vertex of $\mathcal{F}$ is adjacent to each vertex of $\mathcal{K}$, we have that each vertex of $\mathbb{K}$ is adjacent to all vertices in $\mathcal{F}$. For each vertex v of $\mathbb{G}(k)$, let $D(v)$ denote the dwell point corresponding to v.

1. If $|\mathcal{F}| = 1$, then $\mathbb{G}(k)$ is the complete graph in which case rendezvous has already occurred.

2. If $|\mathcal{F}| = 2$, then let a, b be the elements of $\mathbb{F}$. If a and b are adjacent, then $\mathbb{G}(k)$ is again a complete graph. If a and b are not adjacent, then let u be any vertex of $\mathcal{K}$. Since u is adjacent to both a and b, but a and b are not adjacent to each other, that means neither $D(a)$ nor $D(b)$ can be the kth way-point of any agent at $D(v)$. Clearly, $D(a)$ cannot be the kth way-point of any agent at $D(b)$ and $D(b)$ cannot be the kth way-point of any agent at $D(a)$. Hence, neither $D(a)$ nor $D(b)$ are the kth way-points of any agent. On the other hand, since the graph induced by vertices in $\mathcal{K}$ is complete, and a is adjacent to all vertices in $\mathcal{K}$, that means the kth way-point of each agent at $D(a)$ must be some dwell-point $D(v)$, $v \in \mathcal{K}$. Similarly, the kth way-point of each agent at $D(b)$ must be some dwell-point $D(v)$, $v \in \mathcal{K}$. Therefore, the vertex set of $\mathbb{G}(k+1)$ must be a subset of $\mathcal{K}$. Since $\mathcal{K}$ induces a complete graph in $\mathbb{G}(k)$, and $\mathbb{G}(k+1)$ is an induced subgraph of $\mathbb{G}(k)$, we have that $\mathbb{G}(k+1)$ must be a complete graph, in which case rendezvous has occurred.

3. Suppose $|\mathcal{F}| \geq 3$ and no vertex in $\mathcal{F}$ is adjacent to all the other vertices of $\mathcal{F}$. First consider $v \in \mathcal{K}$. As noted previously, v is adjacent to all vertices of $\mathcal{F}$. Since no vertex in $\mathcal{F}$ is adjacent to all the other vertices of $\mathcal{F}$., we have that $D(a)$ for all $a \in \mathcal{F}$ cannot be the kth way-point of any agent at $D(v)$. Now consider $a \in \mathcal{F}$. Let $\mathcal{M}(a)$ denote the vertices adjacent to a in $\mathcal{F}$. By assumption a is adjacent to at least two vertices in $\mathcal{F}$ so there are at least two vertices in $\mathcal{M}(a)$. If any vertex in $\mathcal{M}(a)$, say b, is adjacent to any other vertex in $\mathcal{M}(a)$, say c, then a, b and c must induce a triangle, which contradicts the assumption that the graph induced in $\mathbb{G}(k)$

by $\mathcal{F}$ is triangle free. This implies the kth way-point of any agent at $D(a)$ must be $D(w)$ for some $w \in \mathcal{K}$. Therefore, the vertex set of $\mathbb{G}(k+1)$ must be a subset of $\mathcal{K}$. Since $\mathcal{K}$ induces a complete graph in $\mathbb{G}(k)$, and $\mathbb{G}(k+1)$ is an induced subgraph of $\mathbb{G}(k)$, we have that $\mathbb{G}(k+1)$ must be a complete graph, in which case rendezvous has occurred.

Now suppose $|\mathcal{F}| \geq 3$ and there is exactly one vertex in $\mathcal{F}$ which is adjacent to all the other vertices of $\mathcal{F}$. Let this vertex be s. Note that there cannot be two such vertices because if there were then the two vertices and any other vertex of $\mathcal{F}$ would induce a triangle. Since $\mathcal{F}$ induce a triangle free graph, we have that vertices $u, v \in \mathcal{F}$, where $u, v \neq s$, cannot be adjacent. Therefore, if $\mathcal{F}$ has q vertices, then the graph induced by vertices of $\mathcal{F}$ consists of the q vertices and $q - 1$ edges, each of which is incident on s. Since s is adjacent to all vertices of $\mathcal{K}$, we have that the graph induced by $\mathcal{K} \cup \{s\}$ is complete. Consider an agent positioned at dwell point $D(s)$. Clearly, no $D(u)$, $u \in \mathcal{F}$, $u \neq s$, can be a kth way-point of the agent. Consider an agent positioned at $D(v)$, $v \in \mathcal{K}$. Since v is adjacent to all vertices of $\mathcal{F}$, and each $w \in \mathcal{F} - \{s\}$ is only adjacent to s of $\mathcal{F}$, we have that $D(w)$, $w \in \mathcal{F} - \{s\}$, cannot be the kth way-point of any agent at dwell point $D(v)$. For an agent positioned at $D(w)$, $w \in \mathcal{F} - \{s\}$, its kth way-point must be some dwell point $D(u)$, $u \in \mathcal{K} \cup \{s\}$. The above implies in the $(k+1)$th time period, i.e. after the agents have moved to their kth way-points, all the dwell points $D(w)$, $w \in \mathcal{F} - \{s\}$, will be unoccupied. Hence, $\mathbb{G}(k+1)$ is a graph induced by a subset of $\mathcal{K} \cup \{s\}$. Since the graph induced by $\mathcal{K} \cup \{s\}$ is complete, so is any graph induced by a subset of $\mathcal{K} \cup \{s\}$, hence we get that $\mathbb{G}(k+1)$ is complete and the agents rendezvous in the $(k+1)$th time period. $\square$

Proof of Lemma 3 If $m = 1$, then rendezvous will occur in the next time period at the spoke vertex of $\mathbb{W}_1$. So suppose $m > 1$. This implies $\mathbb{T}(k)$ has at least two vertices. Note that since $\mathbb{T}(k)$ is balanced, it must be the case that r cannot be a leaf vertex of $\mathbb{T}(k)$, for otherwise $l(r, u) = 0$ for all leaf vertices in $\mathbb{T}(k)$, which would imply $\mathbb{T}(k)$ has only one vertex. Hence, r must be adjacent to at least two vertices in $\mathbb{T}(k)$. Hence, that $\mathbb{T}(k)$ is balanced and $m > 1$ imply $m \geq 3$.

For notational convenience, we will say that an agent is positioned at vertex i of $\mathbb{G}(k)$ if the agent is positioned at a dwell-point corresponding to vertex i. We say that wheel $\mathbb{W}_i$, $i \in \{1, \ldots, m\}$, is a leaf wheel of $\mathbb{G}(k)$ if vertex i in $\mathbb{T}(k)$ is a leaf. The following is a direct consequence of assumption 5:

Claim 1: If $\mathbb{W}_i$, $i \in \{1, \ldots, m\}$, is a non-leaf wheel of $\mathbb{G}(k)$, then no agent positioned at any dwell-point corresponding to a vertex in $\mathbb{W}_i$ moves in the kth maneuvering period.

The following is a direct consequence of assumption 4:

Claim 2: If $\mathbb{W}_i$, $i \in \{1, \ldots, m\}$, is a leaf wheel of $\mathbb{G}(k)$ adjacent to $\mathbb{W}_j$, then any any vertex of $\mathbb{W}_i$ not in $\mathbb{W}_j$ will not be in $\mathbb{G}(k+1)$.

Let the non-leaf wheels of $\mathbb{W}_i$, $i \in \{1, \ldots, m\}$, of $\mathbb{G}(k)$ be $\mathbb{W}_{i1}, \ldots, \mathbb{W}_{iq}$. Claims 1 and 2 imply that after

the agents move in the kth maneuvering period, $\mathbb{G}(k+1)$ will be the union of $\mathbb{W}_{i1}, \ldots, \mathbb{W}_{iq}$, and $\mathbb{T}(k+1) = \bar{\mathbb{G}}(k+1)(\mathbb{W}_{i1}, \ldots, \mathbb{W}_{iq})$ will be the graph induced in $\mathbb{T}(k)$ by the non-leaf vertices of $\mathbb{T}(k)$. The latter implies that $l(r,u) = l(r,v)$ for all leaf vertices u and v in $\mathbb{T}(k+1)$ since $\mathbb{T}(k+1)$ is obtained from $\mathbb{T}(k)$ by removing the leaf vertices of $\mathbb{T}(k)$. Hence, assumptions 1, 2 and 3 hold for $\mathbb{G}(k+1)$ and $\bar{\mathbb{G}}(k+1)(\mathbb{W}_{i1}, \ldots, \mathbb{W}_{iq})$.

Now we show that leaf and non-leaf vertices of $\mathbb{T}(k+1)$ satisfy 4, 5 and 6.

$\mathbb{G}(k+1)$ satisfies 6: Let i be a non-leaf of $\mathbb{T}(k+1)$, and suppose i is adjacent to j in $\mathbb{T}(k+1)$ where $l(r,j) = l(r,i) - 1$ in $\mathbb{T}(k+1)$. Note that $l(r,j) = l(r,i) - 1$ in $\mathbb{T}(k)$ as well since $\mathbb{T}(k+1)$ is obtained from $\mathbb{T}(k)$ by removing leaf nodes only. So $\mathbb{W}_j$ is in $\mathbb{G}(k+1)$.

Suppose some vertex v of $\mathbb{W}_i$ not in $\mathbb{W}_j$ is a vertex of a wheel $\mathbb{W}_h$, $h \in \{i1, \ldots, iq\} - (\{i\} \cup \mathcal{C}(r,i,\mathbb{T}(k+1)))$. Note that h is in $\mathbb{T}(k+1)$, and so cannot be a leaf of $\mathbb{T}(k)$. Now we show that $h \in \{1, \ldots, m\} - (\{i\} \cup \mathcal{C}(r,i,\mathbb{T}(k)))$. Since $\mathbb{T}(k+1)$ is obtained from $\mathbb{T}(k)$ by removing the leaf vertices of $\mathbb{T}(k)$, we have that $\mathcal{C}(r,i,\mathbb{T}(k+1)) \subset \mathcal{C}(r,i,\mathbb{T}(k))$, and the vertices in $\mathcal{C}(r,i,\mathbb{T}(k)) - \mathcal{C}(r,i,\mathbb{T}(k+1))$ are leaf vertices of $\mathbb{T}(k)$. This implies $h \notin \mathcal{C}(r,i,\mathbb{T}(k)) - \mathcal{C}(r,i,\mathbb{T}(k+1))$ since h is not a leaf of $\mathbb{T}(k)$.

Since $h \in \{i1, \ldots, iq\} - (\{i\} \cup \mathcal{C}(r,i,\mathbb{T}(k+1)))$, we have that $h \notin (\{i\} \cup \mathcal{C}(r,i,\mathbb{T}(k+1)))$. If $h \in \{i\} \cup \mathcal{C}(r,i,\mathbb{T}(k)))$, then h must necessarily be in $\mathcal{C}(r,i,\mathbb{T}(k)) - \mathcal{C}(r,i,\mathbb{T}(k+1))$. But as noted above, $h \notin \mathcal{C}(r,i,\mathbb{T}(k)) - \mathcal{C}(r,i,\mathbb{T}(k+1))$. Hence, $h \notin \{i\} \cup \mathcal{C}(r,i,k))$, which implies $h \in \{1, \ldots, m\} - (\{i\} \cup \mathcal{C}(r,i,\mathbb{T}(k)))$. From this, we get that v of $\mathbb{W}_i$ not in $\mathbb{W}_j$ is a vertex of wheel $\mathbb{W}_h$, $h \in \{1, \ldots, m\} - (\{i\} \cup \mathcal{C}(r,i,\mathbb{T}(k)))$, which contradicts the assumption that 6 holds for $\mathbb{G}(k+1)$.

$\mathbb{G}(k+1)$ satisfies 4: Let i be a leaf of $\mathbb{T}(k+1)$, and suppose i is adjacent to j in $\mathbb{T}(k+1)$. Since i, j are in $\mathbb{T}(k+1)$, we have that i, j were not leaf vertices in $\mathbb{T}(k)$. Hence, both i and j were in $\mathbb{T}(k)$. This implies both $\mathbb{W}_i$ and $\mathbb{W}_j$ are in $\mathbb{G}(k+1)$.

Now consider all $h \in \bar{\mathcal{C}}(r,i,\mathbb{T}(k))$. If vertex h is not a leaf vertex of $\mathbb{T}(k)$, then h is in $\mathbb{T}(k+1)$. Since $\mathbb{T}(k+1)$ is obtained from $\mathbb{T}(k)$ by removing just the leaf vertices, we have that i cannot be a leaf of $\mathbb{T}(k+1)$. Therefore, $\mathcal{C}(r,i,\mathbb{T}(k)) = \bar{\mathcal{C}}(r,i,\mathbb{T}(k))$, so all wheels $\mathbb{W}_h$, $h \in \mathcal{C}(r,i,\mathbb{T}(k))$ were leaf wheels in $\mathbb{G}(k)$. Since i is not a leaf in $\mathbb{T}(k)$ we have from 6 that each vertex w of $\mathbb{W}_i$ not in $\mathbb{W}_j$ is not a vertex of any wheel not in $\mathbb{W}_h$, $h \in \{i\} \cup \mathcal{C}(r,i,\mathbb{T}(k))$. Since all wheels $\mathbb{W}_h$, $h \in \mathcal{C}(r,i,\mathbb{T}(k))$ were leaf wheels, none of those wheels are in $\mathbb{G}(k+1)$, we have that each vertex of $\mathbb{W}_i$ not in $\mathbb{W}_j$ cannot be in any wheel $\mathbb{W}_l$, $l \neq i$, in $\mathbb{G}(k+1)$. Hence, the leaf wheels of $\mathbb{G}(k+1)$ satisfy 4.

$\mathbb{G}(k+1)$ satisfies 5: Let i be a non-leaf of $\mathbb{T}(k+1)$. This and 3 imply i was not a leaf and nor was it adjacent to any leafs in $\mathbb{T}(k)$. Since $\mathbb{T}(k+1)$ is obtained from $\mathbb{T}(k)$ by removing just the leaf vertices, we have that $\mathcal{P}(r,i,\mathbb{T}(k+1)) = \mathcal{P}(r,i,\mathbb{T}(k))$ and $\bar{\mathcal{C}}(r,i,\mathbb{T}(k+1)) = \bar{\mathcal{C}}(r,i,\mathbb{T}(k))$.

Recall that $\mathcal{M}(u) \subset (\{u\} \cup \mathcal{N}(u,k)) \cap \bigcup_{h \in \mathcal{P}(r,i,\mathbb{T}(k)) \cup \bar{\mathcal{C}}(r,i,\mathbb{T}(k))} V(\mathbb{W}_h)$. Since $\mathcal{M}(u)$ induce

a (broken) wheel with spoke u, it follows that each element of $\mathcal{M}(u)$ in $\bigcup_{h \in \mathcal{P}(r,i,\mathbb{T}(k)) \cup \bar{\mathcal{C}}(r,i,\mathbb{T}(k))} V(\mathbb{W}_h)$ is also in $\mathcal{N}(u,k)$.

Since $\mathcal{P}(r,i,\mathbb{T}(k+1)) = \mathcal{P}(r,i,\mathbb{T}(k))$ and $\bar{\mathcal{C}}(r,i,\mathbb{T}(k+1)) = \bar{\mathcal{C}}(r,i,\mathbb{T}(k))$, we have that $\bigcup_{h \in \mathcal{P}(r,i,\mathbb{T}(k)) \cup \bar{\mathcal{C}}(r,i,\mathbb{T}(k))} V(\mathbb{W}_h) = \bigcup_{h \in \mathcal{P}(r,i,\mathbb{T}(k+1)) \cup \bar{\mathcal{C}}(r,i,\mathbb{T}(k+1))} V(\mathbb{W}_h)$, so $\mathcal{M}(u) \subset \bigcup_{h \in \mathcal{P}(r,i,\mathbb{T}(k+1)) \cup \bar{\mathcal{C}}(r,i,\mathbb{T}(k+1))} V(\mathbb{W}_h)$. Moreover, each element of $\mathcal{M}(u)$ in $\bigcup_{h \in \mathcal{P}(r,i,\mathbb{T}(k+1)) \cup \bar{\mathcal{C}}(r,i,\mathbb{T}(k+1))} V(\mathbb{W}_h)$ is also adjacent to u, and therefore $\mathcal{M}(u) \subset \{u\} \cup \mathcal{N}(u,k+1)$. Hence, $\mathcal{M}(u) \subset (\{u\} \cup \mathcal{N}(u,k+1)) \cap \bigcup_{h \in \mathcal{P}(r,i,\mathbb{T}(k+1)) \cup \bar{\mathcal{C}}(r,i,\mathbb{T}(k+1))} V(\mathbb{W}_h)$.

Since $\mathbb{G}(k+1)$ is an induced subgraph of $\mathbb{G}(k)$, it follows that $\mathcal{M}(u)$ induce the same graph in $\mathbb{G}(k+1)$ as it does in $\mathbb{G}(k)$, namely a proper (broken) wheel with spoke u. Let w be any vertex of $\mathcal{N}(u,k+1)$ not in $\mathcal{M}(u)$. Since $\mathbb{G}(k+1)$ is a subgraph of $\mathbb{G}(k)$, we have that $\mathcal{N}(u,k+1) \subset \mathcal{N}(u,k)$, which implies $w \in \mathcal{N}(u,k)$, so w cannot be adjacent to all vertices of $\mathcal{M}(u)$. So 5 holds for $\mathbb{G}(k+1)$ and $\mathbb{T}(k+1)$.

The above imply that after the agents move in the kth maneuvering period, $\mathbb{G}(k+1)$ will again satisfy assumptions 1-6, and moreover, $\mathbb{G}(k+1)$ is the union of a proper subset of the graphs $\mathbb{W}_i$, $i \in \{1, \ldots, m\}$.

It follows by induction that $\mathbb{G}(k+L)$ must be equal to $\mathbb{W}_r$, where $L = l(r,u)$ for any leaf vertex u of $\mathbb{T}(k)$, in which case rendezvous will occur at the dwell-point corresponding to the spoke vertex of $\mathbb{W}_r$ in the next time period. $\square$

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