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Finite-Time Stability Analysis of Linear Discrete-Time Systems via Polyhedral Lyapunov Functions

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Abstract— **In this paper we consider the finite-time stability problem for discrete-time linear systems. Differently from previous papers, the stability analysis is performed with the aid of polyhedral Lyapunov functions rather than using the classical quadratic Lyapunov functions. In this way we are able to deal with more realistic constraints on the state variables; indeed, in a way which is naturally compatible with polyhedral functions, we assume that the sets to which the state variables must belong to in order to satisfy the finite-time stability requirement are boxes (or more in general polytopes) rather than ellipsoids. The main result, derived by using polyhedral Lyapunov functions, is a sufficient condition for finite-time stability of discrete-time linear systems. Some examples are presented to illustrate the advantages of the proposed methodology over existing methods.**

I. INTRODUCTION

The concept of finite-time stability (FTS), or short-time stability, considered in this paper dates back to the Sixties, when this idea was introduced in the control literature [6], [9]. A system is said to be finite-time stable if, given a bound on the initial condition, its state does not exit a certain domain during a specified time interval. It is important to note that FTS and Lyapunov Asymptotic Stability (LAS) are independent concepts; indeed a system can be FTS but not LAS, and vice versa. While LAS deals with the behavior of a system in a sufficiently long (in principle infinite) time interval, FTS is a more practical concept, useful to study the behavior of the system in a finite (possibly short) interval. Therefore FTS finds application whenever it is desired that the state variables do not exit a given domain (for example to avoid saturations or the excitation of nonlinear dynamics) during the transients.

Recently, in [2] for the discrete-time case and in [3] for the continuous-time case, the authors extended the definition of FTS given in [6] and derived sufficient conditions for FTS and finite-time stabilization via state feedback of linear systems. All these conditions require the solution of feasibility problems involving Linear Matrix Inequalities (LMIs).

The definition of FTS given in [2] and [3] exploits the standard weighted quadratic norm to define both the initial state domain (*initial domain*) and the domain where the trajectory is requested to be confined over a prescribed time interval (*trajectory domain*); therefore such domains turn out to be ellipsoidal. The definition of the above domains is consistent with the fact that quadratic Lyapunov functions are used to derive the main results of [2] and [3].

In this paper we propose a new definition for the initial and trajectory domains that makes use of polytopes rather than of ellipsoidal domains. Polytopic domains naturally arise in many practical problems when, for instance, we consider constraints on the state variables in the form $a_i \leq x_i \leq b_i$.

If the domains are defined by means of polytopes, the FTS analysis based on the ellipsoidal domains introduces conservatism since it is needed to approximate the polytopic initial domain by an appropriate ellipsoidal domain containing it, and the polytopic trajectory domain by another ellipsoidal domain contained in it. For example let us consider the discrete-time system

$$x(t+1) = \begin{pmatrix} 1.0 & 0.4 \\ -0.5 & 1.0 \end{pmatrix} x(t) , \quad t = 0, 1, \ldots \quad (1)$$

and assume that the following constrains on the state variables are imposed

$$-0.7 \leq x_1(0) \leq 0.7 \quad (2a)$$
$$-1.0 \leq x_2(0) \leq 1.0 \quad (2b)$$
$$-2.45 \leq x_1(t) \leq 2.45,\ t \in \{1, \ldots, N\} \quad (2c)$$
$$-2.5 \leq x_2(t) \leq 2.5,\ t \in \{1, \ldots, N\} , \quad (2d)$$

where $N = 7$.

If we analyze this FTS problem using the approach proposed in [2], we need to approximate the initial domain and the trajectory domain by ellipsoidal domains (see Fig. 1); it is therefore clear that the approximation of the domains introduces conservatism in the FTS analysis.

Moreover, as we will see in Section II, in some cases the technique proposed in [2] cannot be applied; this happens when the ellipsoid approximating the trajectory domain does not contain the ellipsoid approximating the initial domain.

To overcome this problem, in this paper we will provide a technique based on polyhedral Lyapunov functions [5] which allows us to take directly into account polytopic domains in the FTS analysis; the main result is a sufficient condition for FTS of linear time-invariant systems. Then we present some numerical examples to show the advantages of the proposed approach over the existing techniques.

The paper is organized as follows: in Section II a definition of FTS in which the initial and trajectory domains are polytopes is given, and some preliminary results are provided. In

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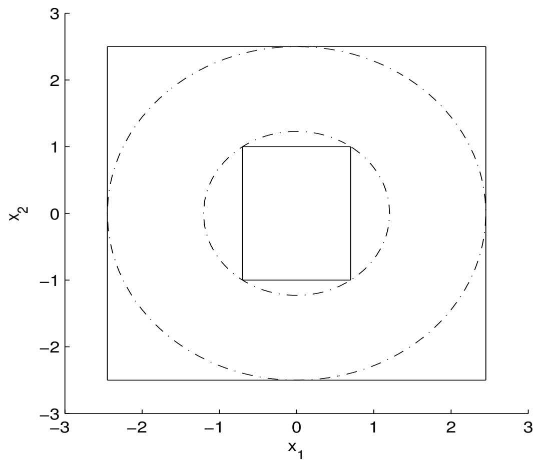

Fig. 1. *Initial domain* and *trajectory domain* for system (1).

Section III the main result of the paper is stated, namely a sufficient condition for FTS based on polyhedral functions theory. In Section IV some examples which illustrate the benefits of the proposed approach are discussed. Finally, some conclusions are drawn in Section V.

Notation

We denote by q_i, $i = 1, ..., m$, the i-th column of a matrix $Q \in \mathbb{R}^{n\times m}$. If $Q \in \mathbb{R}^{n\times m}$ is a full row rank matrix, we indicate with $\wp(Q)$ the polytope defined as (see [8], p. 6)

$$\wp(Q) := \{x \in \mathbb{R}^n : \|Q^T x\|_\infty \le 1\}\,, \tag{3}$$

where, given a vector $v \in \mathbb{R}^m$, $\|v\|_\infty := \max\{|v_1|, \dots, |v_m|\}$ denotes the infinity norm of v. By $\partial\wp(Q)$ we indicate the boundary of the polytope $\wp(Q)$.

II. Problem Statement and Preliminaries

Let us consider the following linear system

$$x(t+1) = Ax(t)\,, \quad t = 0, 1, \dots\,, \tag{4}$$

where $A \in \mathbb{R}^{n\times n}$. Roughly speaking, system (4) is said to be finite-time stable (FTS) if, given a certain initial domain, its state remains, over a finite-time interval, within a prescribed trajectory domain. In particular, we assume that both the domains are polytopes.

Definition 1 (Finite-time stability): System (4) is said to be FTS with respect to (P_0, P, N), where N is a positive number, $P_0 \in \mathbb{R}^{n\times m_0}$ and $P \in \mathbb{R}^{n\times m}$ are two full-row rank matrices with $\wp(P_0) \subset \wp(P)$, if

$$x(0) \in \wp(P_0) \Rightarrow x(t) \in \wp(P) \quad \forall t \in \{1, \dots, N\}. \tag{5}$$

◊

Remark 1: Note that, given a full row rank matrix P, the set $\wp(P)$ is a polytope symmetric with respect to the origin (see (3)). It follows that, by Definition 1, we are restricting our attention to the class of initial and trajectory domains that are symmetric polytopes. ◊

Remark 2: A sufficient condition for system (4) to be FTS with respect to (P_0, P, N) can be derived by using the approach proposed in [2]. The main result of [2] states that system (4) is FTS with respect to (P_0, P, N) if there exist three positive scalars c_1, c_2, α, with $\alpha \ge 1$ and $c_2 > c_1$, and two positive definite matrices $R, Q \in \mathbb{R}^{n\times n}$ such that

$$\wp(P_0) \subseteq \mathcal{E}_1 = \{x \in \mathbb{R}^n : x^T R x \le c_1\} \tag{6a}$$
$$\wp(P) \supseteq \mathcal{E}_2 = \{x \in \mathbb{R}^n : x^T R x < c_2\} \tag{6b}$$
$$A^T Q A - \alpha Q < 0 \tag{6c}$$
$$\frac{c_1}{c_2}\alpha^N I < \tilde{Q} < I\,, \tag{6d}$$

where $\tilde{Q} = R^{-1/2}\, Q\, R^{-1/2}$. First note that this way of proceeding unavoidably introduces conservatism in the FTS analysis. Even worse, there are some cases when it is not possible to find a matrix R and two scalars c_1 and c_2, $c_2 > c_1$, such that conditions (6a) and (6b) are satisfied with $\mathcal{E}_1 \subset \mathcal{E}_2$. In these cases, the procedure derived in [2] cannot be applied. For example assume that the initial and trajectory domains are

$$\wp(P_0) = \{x \in \mathbb{R}^2 :\ |x_1| \le 1,\ |x_2| \le 1\}$$
$$\wp(P) = \{x \in \mathbb{R}^2 :\ |x_1| \le 1 + \epsilon,\ |x_2| \le 2,\ \epsilon > 0\}\,.$$

It is easy to see that, regardless of the system under consideration and the step N, there exists a lower bound $\bar{\epsilon}$ to the value of ϵ for which the approach proposed in [2] cannot be exploited because it is not possible to find two ellipsoidal domains $\mathcal{E}_1$ and $\mathcal{E}_2$ that verify $\wp(P_0) \subseteq \mathcal{E}_1 \subset \mathcal{E}_2 \subseteq \wp(P)$. ◊

In Section III, we will present an alternative sufficient condition which does not suffer from the drawbacks of the approach proposed in [2]. To this end, we will make use of the class of polyhedral Lyapunov functions [5], which are piecewise linear functions of the following form

$$V(x) = \|Q^T x\|_\infty\,,$$

where Q is a full row rank matrix.

A. Some useful results about polytopes

In the following we provide some preliminary definitions and results on linear algebra and polytopes that will be necessary to state the main result of the paper.

If we deal with a finite set, say $K = \{x^{(1)}, \dots, x^{(l)}\} \subset \mathbb{R}^n$, the *convex hull* of K turns out to be a *polytope*, whose *dimension* ([10], p. 5), is given by the dimension of the affine hull of K, i.e.

$$\mathrm{rank}\left[\begin{array}{cccc} x^{(2)} - x^{(1)} & x^{(3)} - x^{(1)} & \dots & x^{(l)} - x^{(1)} \end{array}\right].$$

Moreover, as stated in next lemma, the set of vertices of a given polytope $\mathcal{P}$ is a subset of K.

Lemma 1 ([10]): Given a polytope defined as the convex hull of $K = \{x^{(1)}, \dots, x^{(l)}\} \subset \mathbb{R}^n$, the vertices of the polytope are the points $x^{(i)} \in K$ which satisfy the following property

$$x^{(i)} \notin \mathrm{conv}\left(K - \{x^{(i)}\}\right).$$

◊

Remark 3: Note that, given a collection of symmetric points $K = \{x^{(1)}, \dots, x^{(2l)}\}$, $x^{(i)} = -x^{(l+i)}, i = 1, \dots, l$, if $x^{(i)}$ is a vertex of $\mathrm{conv}(K)$, then also $x^{(l+i)} = -x^{(i)}$ is a vertex of $\mathrm{conv}(K)$. $\diamondsuit$

Remark 4: A given symmetric polytope $\mathcal{P}$ admits two different equivalent descriptions: by means of its vertices, and by means of a suitable matrix (see (3)). The algorithm in [4], implemented in the Matlab routine *convhulln*, enables to find the matrix Q defining a polytope starting from the polytope vertices. $\diamondsuit$

In the following, given a symmetric polytope $\wp(Q)$, we indicate with $x_Q^{(i)}$ with $i = 1 \dots 2l$ the vertices of $\wp(Q)$ and we suppose that $x_Q^{(i)} = -x_Q^{(i+l)}$ for $i = 1 \dots l$.

To conclude this section, we present a lemma that will be used in the proof of the main result.

Lemma 2: Let $P_0 \in \mathbb{R}^{n \times m_0}$ and $P \in \mathbb{R}^{n \times m}$ be two full-row rank matrices. If $\wp(P_0) \subseteq \wp(P)$ then

$$\|P^T x\|_\infty \le \|P_0^T x\|_\infty , \quad \forall x \in \mathbb{R}^n .$$

Proof: Consider a vector $x \in \mathbb{R}^n$. There exist two points $\bar{x} \in \partial\wp(P)$ and $\bar{x}_0 \in \partial\wp(P_0)$, and two positive scalars β and β_0 such that

$$x = \beta\bar{x} = \beta_0\bar{x}_0.$$

Taking into account that $\wp(P_0) \subseteq \wp(P)$, we have that for some $\gamma \ge 1$

$$\bar{x} = \gamma\bar{x}_0 ,$$

which implies

$$\beta \le \beta_0 . \tag{7}$$

Finally, from the definition of boundary point of a polytope, we have

$$\|P^T x\|_\infty = \beta\|P^T \bar{x}\|_\infty = \beta$$
$$\|P_0^T x\|_\infty = \beta_0\|P_0^T \bar{x}_0\|_\infty = \beta_0$$

From the last statement and (7), the proof follows. ■

III. Main result

The following theorem is the main result of the paper.

Theorem 1 (Sufficient condition for FTS): System (4) is finite-time stable with respect to (P_0, P, N) if there exists a polytope $\wp(Q)$ of dimension n such that the following conditions hold

$$\max_i \|Q^T A x_Q^{(i)}\|_\infty \ge 1 , \tag{8}$$

$$\max_i \|Q^T x_{P_0}^{(i)}\|_\infty \max_i \|P^T x_Q^{(i)}\|_\infty \cdot \cdot \left(\max_i \|Q^T A x_Q^{(i)}\|_\infty\right)^N \le 1 , \tag{9}$$

where $x_{P_0}^{(i)}$ are the vertices of the polytope $\wp(P_0)$.

Proof: Consider a polytope $\wp(Q)$ specified as the convex hull of the set $\{x_Q^{(1)}, ..., x_Q^{(2l)}\}$.

Now let us consider the polyhedral Lyapunov function

$$V(x) = \|Q^T x\|_\infty . \tag{10}$$

Assume that there exists a scalar $\alpha \ge 1$ such that the condition

$$V(x(t+1)) \le \alpha V(x(t)) \tag{11}$$

holds for all $t = 0, \dots, N$, where $x(t+1)$ is evaluated along the solution of the system (4).

We will first demonstrate that conditions

$$\max_i \|Q^T x_{P_0}^{(i)}\|_\infty \max_i \|P^T x_Q^{(i)}\|_\infty \, \alpha^N \le 1 \tag{12}$$

and (11) imply that system (4) is FTS with respect to (P_0, P, N). Then, to conclude the proof, we will show that conditions (11) and (12) are implied by (8) and (9).

Applying iteratively (11), we obtain

$$V(x(t)) \le \alpha^t V(x(0)), \quad t = 1, \dots, N . \tag{13}$$

It follows that

$$\|Q^T x(t)\|_\infty \le \|Q^T x(0)\|_\infty \alpha^t \qquad \forall t = 1, \dots, N . \tag{14}$$

Since $x(0) \in \wp(P_0)$ and $\|Q^T x\|_\infty$ enjoys a radial property, an upper bound to the quantity $\|Q^T x(0)\|_\infty$ is attained at one of the vertices of $\wp(P_0)$, i.e.

$$\|Q^T x(0)\|_\infty \le \max_i \|Q^T x_{P_0}^{(i)}\|_\infty . \tag{15}$$

Let us choose $h > 0$ such that

$$\wp(Q) \subseteq \wp(hP) . \tag{16}$$

Taking into account Lemma 2, equation (16) can be equivalently written as

$$\|Q^T x\|_\infty \ge h\|P^T x\|_\infty \qquad \forall x \in \mathbb{R}^n . \tag{17}$$

Since the polytope $\wp(Q)$ is included in the polytope $\wp(hP)$, the vertices of $\wp(Q)$ belong to $\wp(hP)$, therefore we have

$$\max_i h\|P^T x_Q^{(i)}\|_\infty \le 1 . \tag{18}$$

Equation (18) gives an upper bound to the values of h that satisfy (17)

$$h \le h_{max} := \frac{1}{\max_i \|P^T x_Q^{(i)}\|_\infty}. \tag{19}$$

From (17) and (19) we have along the system trajectories

$$\|Q^T x(t)\|_\infty \ge h_{max}\|P^T x(t)\|_\infty = \frac{\|P^T x(t)\|_\infty}{\max_i \|P^T x_Q^{(i)}\|_\infty}. \tag{20}$$

Putting together (14), (15) and (20), we obtain

$$\|P^T x(t)\|_\infty \le \max_i \|Q^T x_{P_0}^{(i)}\|_\infty \max_i \|P^T x_Q^{(i)}\|_\infty \alpha^t , \; t = 1, \dots N . \tag{21}$$

From (21) it readily follows that (12) implies, for all $t = 1, \dots N$, $\|P^T x(t)\|_\infty \le 1$; from this last consideration our first claim follows.

Now we will prove that conditions (8) and (9) guarantee (11) and (12). Condition (11) is guaranteed if [5]

$$\max_j \tilde{q}_j^T A x \le \alpha \max_j \tilde{q}_j^T x, \quad \forall x \in \mathbb{R}^n , \tag{22}$$

where $\tilde{q}_j$ denotes the j-th column of $\tilde{Q} = \begin{pmatrix} Q & -Q \end{pmatrix}$.

For each x, there exist a positive scalar λ and a point $x_b \in \partial\wp(Q)$ such that $x = \lambda x_b$. Since $\max_j \tilde{q}_j^T x_b = 1$, condition (22) can be equivalently rewritten

$$\max_j \tilde{q}_j^T A x \leq \alpha\,, \quad \forall x \in \partial\wp(Q)\,. \tag{23}$$

Notice that the maximum value of the linear function

$$\max_j \tilde{q}_j^T A x$$

on a face of $\wp(Q)$ is attained at the vertices of the face itself. Hence, condition

$$\max_i \|Q^T A x_Q^{(i)}\|_\infty \leq \alpha \tag{24}$$

guarantees (23). Condition (8) guarantees the existence of a scalar $\alpha \geq 1$ such that (24) holds. Now, the proof follows noticing that condition (12) is implied by (9). ∎

In order to find a polyhedral Lyapunov function satisfying the conditions of Theorem 1, the following procedure can be adopted.

Procedure 1 (Implementation of Theorem 1):

1) Fix an initial number $2l \geq 2n$ of symmetric points $x_Q^{(i)}$ on a hypersphere with radius 1. Let indicate with $K_0 = \{x_Q^{(i)}\}_{i=1,\dots,2l}$ the set of such points.
2) Find a set of symmetric points K solving the problem

$$\min_K f(K) \tag{25}$$
$$\text{s.t.} \max_i \|Q^T A x_Q^{(i)}\|_\infty \geq 1$$
$$\text{rank}(Q) = n$$

with initial condition K_0, where

$$f(K) = \max_i \|Q^T x_{P_0}^{(i)}\|_\infty \max_i \|P^T x_Q^{(i)}\|_\infty \cdot$$
$$\cdot \left(\max_i \|Q^T A x_Q^{(i)}\|_\infty\right)^N - 1$$

3) Let $M = \min_K f(K)$. If $M < 0$ then set

$$K_{opt} = \arg M,$$

and go to step 4, else set

$$K_0 = K \cup \left\{x_Q^{(l+1)}, -x_Q^{(l+1)}\right\},\ x_Q^{(l+1)} \in \mathbb{R}^n$$
$$l = l+1\,,$$

and go to step 2.
4) The polyhedral Lyapunov function which proves the FTS of system (4) wrt (P_0, P, N) is

$$V(x) = \|Q^T x\|_\infty$$

where Q describes the polytope of vertices K_{opt}. ◊

Remark 5: To solve problem (25), we have made use of the Matlab Optimization Toolbox routine *fminimax* [1], with variables $\pm x_Q^{(i)}, i = 1, \dots, l$. ◊

Remark 6: The choice of $x_Q^{(l+1)}$ in step 3 is done putting such point on one of the faces of $\wp(Q)$. In this way, since at each step the algorithm begins from the solution found in the previous step, the value M decreases (or, at least, does not increase) at each step. ◊

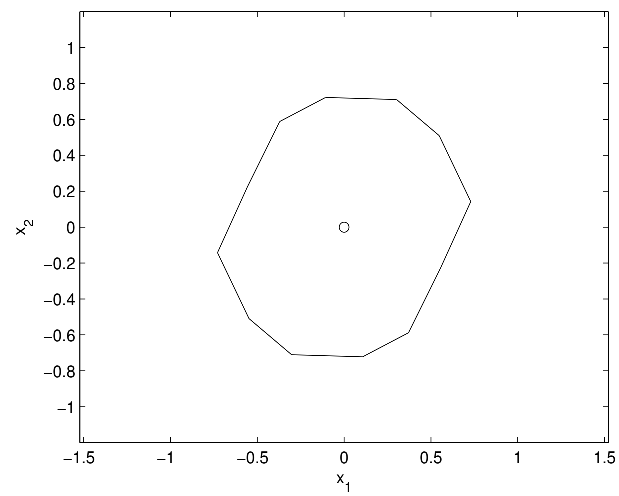

Fig. 2. Polyhedral Lyapunov function with 12 vertices.

IV. Examples and applications

A. Comparison with the previous literature

Let us reconsider the FTS problem presented in the Introduction. We will show that Theorem 1 enables to prove that system (1), under the constraints (2), is FTS while the sufficient condition proposed in [2] does not enable to draw any conclusion.

Our goal is to check whether system (1) is FTS with respect to (P_0, P, N), where P_0 and P are the 4-vertices polytopes selected accordingly to the constraints (2) (see Fig. 1), and $N = 7$.

We first tried to verify the FTS stability of system (1) by using the approach described in Remark 2. To this end, we selected R and c_2 imposing the ellipsoid $\mathcal{E}_2$ to be symmetric with respect to the coordinate axis and with maximum possible volume. Consequently, c_1 was computed by a scaling operation (see Fig. 1). The conditions (6c)–(6d) were evaluated with the aid of the Matlab LMI Toolbox [7] and the derived problem was found unfeasible for all $\alpha \geq 1$.

Next, we tried to solve the problem with the application of Theorem 1. Using Procedure 1, we verified that system (1) is FTS with respect to (P_0, P, N), by using the polyhedral Lyapunov function of 12 vertices shown in Fig. 2.

Next, consider the third order system with the dynamical matrix

$$A = \begin{pmatrix} 0.5 & 1 & 0 \\ 0 & 1.05 & 0 \\ 0 & 1 & 0.9 \end{pmatrix}\,. \tag{26}$$

Consider the following boxes in $\mathbb{R}^3$

$$P_0 = \{x \in \mathbb{R}^3 : |x_1| \leq 0.5, |x_2| \leq 0.5, |x_3| \leq 0.5\}$$
$$P = \{x \in \mathbb{R}^3 : |x_1| \leq 6, |x_2| \leq 6, |x_3| \leq 6\}\,;$$

moreover let $N = 5$. We found that the system is FTS with respect to (P_0, P, N) the polyhedral Lyapunov function of 12 vertices shown in Fig. 3.

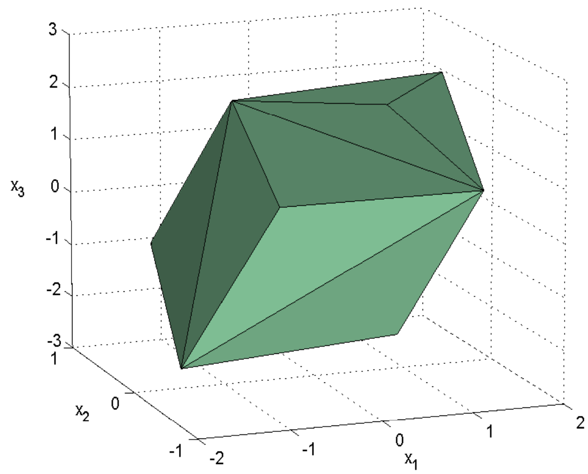

Fig. 3. Polyhedral Lyapunov function with 12 vertices.

B. Discussion

From the above results we can conclude that the method proposed in this paper improves the existing literature when, as often it happens in practical engineering problems, the initial and trajectory domains, to which the state variables are constrained to belong to, are boxes or, more in general, polytopes in the state space.

Indeed, in this case, the problem data may be such that the method proposed in [2] cannot be applied for the FTS analysis of the system under consideration (see Remark 2).

On the other hand the approach proposed in this paper suffers from the fact that the feasibility problem with constraints given by (9) and (12) is, in general, not convex, and therefore the convergence to the optimal solution is not guaranteed. Conversely, the approach of [2] is based on LMIs conditions which lead to a convex optimization problem. However, as shown in Section IV-A, even when the approach of [2] is applicable, Theorem 1 may result a less conservative FTS analysis tool than the main theorem in [2].

Therefore it is mandatory to use the approach of this paper when we are in the situation described in Remark 2 (the ellipsoid approximating the trajectory domain does not contain the ellipsoid approximating the initial domain); in the other cases the proposed methodology can be considered a viable, possibly less conservative, alternative to the main result in [2].

V. Conclusions

In this paper we have considered the finite-time stability problem for discrete-time linear systems, where the initial and trajectory domains are polytopes. In this case the classical approach based on quadratic Lyapunov functions may result overly conservative. It is shown in this paper that a more effective methodology is the one based on polyhedral Lyapunov functions arguments. Indeed, the main result is a sufficient condition for finite-time stability obtained by using this class of functions. Some numerical examples illustrate the effectiveness of the proposed approach.

References

[1] *Optimization Toolbox 3, User's Guide*. The Mathworks, Inc., Natick, MA, 2007.

[2] F. Amato and M. Ariola. Finite-time control of discrete-time linear systems. *IEEE Trans. on Aut. Contr.*, 50:724–729, 2005.

[3] F. Amato, M. Ariola, and P. Dorato. Finite-time control of linear systems subject to parametric uncertainities and disturbances. *Automatica*, 37:1459–1463, 2001.

[4] C. B. Barber, D. P. Dobkin, and H. T. Huhdanpaa. The Quickhull Algorithm for Convex Hulls. *ACM Transactions on Mathematical Software*, 22:469–483, 1996.

[5] F. Blanchini. Nonquadratic Lyapunov functions for robust control. *Automatica*, 31(3):451–461, March 1995.

[6] P. Dorato. Short time stability in linear time-varying systems. In *Proc. IRE International Convention Record Part 4*, pages 83–87, 1961.

[7] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali. *LMI Control Toolbox*. The Mathworks Inc, Natick, MA, 1995.

[8] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, NJ, 1972.

[9] L. Weiss and E. F. Infante. Finite time stability under perturbing forces and on product spaces. *IEEE Trans. Auto. Contr.*, 12:54–59, 1967.

[10] G. M. Ziegler. *Lectures on Polytopes*. Springer, NY, 1998.