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Coverage control with unicycles via hybrid modeling

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Abstract— **This paper presents a gradient-descent coverage algorithm for a group of nonholonomic vehicles. Similar to previous approaches, the deployment strategy relies on Locational Optimization techniques to assign Voronoi regions to vehicles. The algorithm is distributed in the sense of the Delaunay graph. In order to deal with unicycle dynamics and guarantee performance, we introduce several vehicle modes; i.e., forward movement, rotation in place and resting. The analysis of the algorithm relies on a recently introduced invariance principle for hybrid systems.**

I. Introduction

Mobile sensor networks and multi-vehicle systems hold the promise to impact a wide variety of application areas; see [1]. The ability to autonomously deploy across a spatial region, as well as dynamically adjust to single-point failures gives mobile networks an advantage over static ones. This leads to the question of how to design effective motion coordination algorithms for their unsupervised control [2]. In a first approximation and due to the complexity that systems interacting over networks possess, it is reasonable to consider simple dynamical models for each vehicle. However, many current unmanned systems have non-trivial dynamics, which can invalidate the performance of the proposed algorithms. This work tries to contribute to this aspect by proposing a motion coordination strategy for the deployment of a nonholonomic mobile sensor network.

Although each robotic agent in a network may be controllable and the interaction among them can even be fixed, the consideration of non-trivial vehicle dynamics needs special treatment to avoid destabilizing effects. This has motivated a large number of papers on the design of coordination algorithms for multi-agent systems with fixed interaction topologies; see e.g., [3], [4], [5], [6] on formation stabilization and synchronization. In particular, the stability analysis of this class of algorithms can be approached via Lyapunov methods and the classical LaSalle invariance principle as from [7]. On the other hand, when the inter-vehicle interaction topology is not fixed, even the consideration of first-order integrator dynamics may require hybrid-systems or switched-systems techniques for analysis.

The most complex situation is one where vehicles' dynamics are not first order and the group interaction topology is not fixed. A possible approach to deal with this situation is to stabilize the emerging behavior of the network by introducing several vehicle dynamical modes. This makes unavoidable the use of hybrid system theory to design an analyze the multi-vehicle system; posing a significant difficulty.

The use of multiple Lyapunov functions has been a predominant method for proving stability of a hybrid system, see [8], [9] and references therein. When dealing with multi-agent systems, however, much of our previous work, [10], [11], relied on LaSalle's invariance principle instead. The work of Lygeros et al. [12] provides an extension of LaSalle's invariance principle to hybrid systems. More recently, the work of Sanfelice, Goebel, and Teel [13], [14] revisits the notion of hybrid (time) trajectories and develops a LaSalle invariance principle based on graphical convergence of set-valued maps. In this paper we choose the latter framework to present and analyze our system.

With respect to previous work, this paper contributes to current research on the control of nonholonomic vehicle networks. References include; e.g., obstacle avoidance [15], cyclic pursuit [16], [17], and path-planning for Dubins vehicles [18]. Here, we address a problem posed in an earlier work [10] regarding convergence of a coverage control problem using unicycle type dynamics. In [10] convergence to these configurations was proved for omni-directional vehicles. Wheeled vehicles were also considered, but the control algorithm was designed so that vehicles converged to a fixed target point as in [19], which was updated at discrete-time intervals. Here we lift this simplification allowing for target points (which depend on neighboring vehicles' positions) to vary continuously with time. This paper also presents an application of the results in [13] and how these can be useful in the context of multi-vehicle motion coordination.

The paper is organized as follows. In Section II, we review topics from Locational Optimization as well as introduce the nonholonomic vehicle dynamics that we consider. In Section III, we provide a brief review of the work in [14] and [13] to describe the hybrid model framework. We then apply those results to our system of wheeled vehicles in Section IV and verify that our network of vehicles works within this framework. In Section V we prove convergence to centroidal configurations. We also provide simulations of this hybrid system and show that the algorithms perform as intended. Finally Section VI presents future lines of research regarding hybrid systems and nonholonomic vehicles.

II. Problem setup and notation

In this section, we introduce the basic notation of the paper and some background on Locational Optimization theory pertinent to our application. For additional topics regarding Locational Optimization, see [20]. We conclude the section by describing the unicycle-type vehicle dynamics that we consider throughout the paper.

In the following we denote by $\mathbb{R}_{\geq 0}$ be the set of non-negative real numbers, and $\mathbb{N}$ will be the set of non-negative integers. Let $\mathrm{SE}_X(2)$ be the special Euclidean group for a given set $X \subseteq \mathbb{R}^2$. That is, $(x, y, \theta) \in \mathrm{SE}_X(2)$ describes the position and orientation of a vehicle with respect to a fixed global coordinate frame, with $(x, y) \in X$.

Given a set $S \subset \mathbb{R}^d$, we denote by $\mathrm{Int}\, S$ and by $\partial(S)$ the interior and the boundary of S, respectively. We also use S^c to denote the complementary set of S, $S^c = \mathbb{R}^d \setminus S$.

Let Q_0 be a convex polygon in $\mathbb{R}^2$ including its interior, and let $v \cdot w$ denote the inner product between $v, w \in \mathbb{R}^2$. Although we define Q_0 to be a convex polygon, for the sake of having a unique well-defined normal along the boundary ∂Q_0 we will replace the vertexes of Q_0 with an arc of radius ϵ, where ϵ is arbitrarily small. Let Q denote the approximated Q_0. These "rounded corners" guarantee continuity of the following functions, facilitating the analysis later on. Let $t \colon \partial Q \to \mathbb{R}^2$ be the unit positively oriented tangent vector along the boundary of Q, and $t(x) = (t_1(x), t_2(x))$. We define the normal vector to be a function, $n \colon Q \to \mathbb{R}^2$ as:

$$n(x) = \begin{cases} (-t_2(x), t_1(x)) & , x \in \partial Q \\ 0 & , x \in \mathrm{Int}(Q) \end{cases}$$

Note that the resulting vector points towards the interior of Q for all $x \in \partial Q$.

A. Locational Optimization

Let $\phi : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be a scalar field with bounded support Q. Here, ϕ represents an *a priori* measure of information that help distinguish areas of Q which are more important than others (in other words, the higher the value of $\phi(q)$ the more attention the group has to pay to q). Let $P = (p_1, \dots, p_n)$ be the *location of n sensors*, each moving in the space Q. We will consider the Locational Optimization [20] function:

$$\mathcal{H}(P) = \int_Q \min_{i \in \{1,\dots,n\}} \|q - p_i\|^2 \phi(q) dq \,. \qquad (1)$$

as a measure of how poor the coverage provided by the mobile sensing network in Q is. Smaller $\mathcal{H}$ has the interpretation of better coverage, thus we are interested in minimizing it.

By introducing Voronoi partitions as in [11], the gradient of the cost function may be computed in a distributed fashion. This enables individual agents of the network to compute a control law based only on knowledge of their Voronoi neighbors. The ordinary Voronoi partition of $\mathbb{R}^2$ is $\mathcal{V} = (V_1, \dots, V_n)$ where for all $i \in \{1, \dots, n\}$,

$$V_i = \{q \in \mathbb{R}^2 \mid \|q - p_i\| \leq \|q - p_j\|, \forall\, i \neq j\} \,.$$

Each Voronoi region has mass M_{V_i} and centroid C_{V_i}, where

$$M_{V_i} = \int_{V_i} \phi(q) dq \,, \qquad C_{V_i} = \frac{1}{M_{V_i}} \int_{V_i} q\phi(q) dq \,.$$

Using these partitions, the cost function can be rewritten as

$$\mathcal{H}(P) = \sum_{i=1}^{n} \int_{V_i} \|q - p_i\|^2 \phi(q) dq \,. \qquad (2)$$

The use of omni-directional vehicles in [10] allows the minimization of (2) via a Lloyd-like gradient descent control law. Under this control law, individual agents move directly towards the centroid of their Voronoi regions. The control law is also distributed in the sense of the Delaunay graph. That is, an agent only requires position knowledge of Voronoi neighbors to compute its own Voronoi region, and the corresponding centroid. We will develop an analogously distributed control law for nonholonomic vehicles. Although it may not be possible for a nonholonomic vehicle to move directly towards the centroid, we will introduce several dynamical modes to guarantee that vehicles still propel toward these centroidal configurations.

B. Vehicle dynamics

Here, we present the different dynamical modes under which vehicles in the network can evolve, and the intuition behind them. This will be made more formal in Section IV.

Referencing Figure 1, each vehicle has configuration variables $(\theta_i, p_i) \in \mathrm{SE}_Q(2)$, and a body coordinate frame with basis $e_{x_i} = (\cos\theta_i, \sin\theta_i)$ and $e_{y_i} = (-\sin\theta_i, \cos\theta_i)$. We define the angle $\alpha_i \in [-\pi, \pi]$ to be the angle between e_{x_i} and a target point, in this case the region centroid C_{V_i}. As it will be clear later, in order to decrease $\mathcal{H}$, we will require $e_{x_i} \cdot (C_{V_i} - p_i) \geq 0$. For notational reasons, we denote $d_i = (C_{V_i} - p_i)$ as in Figure 1.

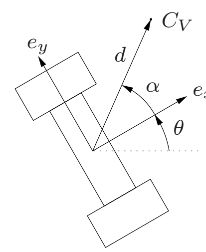

Fig. 1. Vehicle with wheeled mobile dynamics.

In forward motion, each vehicle flows according to

$$\dot{\theta}_i = \omega \,, \quad \dot{p}_i^1 = v\cos\theta_i \,, \quad \dot{p}_i^2 = v\sin\theta_i \,, \qquad (3)$$

where (ω, v) are the control inputs. Note that the definition of (θ_i, v) is unique up to the discrete action $(\theta_i, v) \mapsto (\theta_i + \pi, -v)$. A possibility is to use this symmetry to require $e_{x_i} \cdot d_i \geq 0$ for all $i \in \{1, \dots, n\}$. Should the equality be violated at some time $t = t_0$, i.e., the centroid is behind the vehicle, then we could redefine $\theta_i(t^+) = \theta_i(t_0) + \pi$. The vehicle would instantaneously reverse directions leading to a fast adjustment. However, to provide a technical proof of correctness, we modify these proposed dynamics in many cases by a rotation in place.

A rotation in place introduces the new set of dynamics:

$$\dot{\theta}_i = \omega \,, \quad \dot{p}_i = 0 \,, \qquad (4)$$

where ω is the only control input.

In order to stabilize the vehicle to the target C_{V_i}, we will employ a discontinuous stabilizing law similar to that of [19]. This law relies on the angle α to both stabilize the position and orientation of the unicycle; see Figure 1. However, as the vehicle approaches C_V, the angle α will become ill-defined. To avoid this problem, we will make vehicles switch their dynamics to rest when they are within an ϵ-neighborhood of their targets. That is, the dynamics will be:

$$\dot{\theta}_i = \dot{p}_i = 0\,. \tag{5}$$

In the following, an additional discrete variable, $l \in \{1,2,3\}$, will be used to describe which of the three modes (forward, rotation, and rest) a vehicle is in. Each agent can then be described by a state variable, $x_i \in \{1,2,3\} \times \mathrm{SE}_Q(2)$. We will denote the state of the group of n vehicles by $x = (x_1, \ldots, x_n) \in \mathbb{R}^{4n}$.

III. Hybrid automata

Here we gather useful results on the modeling and the stability analysis of hybrid automata. The exposition is taken from [14], [13] and we include it here for completeness.

Definition 3.1 (Hybrid time domain): A subset $D \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if

$$D = \bigcup_{j=0}^{J-1} \left([t_j, t_{j+1}], j\right)\,,$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \cdots \leq t_J$. It is a hybrid time domain if for all $(T, J) \in D$, $D \cap ([0,T] \times \{0,1,\ldots,J\})$ is a compact hybrid domain.
These hybrid time domains can be ordered, and we say that $(t_i, j_i) \preceq (t_{i+1}, j_{i+1})$ iff $t_i + j_i \leq t_{i+1} + j_{i+1}$, $j \in \{1, \ldots, J\}$.

Definition 3.2 (Generalized solution): A generalized solution is a function $x(t,j) \in O$ defined on a hybrid time domain $\operatorname{dom} x$ such that:

1) on each interval $[t_j, t_{j+1}] \times \{j\} \subset \operatorname{dom} x$ of positive length (so that $t_j < t_{j+1}$) we have

$$\dot{x}(t,j) \in F(x(t,j))\,, \quad x(t,j) \in A\,,$$

2) for each $(t,j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$, we have

$$x(t,j+1) \in G(x(t,j))\,, \quad x(t,j) \in B\,.$$

The set-valued maps $F : O \rightrightarrows \mathbb{R}^n$ and $G : O \rightrightarrows \mathbb{R}^n$ are the flow map and jump map, respectively. The sets $A \subset O$ and $B \subset O$ denote where the state may flow in continuous time, and where the state may make a discontinuous jump, respectively. It is possible for $A \cap B \neq \emptyset$, and in this case, both flowing and jumping may occur. Together, F, G, A, B define a hybrid system, $\mathcal{S} = (F, G, A, B)$.

Definition 3.3 (Complete): A solution $x : \operatorname{dom} x \to \mathbb{R}^n$ to a hybrid system $\mathcal{S}$ is complete if $\operatorname{dom} x$ is unbounded.
As an example, a solution that is defined for all time, $(t,j) \in [0,\infty) \times \{0,1,\ldots,J\}$ is complete. In addition a solution that exhibits an infinite number of switches in a finite time, $(t,j) \in [0,T] \times \mathbb{N}$ is also complete.

Definition 3.4 (Precompact): A solution $x : \operatorname{dom} x \to \mathbb{R}^n$ to a hybrid system $\mathcal{S}$ is precompact if it is complete and the closure of the range of x, $\overline{\operatorname{rge} x} \subset O$.

Definition 3.5 (Locally bounded): A set-valued map $M : O \rightrightarrows \mathbb{R}^n$ is locally bounded if for all $x \in O$, there exists a neighborhood U of x such that $M(U)$ is a bounded set.

Definition 3.6 (Outer semicontinuous): A set-valued map $M : O \rightrightarrows \mathbb{R}^n$ is outer semicontinuous (osc) at x if for each convergent sequence $x_k \to x$, and $y_k \to y$ such that $y_k \in M(x_k)$, then $y \in M(x)$. Equivalently, if

$$\limsup_{z \to x} M(z) \subset M(x)\,.$$

Definition 3.7 (Weakly invariance): For a hybrid system $\mathcal{S}$, the set $\mathcal{M} \subset O$ is said to be:

(i) *weakly forward invariant* (with respect to $\mathcal{S}$) if for each $x_0 \in \mathcal{M}$ there exists at least one complete solution x with $x(t,j) \in \mathcal{M}$ for all $(t,j) \in \operatorname{dom} x$;

(ii) *weakly backward invariant* (with respect to $\mathcal{S}$) if for each $q \in \mathcal{M}, N > 0$ there exists $x_0 \in \mathcal{M}$ and at least one solution x such that for some $(t^*, j^*) \in \operatorname{dom} x$, $t^* + j^* > N, x(t^*, j^*) = q$ and $x(t,j) \in \mathcal{M}$ for all $(t,j) \preceq (t^*, j^*), (t,j) \in \operatorname{dom} x$;

(iii) *weakly invariant* (with respect to $\mathcal{S}$) if it is both weakly forward invariant and weakly backward invariant.

Assumption 3.8 (Basic Conditions): In addition, a hybrid system $\mathcal{S} = (F, G, A, B)$ on a state space $O \subset \mathbb{R}^n$ satisfy the following basic conditions:

(i) $O \subset \mathbb{R}^n$ is an open set,

(ii) A and B are relative closed sets in O,

(iii) F is outer semicontinuous, locally bounded on O, and convex for all $x \in A$,

(iv) G is outer semicontinuous, locally bounded on O, and satisfies $G(x) \subset O$ for all $x \in B$.

We now state the result from [13], (Corollary 4.3).

Theorem 3.9 (Hybrid LaSalle invariance principle): Given a hybrid system $\mathcal{S} = (F, G, A, B)$ that satisfies the Basic Conditions of Assumption 3.8, suppose that:

- $V : O \to \mathbb{R}$ is continuous on O and locally Lipschitz on a neighborhood of A,
- $\mathcal{U} \subset O$ is nonempty,
- $u_A(x) = \max_{f \in F(x)} \mathcal{L}_f V(x) \leq 0$ for all $x \in A$,
- $u_B(x) = \max_{x^+ \in G(x)} \{V(x^+) - V(x)\} \leq 0$ for all $x \in B$.

Let x be precompact with $\overline{\operatorname{rge} x} \subset \mathcal{U}$. Then, for some constant $r \in V(\mathcal{U})$, x approaches the largest weakly invariant set in $V^{-1}(r) \cap \mathcal{U} \cap \left(\overline{u_A^{-1}(0)} \cup u_B^{-1}(0)\right)$.

IV. Hybrid modeling of unicycle network

Here we formally define the hybrid system sketched in Section II-B, so that it satisfies the Basic Conditions of Assumption 3.8 and then Theorem 3.9 becomes applicable.

To begin, we would like to specify some $O \subset \mathbb{R}^{4n}$ for which $(\{1,2,3\} \times \mathrm{SE}_Q(2))^n \subset O$. To satisfy condition (i) of Assumption 3.8 O must be open. Then we can just define $O = \mathbb{R}^{4n}$ so that $x \in O$ and O is open.

We now define the hybrid system that models the nonholonomic vehicles, $\mathcal{S} = (F, G, A, B)$. In Section II-B, we described three different types of dynamics. Here we specify the set $A \subseteq O$ where these dynamics may occur such that A is relatively closed in O to satisfy (ii) of Assumption 3.8. To begin, we examine the configurations when a particular agent can flow, A_i. The set A_i is composed of three disjoint subsets, one for each defined flow (3), (4), and (5), respectively:

$$
\begin{aligned}
A_i^1 &= \{x \in O \mid x_i \in \{1\} \times \mathrm{SE}_Q(2), e_{x_i} \cdot d_i \geq \underline{\epsilon}, \\
&\qquad \|d_i\| \geq \epsilon\}\,, \\
A_i^2 &= \{x \in O \mid x_i \in \{2\} \times \mathrm{SE}_Q(2), \|d_i\| \geq \epsilon, \\
&\qquad e_{x_i} \cdot d_i \leq \epsilon\} \\
&\quad \cup \{x \in O \mid x_i \in \{2\} \times \mathrm{SE}_{\partial Q}(2), \|d_i\| \geq \epsilon, \\
&\qquad e_{x_i} \cdot n \leq 0\}\,, \\
A_i^3 &= \{x \in O \mid x_i \in \{3\} \times \mathrm{SE}_Q(2), \|d_i\| \leq \epsilon\}\,,
\end{aligned}
$$

where $0 < \underline{\epsilon} < \epsilon$ and ϵ is arbitrarily small. Therefore, each agent can flow if it is in the disjoint union of sets $A_i = A_i^1 \cup A_i^2 \cup A_i^3$. Since A is the set of all configurations where continuous flow occurs, we have that,

$$
A = \bigcap_{i=1}^{n} A_i\,. \tag{6}
$$

Since each $A_i^k \in A_i$ is relatively closed in O, A is also relatively closed in O.

Under normal, forward motion, consider the following control law for an individual agent:

$$
F_i^1(x) = \begin{cases} \dot{l}_i = 0\,, \\ \dot{\theta}_i = k_{\theta_i} \alpha_i\,, \\ \dot{p}_i = \begin{pmatrix} k_{p_i} \cos\theta \\ k_{p_i} \sin\theta \end{pmatrix}\,. \end{cases}
$$

for each $i \in \{1, \ldots, n\}$, where $0 \leq k_{\theta_i}, k_{p_i} < \infty$. Under the evolution of F_i^1, it may happen that a vehicle at $p_i \in \partial Q$ moves outside of Q or the centroid C_{V_i} moves behind the agent. This cannot be allowed, so we will implement the following control law for these situations:

$$
F_i^2(x) = \begin{cases} \dot{l}_i = 0\,, \\ \dot{\theta}_i = k_{\theta_i} \operatorname{sgn}(\alpha_i)\,, \\ \dot{p}_i = 0\,. \end{cases}
$$

Note that $F_i^2(x)$ describes a rotation that will make $|\alpha_i|$ decrease. To resolve the ill-defined equations of motion for sensors near their Voronoi region centroids, we introduce:

$$
F_i^3(x) = \begin{cases} \dot{l}_i = 0\,, \\ \dot{\theta}_i = 0\,, \\ \dot{p}_i = 0\,. \end{cases}
$$

From the three cases, we can define the flow map $F : O \rightrightarrows O$. When $x \notin A$, $F(x) = \emptyset$, and when $x \in A$,

$$
\begin{aligned}
F(x) &= \begin{pmatrix} F_1(x) \\ \vdots \\ F_n(x) \end{pmatrix}\,, \\
F_i(x) &= \begin{cases} F_i^1(x) & \text{if } l_i = 1\,, \\ F_i^2(x) & \text{if } l_i = 2\,, \\ F_i^3(x) & \text{if } l_i = 3\,. \end{cases}
\end{aligned} \tag{7}
$$

We now study the cases where transitions from flowing to jumping may occur. These cases are:

1) switching direction of travel,
2) forward motion to rotation (when the centroid is almost, perpendicular to the direction of travel or when the agent is on the boundary),
3) rotation to forward motion,
4) forward motion or rotation to resting near a centroid,
5) resting to forward motion.

We examine the set of configurations, B_i, $i \in \{1, \ldots, n\}$, where such jumps can occur for a particular agent. If we let:

$$
\begin{aligned}
B_i^1 &= \{x \in O \mid x_i \in \{1\} \times \mathrm{SE}_Q(2), e_{x_i} \cdot d_i \leq -\epsilon\}\,, \\
B_i^2 &= \{x \in O \mid x_i \in \{1\} \times \mathrm{SE}_{\partial Q}(2), e_{x_i} \cdot n \leq 0\}, \\
&\quad \cup \{x \in O \mid x_i \in \{1\} \times \mathrm{SE}_Q(2), -\epsilon \leq e_{x_i} \cdot d_i \leq \underline{\epsilon}\}\,, \\
B_i^3 &= \{x \in O \mid x_i \in \{2\} \times \mathrm{SE}_Q(2), e_{x_i} \cdot d_i \geq \epsilon, \\
&\qquad e_{x_i} \cdot n \geq 0\}\,, \\
B_i^4 &= \{x \in O \mid x_i \in \{1, 2\} \times \mathrm{SE}_Q(2), \|d_i\| \leq \epsilon\}\,, \\
B_i^5 &= \{x \in O \mid x_i \in \{3\} \times \mathrm{SE}_Q(2), \|d_i\| \geq \overline{\epsilon}\}\,,
\end{aligned}
$$

where $\overline{\epsilon} > \epsilon$ and $\overline{\epsilon}$ is arbitrarily small. Then $B_i = \bigcup_{k=1}^{5} B_i^k$ and the set B of configurations where jumping can occur is

$$
B = \bigcup_{i=1}^{n} B_i\,. \tag{8}
$$

Observe again that each B_i^k is relatively closed in O, and so B is also relatively closed in O, satisfying (ii) of 3.8. A jump can occur if the state is in any of the five regions for a given i. The corresponding set of configurations, $G_i(x)$, where x might jump to are:

$$
\begin{aligned}
G_i(x) &= \{(x_1, \ldots, g_i^k(x), \ldots, x_n) \mid x \in B_i^k, \\
&\qquad \forall\, k \in \{1, \ldots, 5\}\}\,, \\
g_i^1(x) &= (1, \theta_i + \pi, p_i)\,, \\
g_i^2(x) &= (2, \theta_i, p_i)\,, \\
g_i^3(x) &= (1, \theta_i, p_i)\,, \\
g_i^4(x) &= (3, \theta_i, p_i)\,, \\
g_i^5(x) &= (1, \theta_i, p_i)\,.
\end{aligned}
$$

The overall jump map $G : O \rightrightarrows O$ is

$$
G(x) = \bigcup_{i=1}^{n} G_i(x)\,, \tag{9}
$$

when $x \in B$, otherwise $G(x) = \emptyset$.

Remark 4.1: The jump map G takes the state $x(t, j) \in B_i^k$ to another set, $x(t, j+1) \in A \cup B$. For example:

1) If $k = 1$ then $G(x) \in A_i^1$,
2) If $k = 2$ then $G(x) \in A_i^2 \cup B_i^4$,
3) If $k = 3$ then $G(x) \in A_i^1 \cup B_i^4$,
4) If $k = 4$ then $G(x) \in A_i^3$,
5) If $k = 5$ then $G(x) \in A_i^1 \cup B_i^1$.

The state may also be in more than one jump set, such as $x \in B_i^2 \cup B_i^4$. When this happens, the state may jump according to $g_i^2(x)$ or $g_i^4(x)$, making this process non-deterministic. •

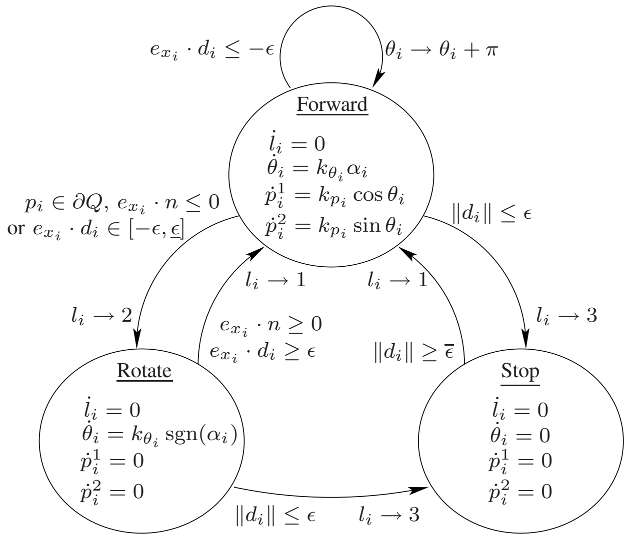

Fig. 2. State transition diagram for each vehicle in the network.

Remark 4.2: If we only implemented direction flipping, there exists a trajectory such that when $e_{x_i} \cdot d_i = 0$, the hybrid time domain (t, j) grows unbounded in j for fixed t. We include $\underline{\epsilon}, \epsilon, \overline{\epsilon}$, and the careful definition of A and B to prevent this and other similar situations. Our choice of A, B is not unique, and other possibilities exist. •

Proposition 4.3: The hybrid system defined in equations (6), (7), (8), (9) satisfies the basic conditions of Assmption 3.8.

Proof: We omit the proof for brevity, the complete analysis can be found at [21]. ■

V. Asymptotic stability

Our system satisfies the Basic Conditions, so we can apply the hybrid LaSalle invariance principle in Theorem 3.9. We now state our main result.

Theorem 5.1: Let $\mathcal{U} = O$. Given the hybrid system defined in equations (6), (7), (8), (9), any precompact trajectory $x(t, j)$, with $\mathrm{rge}\, x \in \mathcal{U}$, will approach the set of points

$$\mathcal{M} = \{x \in O \mid x \in A_i^3, \forall i \in \{1, \ldots, n\}\}. \tag{10}$$

Proof: We choose V to be the cost function (2). It can be shown that (2) is locally Lipschitz on O [10].

For all x in A, $u_A(x) = \mathcal{L}_F\mathcal{H}$. We now compute the derivative, see [10] for the complete derivation.

$$\mathcal{L}_F\mathcal{H} = \sum_{i=1}^{n}\left[\frac{\partial\mathcal{H}}{\partial p_i}\dot{p}_i + \frac{\partial\mathcal{H}}{\partial\theta_i}\dot{\theta}_i + \frac{\partial\mathcal{H}}{\partial l_i}\dot{l}_i\right] = \sum_{i=1}^{n}\left[\frac{\partial\mathcal{H}}{\partial p_i}\dot{p}_i\right]$$
$$= \sum_{i=1}^{n}\left[2M_{V_i}(p_i - C_{V_i})^T\dot{p}_i\right].$$

Note that when an agent is in a rotating or rest mode, $x_i \in A_i^2 \cup A_i^3$ and $\dot{p}_i = 0$. When $x_i \in A_i^1$, we have

$$\frac{\partial\mathcal{H}}{\partial p_i}\dot{p}_i = \left[2M_{V_i}(p_i - C_{V_i})^T\begin{pmatrix}k_{p_i}\cos\theta \\ k_{p_i}\sin\theta\end{pmatrix}\right]$$
$$= \left[2k_{p_i}M_{V_i}(p_i - C_{V_i}) \cdot e_{x_i}\right].$$

Recall from the definition of A_i^1 that $e_{x_i} \cdot (C_{V_i} - p_i) \geq \underline{\epsilon}$, then $\frac{\partial\mathcal{H}}{\partial p_i}\dot{p}_i < 0$. Therefore, for all $x \in A$, $u_A(x) \leq 0$.

Since G is set-valued, $u_B(x) = \max_{x^+ \in G(x)}\{\mathcal{H}(x^+) - \mathcal{H}(x)\}$. The cost function (2), does not have any dependence on l_i or θ_i. In addition, the jump map (9) does not create discontinuities in position. Therefore, the cost function does not change in value over jumps, $u_B(x) = 0$ for all $x \in B$.

All conditions of the hybrid LaSalle invariance principle have been satisfied. The precompact trajectories x will approach the largest weakly invariant set in

$$L = V^{-1}(r) \cap \mathcal{U} \cap \left(\overline{u_A^{-1}(0)} \cup u_B^{-1}(0)\right)$$
$$= \mathcal{H}^{-1}(r) \cap \left(\overline{u_A^{-1}(0)} \cup B\right),$$

for some $r \in \mathcal{H}(\mathcal{U})$. Note that $\mathcal{H}^{-1}(r)$ represents some level set of the cost function (2). Now we must identify the largest weakly invariant set, $\mathcal{M}$ in L. Since our system is autonomous, the largest weakly forwards invariant set is also the largest weakly invariant set.

We now check for weakly invariant trajectories. We do this by assuming that one vehicle is in a switching state, and show that it must switch to a flowing state, and remain there for a non-zero amount of time. Then we show that the only flowing state which remains in a level set for all time is the stationary state, $x \in A_i^3$ for all $i \in \{1, \ldots, n\}$.

Suppose there exists a trajectory $\tilde{x}(t, j)$ with $\mathcal{H}(\tilde{x}) = r$ for all $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ such that $\tilde{x}(t_0, j_0) \in B$. This implies that there exists i^* and k^* such that $\tilde{x}(t_0, j_0) \in B_{i^*}^{k^*}$. We consider the case $\tilde{x}(t_0, j_0) \in B_{i^*}^{k^*} \setminus A$ and $\tilde{x}(t_0, j_0) \in A_j \setminus B$ for all $j \neq i^*$. If there are multiple vehicles in a switching state, it will eventually happen that they all will have switched. From this trajectory $\tilde{x}$, we have the following implications:

1) If $k^* = 1$ then a switch must occur since $B_i^1 \cap A = \emptyset$. This switch results in $x(t_0, j_0 + 1) \in A_i^1 \setminus B$.
2) If $k^* = 2$ then the system jumps and $x(t_0, j_0 + 1) \in A_i^2 \cup B_i^4$.
3) If $k^* = 3$ then a jump is forced and $x(t_1, j_0 + 1) \in A_i^1 \cup B_i^4$.
4) If $k^* = 4$ then again a jump is forced and $x(t_0, j_0 + 1) \in A_i^3$.
5) If $k^* = 5$ then the system jumps according to $g_i^5(x)$. This results in $x(t_0, j_0 + 1) \in A_i^1 \cup B_i^1$.

In the cases where $k_i^* \in \{1, 4\}$, $G(x) \notin B$. For $k_i^* \in \{2, 3, 5\}$, if it happens that $G(x) \in B$, note that $G(G(x)) \notin B$. Therefore, there exists $t_k \geq t_0$ and $j_k > j_0$ such that $\tilde{x}(t_k, j_k) \notin B$, and the system must evolve under F. We have shown that all configurations $x \in B$ return to flowing states. Now we examine the case where $\tilde{x}(t, j) \in A$ to arrive at the final result.

Suppose there exists a trajectory $\tilde{x}(t, j)$ with $\mathcal{H}(\tilde{x}) = r$ for all $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$ and $\tilde{x}(t_0, j_0) \in A$ for some $t_0 + j_0 \geq 0$. Since $\frac{\partial\mathcal{H}}{\partial p_i}\dot{p}_i < 0$ for any $x \in A_i^1$, this implies that $\tilde{x}(t_0, j_0) \in A_i^2 \cup A_i^3$ for all $i \in \{1, \ldots, n\}$. If this is true, then $\dot{p}_i = 0$ for all $i \in \{1, \ldots, n\}$. Suppose there exists an i^* such that $\tilde{x}(t_0, j_0) \in A_{i^*}^2$. Because $\dot{p}_i = 0$, C_{V_i} is constant for all $i \in \{1, \ldots, n\}$, and under the flow F_i^2, $|\alpha_{i^*}|$ decreases. Then,

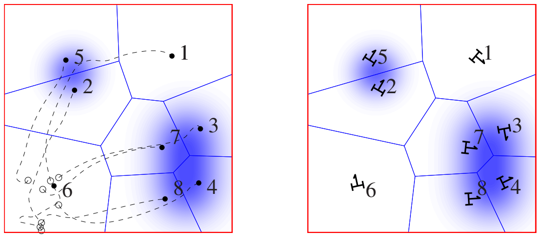

Fig. 3. Wheeled vehicle deployment simulation. The agents start in the lower left corner, denoted by the 'o'. Path lines are shown in the left figure, and final positions and orientations in the right figure.

for some t_1, such that $t_0 \leq t_1 < \infty$, $\tilde{x}(t_1, j_0) \in B^3_{i*}$ where a jumped is forced such that $\tilde{x}(t_1, j_0 + 1) \in A^1_{i*}$. This implies that $u_A(\tilde{x}) < 0$, and the trajectory $\tilde{x}(t, j)$ leaves the level set $\mathcal{H}^{-1}(r)$.

Therefore, in order to remain in the level set $\mathcal{H}^{-1}(r)$, trajectories $x(t, j)$ must satisfy $x \in A^3_i$ for all $i \in \{1, \ldots, n\}$. This also satisfies $x \in \overline{u_A^{-1}(0)}$. ∎

A. Simulations

Here we provide a simulation result for a network of wheeled vehicles. We simulate $n = 8$ unicycles in $Q \subset \mathbb{R}^2 = [0, 10] \times [0, 10]$. The density function, ϕ, is composed of 3 Gaussian distributions (see Figure 3),

$$\phi(q) = 0.05 + 3\left[e^{-\frac{\|q-r_1\|^2}{2}} + e^{-\frac{\|q-r_2\|^2}{2}} + e^{-\|q-r_3\|^2}\right],$$

where $r_1 = (8, 2)$, $r_2 = (8, 4)$ and $r_3 = (3, 7)$. The agent positions and orientations were randomly distributed in the bottom left corner, $l_i = 1$ for all $i \in \{1, \ldots, n\}$. We chose the control gains to be $k_{\theta_i} = 5$ and $k_{p_i} = \text{sat}\,\|C_{V_i} - p_i\|$. Note that any positive k_{θ_i} and k_{p_i} will work. Figure 3 shows that the wheeled vehicles do in fact converge to near-centroidal configurations.

VI. Conclusions

We have introduced a system of wheeled vehicles with unicycle dynamics undergoing deployment. We have also shown that the nonholonomic vehicles converge to centroidal configurations via a hybrid invariance principle. We plan to develop similar deployment algorithms for different nonholonomic dynamics, such as UAVs, where a minimum forward velocity must be maintained and rotation in place is not possible. Additional vehicle modes may also be introduced to handle obstacle or collision avoidance as well.

Even though we performed our analysis using regular Voronoi regions, a very similar convergence result should apply for different spatial tessellations. Other partitions, such as the range-limited partitions from [11] or the power-aware partitions from [22], should not affect the convergence of wheeled vehicles.

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