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# On Spectral Conditions for Positive Realness of Transfer Function Matrices

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*Abstract*— **Necessary and sufficient conditions for positive realness of general transfer function matrices are derived. These conditions can be checked using eigenvalue solvers for both proper and strictly proper transfer function matrices.**

**Keywords:** Positive Real (PR) Conditions; Passivity; Spectral methods.

## I. Introduction

The concept of Positive Realness (PR) and Strict Positive Realness (SPR) of a rational function appears frequently in various aspects of engineering [1], [2], [3], [4]. Efficient techniques for checking whether a given transfer function matrix is positive real has also been the subject of much interest in the numerical linear algebra community. While for the case of proper transfer function matrices, $H(s) = D + C^T(sI - A)^{-1}B$, with $D + D^T > 0$, robust numerical methods exist, [5], [6], [7], [8], [9], the case when $D + D^T \geq 0$ is still problematic. Usually, these situations are resolved using generalised eigenvalue solvers [10]. Our objective in this note is to demonstrate that one need never resort to such solvers and that such problems can always be solved as eigenvalue problems (even when $D + D^T$ is singular). Specifically, this short paper summarises the results derived in [11] and presents some generalisations. Full proofs can be found in the aforementioned reference.

## II. Background

We first present some background material.

*Definition 1:* A rational function in a complex variable $H(s)$ is PR if, and only if, $H(s)$ is real for real values of $s$, and $H(s)$ satisfies

$$\text{Real } [H(s)] \geq 0 \quad \text{for} \quad \text{Real } [s] \geq 0 \,. \tag{1}$$

*Definition 2:* A rational function in a complex variable $H(s)$ is SPR if, and only if,

$$\exists\, \epsilon > 0 \quad \text{such that} \quad H(s - \epsilon) \text{ is PR} \,. \tag{2}$$

The following equivalent definition of positive realness of a rational function is also well known [12].

*Definition 1(a):* A rational function of a complex variable $H(s)$ is PR if and only if $H(s)$ is real for real values of $s$, and all poles of $H(s)$ are in the closed left half plane of $s$. If there are imaginary-axis poles, they are simple with real positive residues, and

$$\text{Real } [H(j\omega)] \geq 0 \qquad \forall\, \omega \in \mathbb{R} \tag{3}$$

where $s = \sigma + j\omega$.

*Definition 3:* A matrix $H(s)$ is a positive real matrix, termed a PR matrix, if and only if the rational function defined by

$$F(s) = x^T H(s) x \tag{4}$$

is a positive real function for every complex $m$-dimensional vector $x$.

*Definition 4:* A matrix $H(s)$ is a strictly positive real matrix, termed a SPR matrix, if and only if there exists an $\epsilon > 0$ such that $F(s - \epsilon)$ is a positive real function for every complex $m$-dimensional vector $x$ where $F(s)$ is defined in (4).

It is well known that checking strict positive realness of a strictly proper transfer function matrix reduces to checking a number of conditions. See [13], [14], and in particular [15]. Basically these amount to checking a number of point conditions and to verifying whether $H(j\omega) + H(j\omega)^* > 0$ for all $\omega \in R$. Our objective in this particular paper is focus on the latter condition.

## III. MAIN RESULTS

Frequently one exploits the Hamiltonian based methods to test for SPR by making use of the following theorem.

*Theorem 1:* [16], [1], [6] Let $A$ be a stable $[n \times n]$ real matrix. Let $B \in R^{n\times m}$, $C \in R^{n\times m}$, and let $D \in R^{m\times m}$, with $D + D^T > 0$. Then, the transfer function matrix

$$H(j\omega) = D + C^T(j\omega I - A)^{-1}B \qquad (5)$$

is SPR if and only if the matrix

$$N = \begin{bmatrix} -A + BQ^{-1}C^T & BQ^{-1}B^T \\ -CQ^{-1}C^T & A^T - CQ^{-1}B^T \end{bmatrix},$$

with $Q = D + D^T$, has no eigenvalues on the imaginary axis. The matrix $N$ is called the Hamiltonian matrix.

In many applications the assumption that $D + D^T > 0$ does not hold. The following known observation is useful in this context.

*Observation: Let $\mathcal{G}$ denote the locus of eigenvalues of the matrix $H(j\omega) + H(j\omega)^*$ for all $\omega \in [-\infty, \infty]$. Let $\mathcal{G}^r$ denote the locus of eigenvalues of the matrix $H(\frac{1}{j\omega}) + H(\frac{1}{j\omega})^*$ for all $\omega \in [-\infty, \infty]$. Then, $\mathcal{G}$ and $\mathcal{G}^r$* coincide.

In fact: any mapping that maps the $\omega$ axis to itself will have this property and we shall explore one such mapping when discussing Positive Realness. Thus, if we can establish $H(\frac{1}{j\omega}) + H(\frac{1}{j\omega})^*$ is positive definite for all $\frac{1}{\omega}$, then it automatically follows that $H(j\omega) + H(j\omega)^*$ is such as well. To this end we note the following easily established result [17].

*Theorem 2:* Let $H(j\omega) = D + C^T(j\omega I - A)^{-1}B$ be a strictly proper SPR transfer function matrix. Then, $H(\frac{1}{j\omega}) = \bar{D} + \bar{C}^T(j\omega I - \bar{A})^{-1}\bar{B}$, with $\bar{A} = A^{-1}$, $\bar{B} = -A^{-1}B$, $\bar{C}^T = C^T A^{-1}$, $\bar{D} = D - C^T A^{-1}B$, and $\bar{D} + \bar{D}^T > 0$.

Theorem 2 implies that the locus of eigenvalues of a transfer function matrix with $D + D^T \geq 0$, is equivalent to the locus of eigenvalues of a transfer function matrix with $\bar{D} + \bar{D}^T > 0$. The fact that $\bar{D} + \bar{D}^T > 0$ ensures that Hamiltonian methods can be applied to this latter problem.

### A. Strict positive realness

*Theorem 3:* [11] Let $A$ be a stable $[n \times n]$ real matrix. Let $B \in R^{n\times m}$, $C \in R^{n\times m}$, and let $D \in R^{m\times m}$, with $D + D^T$ singular. Then the transfer function matrix $H(j\omega) + H(j\omega)^* > 0$, $H(j\omega) = D + C^T(j\omega I - A)^{-1}B$, for all finite $\omega$, if and only if $H(0) + H(0)^* > 0$ and the matrix

$$\bar{N} = \begin{bmatrix} -\bar{A} + \bar{B}\bar{Q}^{-1}\bar{C}^T & \bar{B}\bar{Q}^{-1}\bar{B}^T \\ -\bar{C}\bar{Q}^{-1}\bar{C}^T & \bar{A}^T - \bar{C}\bar{Q}^{-1}\bar{B}^T \end{bmatrix},$$

with $\bar{Q} = \bar{D} + \bar{D}^T$, has no eigenvalues on the imaginary axis except at the origin, with $\bar{A} = A^{-1}$, $\bar{B} = -A^{-1}B$, $\bar{C}^T = C^T A^{-1}$ and $\bar{D} = D - C^T A^{-1}B$.

*Comment:* Note that the matrix $A$ is replaced with $A^{-1}$, $B$ by $-A^{-1}B$, $C$ by $\bar{C}^T = C^T A^{-1}$ and $D$ by $\bar{C}^T = C^T A^{-1}$. In the scalar case strict positive realness of $A, b, c, d$ is equivalent to determining whether $\dot{x} = Ax$ and $\dot{x} = (A - \frac{1}{d}bc^T)x$ have a common quadratic Lyapunov function. It is well known (via a simple congruence argument) that the CQLF existence problem for $\dot{x} = Ax$ and $\dot{x} = (A - \frac{1}{d}bc^T)x$ is identical to the CQLF existence problem for $\dot{x} = A^{-1}x$ and $\dot{x} = (A - \frac{1}{d}bc^T)^{-1}x$. It is easily verified that the system obtained via the transformation $\omega \to \frac{1}{\omega}$ corresponds to this CQLF existence problem in the scalar case when $d > 0$.

Now we ask if there are efficient methods to check positive realness. To this end the following Lemma is useful.

### B. Positive realness

*Lemma 1:* Let $A$ be a stable matrix. Let $H(j\omega) = D + C^T(j\omega I - A)^{-1}B$, with $D + D^T > 0$. Then,

$$det\Big[H(j\omega) + H(j\omega)^*\Big] = S(\omega)det\Big[j\omega I + N\Big],$$

where $S(\omega)$ is a scalar function of $\omega$ such that $S(\omega) < 0$ for all $\omega \in (-\infty, \infty)$.

*Comment:* It follows that $H(j\omega)$ is SPR if and only if $N$ has no eigenvalues on the imaginary axis.

If we assume that $det[H(j\omega) + H(j\omega)^*] \neq 0$ for all frequencies then two special cases need to be discerned.

(i) *Hamiltonian methods apply directly :* This is the case when $D + D^T$ and/or $\bar{D} + \bar{D}^T$ are invertible.

(ii) *$H(0) + H(0)^*$ and $D + D^T$ are both singular.*

Case (i): (Hamiltonian methods apply directly) Let us assume without any loss of generality that $D + D^T > 0$.

*Theorem 4:* Let $\Omega$ be the distinct set of frequencies for which $det\Big[H(j\omega) + H(j\omega)^*\Big] = 0$, with the elements of $\Omega = \Big\{\omega_1, \omega_2, ..., \omega_s\Big\}$, $s \leq n$ listed in strictly increasing order. These frequencies are the eigenvalues of $N$ that are on the imaginary axis. $H(j\omega)$ is PR if and only if: (i) $N$ has no eigenvalues on the imaginary axis of odd multiplicity; (ii) $H(j\Delta_i) + H(j\Delta_i)^*$ has only positive real eigenvalues for all $\Delta_i = \frac{\omega_i + \omega_{i+1}}{2}$, $i \in \Big\{1, s-1\Big\}$.

Case (ii): ($H(0) + H(0)^*$ singular) Suppose now that both $D + D^T$ and $H(0) + H(0)^*$ are both singular. Then, the Hamiltonian cannot be used to test for positive realness. However, if we assume that $det\Big[H(j\omega) + H(j\omega)^*\Big] \neq 0$ for some $\omega \in (-\infty, \infty)$, then there must exist a $\omega_0$, such that $H(j\omega_0) + H(j\omega_0)^* > 0$ (a necessary condition for positive realness). We can then make use of the following observation.

*Observation: Let $\mathcal{G}$ denote the locus of eigenvalues of the matrix $H(j\omega) + H(j\omega)^*$ for all $\omega \in [-\infty, \infty]$. Let $\mathcal{G}^s$ denote the locus of eigenvalues of the matrix $\tilde{H}(j\delta) + \tilde{H}(j\delta)^*$ for all $\delta \in [-\infty, \infty]$, with $\delta = \omega - \omega_0$, and $\tilde{H}(j\delta) = H\Big(j(\delta + \omega_0)\Big)$. Then, $\mathcal{G}$ and $\mathcal{G}^s$* coincide.

Note that $\tilde{H}(j\delta) = D + C^T(j\delta I - \tilde{A})^{-1}B$ where $\tilde{A} = A - j\omega_0 I$. By definition, $\tilde{H}(0) + \tilde{H}(0) > 0$, and Hamiltonian methods can now be applied.

## IV. A Hamiltonian equivalence class

The test for positive realness used a simple observation on the eigenvalue locus of a family of matrices. In the main results of this paper we used the fact that $j\omega$ can be replaced with its reciprocal, but in the section on positive realness, we noted that other transformations can be used as well. In some situations this latter observation is useful as it can be used to improved the conditioning on some of the matrices $A, B, C, D$. In this section we identify entire classes of linear systems that are equivalent from the spectral locus perspective. Checking whether $G(j\omega) + G(j\omega)^* > 0$ for any of these systems immediately implies this statement for any of the others. Thus one may choose a system from this entire equivalence class based on numerical conditioning considerations.

Consider four complex matrices $A, B, C, D$, where $j\omega I - A$ is invertible for all real $\omega$. For $z \in C$ and $zI - A$ invertible, define

$$\sigma(A, B, C, D; z) = \mathrm{Spec}\Big[D + C^*(zI - A)^{-1}B + (D + C^*(zI - A)^{-1}B)^*\Big]$$

where $C^*$ is the Hermitian conjugate of $C$, and where Spec is the spectrum, that is the set of eigenvalues. Define the *spectral locus* corresponding to $A, B, C, D$ to be

$$\rho(A, B, C, D) = \overline{\cup_{\omega \in (-\infty, \infty)} \sigma(A, B, C, D; j\omega)}$$

where $\overline{S}$ denotes the closure of $S$. Now let $a, b, c, d$ be real numbers, where we assume that $b, d$ are not simultaneously zero. Define the following matrices:

$$\begin{aligned} \overline{A} &= (cA - jaI)(bI - jdA)^{-1} \\ \overline{B} &= (bI - jdA)^{-1}B \\ \overline{C} &= (ad + bc)(bI + jdA^*)^{-1}C \\ \overline{D} &= D + jdC^*(bI - jdA)^{-1}B \end{aligned}$$

*Theorem 5:*

$$\rho(A, B, C, D) = \rho(\overline{A}, \overline{B}, \overline{C}, \overline{D}) \qquad (6)$$

*Proof:* Define the complex variable $u$ by the following fractional linear transformation:

$$z = \frac{ja + bu}{c + jdu}$$

It follows that $z$ is pure imaginary if and only if $u$ is pure imaginary. Direct substitution shows that

$$D + C^*(zI - A)^{-1}B = \overline{D} + \overline{C}^*(uI - \overline{A})^{-1}\overline{B}$$

and hence

$$\sigma(A, B, C, D; z) = \sigma(\overline{A}, \overline{B}, \overline{C}, \overline{D}; u).$$

The mapping $z \mapsto u$ is one-to-one on the extended imaginary axis (where the point at infinity is included) and hence

$$\cup_{\mathrm{Re}(z)=0 \cup \{\infty\}} \sigma(A, B, C, D; z) = \cup_{\mathrm{Re}(u)=0 \cup \{\infty\}} \sigma(A, B, C, D; u)$$

The proof is completed by noting that

$$\cup_{\mathrm{Re}(z)=0\cup\{\infty\}}\, \sigma(A,B,C,D;z) = \overline{\cup_{\mathrm{Re}(z)=0}\, \sigma(A,B,C,D;z)}$$

With these observations Theorem 1 can be refined.

*Theorem 6:* Let $A$ be a stable $[n \times n]$ real matrix. Let $B \in R^{n\times m}$, $C \in R^{n\times m}$, and let $D \in R^{m\times m}$, with $D+D^T \geq 0$. Then the transfer function matrix $H(j\omega)+H(j\omega)^* > 0$, $H(j\omega) = D + C^T(j\omega I - A)^{-1}B$, for all finite $\omega$, if and only if for every $a,b,c,d$ with $\overline{Q} = \overline{D} + \overline{D}^* > 0$, the matrix

$$\bar{N} = \begin{bmatrix} -\bar{A} + \bar{B}\bar{Q}^{-1}\bar{C}^T & \bar{B}\bar{Q}^{-1}\bar{B}^T \\ -\bar{C}\bar{Q}^{-1}\bar{C}^T & \bar{A}^T - \bar{C}\bar{Q}^{-1}\bar{B}^T \end{bmatrix},$$

with $\bar{Q} = \bar{D} + \bar{D}^T$, has no eigenvalues on the imaginary axis except possibly at the image of the point $\omega = \infty$ under the mapping $z \mapsto u$ has no eigenvalues on the imaginary axis for any $a,b,c,d$ with $\bar{Q} > 0$, except at the mapping of $\omega = \infty$.

## V. Conclusion

Necessary and sufficient conditions for positive realness of general transfer function matrices are derived.


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