2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Global stochastic synchronization of chaotic oscillators

Maurizio Porfiri, Francesca Fiorilli
Department of Mechanical Engineering
Polytechnic University
Brooklyn, NY 11201, USA
{mporfiri@, ffiori01@students.}poly.edu

***Abstract*—We study synchronization of two chaotic oscillators in a master-slave configuration. The two dynamic systems are coupled via a directed feedback that randomly switches among a finite set of given constant functions at a prescribed time rate. We use stochastic Lyapunov stability theory and partial averaging techniques to show that global synchronization is possible if the switching period is sufficiently small and if the two systems globally exponentially synchronize under an average feedback coupling. The approach is applied to the synchronization of two Chua's circuits.**

***Index Terms*—Global synchronization, master-slave synchronization, stochastic synchronization, chaos, Chua's circuit, exponential synchronization**

I. Introduction

Chaos synchronization is a topic of great interest, due to its observation in a huge variety of phenomena of different nature. In many biological systems, synchronization plays an important role in self-organization of organisms' groups [1]. Examples of synchronization include communication among fireflies [2], locomotion of animals [3], molecular and cellular activity [4], and cardiac stimulation [5]. The study of neural activity [6] is a correlated issue as well. Other examples and applications can also be found in meteorology [7], chemistry [4], and optics [8]. Many reviews on chaos synchronization are currently available [9], [10], [11], [12], [13].

In the literature, different paradigms have been proposed to employ synchronization of two or more chaotic oscillators. We mention, among the others, peer-to-peer coupling [14], [15], [16], back-stepping [17], generalized synchronization [18], phase synchronization [10], and master-slave synchronization [19], [20], [21]. In this work, we focus on master-slave synchronization. In this case, one system acts as a "master" by driving the other system that behave consequently as a "slave".

Most of the research efforts on master-slave synchronization focus on time invariant coupling, see [19], [20], [22] among the others. Nevertheless, experimental and numerical evidences show that synchronization can also be achieved using intermittent time-varying master-slave coupling [23], [24], [25]. In [23], experimental results on synchronization of two periodically coupled chaotic circuits are presented. In [24], the slave system is driven by a sequence of samples of the master's state (*impulsive synchronization*). In [25], the signal transmission from the master to the slave system is adaptively controlled (*selective synchronization*).

The main goal of this work is to establish sufficient conditions for global synchronization of master-slave coupled chaotic systems with time-varying coupling. We focus on the general case where the intermittent coupling changes randomly over time. Intermittent coupling is made possible through a switching function, that changes randomly over time, assuming values among a finite set of constant functions. The synchronization problem is transformed into a nonlinear stochastic stability problem, and it is studied using partial averaging techniques [26], [27], [28], nonlinear system theory [29], and stochastic stability theory [30], [31]. We associate to the stochastic system a partially averaged system characterized by a constant coupling. This auxiliary system can be studied using well-established and manageable Lyapunov based techniques as those presented in [20]. Under suitable regularity conditions, we show that if the partially averaged system is globally exponentially stable and the switching period is sufficiently small, the stochastic system globally synchronizes.

The type of intermittent coupling considered in this paper has been also analyzed in the framework of consensus theory [32], [33]. We note that, in consensus theory, the individual systems' dynamics is linear while in the present case the coupled systems are strongly nonlinear. Partial averaging techniques have been used in the synchronization literature by [15], [16]. Both these efforts deal with peer-to-peer coupling and only local asymptotic synchronization results based on linearized dynamics are presented. In this paper, the inherent non linear nature of the coupled systems is retained and global results are presented.

The system studied in this paper finds many practical applications. For example, in communication and signal processing, chaotic behavior can be used for message encryption and secure communication [34], [35]. Higher communication efficiency can potentially be achieved through sporadic transmission of the driving signal. This is particularly useful when the access to a communication network is limited to assigned time-slots. Furthermore, in many biological systems, interactions happen only sporadically and randomly [36] of neurons and fireflies.

We organize the paper as follows. In Section II, we define the master-slave synchronization problem in the case of a stochastic linear feedback. In Section III, we present a few general results on stability of nonlinear stochastic systems. In Section IV, we apply these results to the synchronization problem. As a sample case, in Section V we consider the case of two stochastically coupled Chua's circuits. Section VI is left for conclusions.

978-1-4244-2079-7/08/$25.00 ©2008 AACC.

II. Problem Statement

We consider the master system

$$\dot{x}(t) = Ax(t) + g(x(t)) + u(t) \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^n$ is the input vector, $A \in \mathbb{R}^{n\times n}$ is a constant matrix, g is a non linear function, n is a positive integer, and $t \in \mathbb{R}^+$ indicates the time variable. We construct a slave system for (1)

$$\dot{\widetilde{x}}(t) = A\widetilde{x}(t) + g(\widetilde{x}(t)) + u(t) + K(t)(x(t) - \widetilde{x}(t)) \quad (2)$$

System (2) is unidirectionally coupled to the master system (1) through the feedback matrix function $K(t)$. We consider the case where $K(t)$ is a piecewise constant signal that in every time interval $[\sigma_k, \sigma_{k+1})$, with $k \in \mathbb{Z}^+$ and $\sigma_0 = 0$, equals the discrete random variable $\mathfrak{K}_k$. We assume that the random variables $\mathfrak{K}_k$ are finite-state, independent, identically distributed, and take values in the finite set $\{K_1, K_2, ..., K_N\}$, with $N \in \mathbb{Z}^+$. We further assume that switching occurs at equally spaced time instants σ_k, with $|\sigma_{k+1} - \sigma_k| = \varepsilon$, where $\varepsilon > 0$ is a fixed time duration.

Following [20], we assume that $g(x) - g(\widetilde{x}) = M_{x,\widetilde{x}}(x - \widetilde{x})$ for some bounded matrix $M_{x,\widetilde{x}}$, whose elements depend on x and $\widetilde{x}$. As discussed in [20], this condition applies to a large variety of chaotic systems.

We express the system of equations (1) and (2) in terms of the error function $e = x - \widetilde{x}$

$$\dot{e}(t) = (A - K(t))e(t) + M_{x(t),x(t)-e(t)}e(t) \quad (3)$$

Equation (3) can be compactly rewritten as

$$\dot{e}(t) = y(e(t), t) - K^\diamond(t/\varepsilon)e(t) \quad (4)$$

where $y(e(t), t) = (A - M_{x(t),x(t)-e(t)})e(t)$ and $K^\diamond(t/\varepsilon) = K(t)$. We note that the matrix function $K^\diamond$ switches at a unit rate. Equation (4) shows that two different time scales are involved in the problem: a fast time scale t/ε representing the switching process and a slow time scale t describing the chaotic dynamics. By considering the error function e, the synchronization problem reduces to the stability analysis of the stochastic and nonautonomous nonlinear system in equation (3).

We associate to the stochastic system (4) a partially averaged deterministic system whose synchronizability can be assessed through well-established Lyapunov stability techniques [20]. We show that if the switching rate is sufficiently fast and the Lyapunov function of the deterministic system is sufficiently regular, the stability properties of the partially averaged system are inherited by the stochastic system.

III. Preliminary results on global stability through fast switching

We consider the integral equation in $\mathbb{R}^n$

$$x(t) = x(\sigma_k) + \int_{\sigma_k}^{t} f(x(\xi), \xi, \Omega)d\xi \quad (5)$$

where $t \in [\sigma_k, \sigma_{k+1})$, $\sigma_0 = 0$, $|\sigma_{k+1} - \sigma_k| = \varepsilon$, and $k \in \mathbb{Z}^+$. The function f is defined in $\mathbb{R}^n \times \mathbb{R}^+ \times \Theta$. Here, Ω is a finite-state random variable taking values in $\Theta = \{\omega_1, ..., \omega_N\}$, with $N \in \mathbb{Z}^+$. We assume that $f(0, t, \omega_j) = 0$, $\forall t \in \mathbb{R}^+$, $j = 1, ..., N$, and that for every $\omega \in \Theta$ the function $f_\omega(x, t) = f(x, t, \omega)$ is globally Lipschitz in $\mathbb{R}^+$, with Lipschitz constant $L_{\omega,\varepsilon}$. We further require that $L_{\omega,\varepsilon} < L$, where L is a constant independent of ω and ε. We note that (5) describes a Markovian nonlinear, nonhomogeneous jump system [37]. We look for a solution of (5) for $t \geq t_0 \in \mathbb{R}^+$ and initial condition $x(t_0) = x_0$. In the sequel, we use $\mathsf{E}[\bullet]$ to indicate expectation and we denote probability with $P\{\bullet\}$.

In this section, we establish sufficient conditions for global stability of the stochastic system (5). First we recall the definition of global mean square exponential stability, [38].

Lemma 1: The system (5) is globally mean square exponentially stable if there exist $\alpha \geq 0$ and $\beta > 0$ such that for any $t_0 \in \mathbb{R}^+$, and any $x_0 \in \mathbb{R}^n$

$$\mathsf{E}[\|x(t)\|^2] \leq \alpha \|x_0\|^2 e^{-\beta(t-t_0)} \quad (6)$$

From classical Lyapunov stability theory, it is well known that a deterministic dynamical system is asymptotically stable if there exists a Lyapunov function whose time derivative along the solutions of the system is negative definite [29]. In [39], this condition is relaxed and it is shown that if the Lyapunov function decreases when evaluated at a discrete sequence of time instants, the system is asymptotically stable. In this case, the time derivative of the Lyapunov function can assume positive and negative values. The following Theorem extends the results of [39] from the deterministic to the stochastic case and it is used in what follows to establish our main claim.

Theorem 1: Consider the system (5) and suppose that there exists a function $V : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ such that $\forall (x, t) \in \mathbb{R}^n \times \mathbb{R}^+$

$$\lambda_{min}\|x\|^2 \leq V(x, t) \leq \lambda_{max}\|x\|^2 \quad (7)$$

with λ_{min} and λ_{max} positive nonzero real constants. Assume also that there exists ν, with $0 < \nu \leq 1$, such that

$$\mathsf{E}[V(x(\sigma_{k+1}), \sigma_{k+1})|x(\sigma_k)] - V(x(\sigma_k), \sigma_k) \leq -\nu V(x(\sigma_k), \sigma_k) \quad (8)$$

for every $k \in \mathbb{Z}^+$. Then (5) is globally mean square exponentially stable.

Proof: Consider arbitrary initial time $t_0 \in \mathbb{R}^+$ and initial condition $x_0 \in \mathbb{R}^n$. We define the index $\hat{k}$ so that $t_0 \in [\sigma_{\hat{k}-1}, \sigma_{\hat{k}})$. Specifying equation (8) at the $\hat{k}$-th and $(\hat{k}+1)$-th switching instants, we have

$$\mathsf{E}[V(x(\sigma_{\hat{k}+1}), \sigma_{\hat{k}+1})|x(\sigma_{\hat{k}}))] \leq (1 - \nu)V(x(\sigma_{\hat{k}}), \sigma_{\hat{k}}) \quad (9)$$

$$\mathsf{E}[V(x(\sigma_{\hat{k}+2}), \sigma_{\hat{k}+2})|x(\sigma_{\hat{k}+1})] \leq (1 - \nu)V(x(\sigma_{\hat{k}+1}), \sigma_{\hat{k}+1}) \quad (10)$$

By taking the conditional expected value of (10) we obtain

$$\begin{aligned}\mathsf{E}[\mathsf{E}[V(x(\sigma_{\hat{k}+2}),\sigma_{\hat{k}+2})|x(\sigma_{\hat{k}+1})]|x(\sigma_{\hat{k}})] \\ \leq (1-\nu)\mathsf{E}[V(x(\sigma_{\hat{k}+1}),\sigma_{\hat{k}+1})|x(\sigma_{\hat{k}})] \quad (11)\end{aligned}$$

Using the smoothing lemma (see for example [40] Lemma 1.1 p. 474) and substituting (9) in (11), we find

$$\mathsf{E}[V(x(\sigma_{\hat{k}+2}),\sigma_{\hat{k}+2})|x(\sigma_{\hat{k}})] \leq (1-\nu)^2 V(x(\sigma_{\hat{k}}),\sigma_{\hat{k}}) \quad (12)$$

Recalling that equation (5) describes a Markov process and iterating the argument above for any $n > \hat{k}$, we obtain

$$\mathsf{E}[V(x(\sigma_n),\sigma_n)|x(\sigma_{\hat{k}})] \leq (1-\nu)^{n-\hat{k}} V(x(\sigma_{\hat{k}}),\sigma_{\hat{k}}) \quad (13)$$

By using the bounds in (7), equation (13) gives

$$\mathsf{E}[\|x(\sigma_n)\|^2|x(\sigma_{\hat{k}})] \leq \lambda_{max}/\lambda_{min}(1-\nu)^{n-\hat{k}}\|x(\sigma_{\hat{k}})\|^2 \quad (14)$$

Equation (14) can be used to derive an upper bound for the unconditioned expected value, that is needed to assess the global mean square exponential stability according to Definition 1. Since $\hat{k}$ is a given instant of time and x_0 is a prescribed initial condition, $x(\sigma_{\hat{k}})$ is a finite-state random variable taking values in $\{x_1(\sigma_{\hat{k}}),...,x_N(\sigma_{\hat{k}})\}$. From the definition of conditional expectation, see for example [41], we have

$$\mathsf{E}[\|x(\sigma_n)\|^2] = \sum_{i=1}^{N} \mathsf{E}[\|x(\sigma_n)\|^2|x_i(\sigma_{\hat{k}})]P\{x_i(\sigma_{\hat{k}})\} \quad (15)$$

where $P\{x_i(\sigma_{\hat{k}})\}$ is the probability that $x_i(\sigma_{\hat{k}})$ is the realization of the random variable $x(\sigma_{\hat{k}})$. Hence, using inequality (14), equation (15) yields

$$\mathsf{E}[\|x(\sigma_n)\|^2] \leq \sum_{i=1}^{N} \frac{\lambda_{max}}{\lambda_{min}}(1-\nu)^{n-\hat{k}}\|x_i(\sigma_{\hat{k}})\|^2 P\{x_i(\sigma_{\hat{k}})\} \quad (16)$$

In order to assess global mean square exponential stability we need to analyze the system dynamics inside every switching interval. Given a generic switching interval $[\sigma_k, \sigma_{k+1})$ and an instant $\bar{t} \in [\sigma_k, \sigma_{k+1})$, using the triangle inequality, $\forall t \geq \bar{t}$ in $[\sigma_k, \sigma_{k+1})$, equation (5) yields

$$\|x(t)\| \leq \|x(\bar{t})\| + \int_{\bar{t}}^{t} \|f(x(\xi),\xi,\Omega)\|d\xi \quad (17)$$

Since the functions f_ω are globally Lipschitz in $\mathbb{R}^+$ and all the corresponding Lipschitz constants are bounded by a constant L, equation (17) yields

$$\|x(t)\| \leq \|x(\bar{t})\| + \int_{\bar{t}}^{t} L\|x(\xi)\|d\xi \quad (18)$$

Using the Gronwall-Bellman inequality, see for example [29], we have

$$\|x(t)\| \leq \gamma\|x(\bar{t})\| \quad (19)$$

with $\gamma = e^{L\varepsilon}$. Therefore, using (19) in (16), we find that $\forall t \in [\sigma_n, \sigma_{n+1})$

$$\mathsf{E}[\|x(t)\|^2] \leq \gamma \sum_{i=1}^{N} \frac{\lambda_{max}}{\lambda_{min}}(1-\nu)^{n-\hat{k}}\|x_i(\sigma_{\hat{k}})\|^2 P\{x_i(\sigma_{\hat{k}})\} \quad (20)$$

Inequality (19) can also be used to find an upper bound for $\|x(\sigma_{\hat{k}})\|$ in terms of the initial conditions. In fact, since $t_0 \in [\sigma_{\hat{k}-1}, \sigma_{\hat{k}})$ according to the definition of $\hat{k}$, (19) holds for $\bar{t} = t_0$ and $t \geq t_0$. Since $\sigma_{\hat{k}} \geq t_0$, we obtain

$$\|x(\sigma_{\hat{k}})\| \leq \gamma^2\|x(t_0)\| \quad (21)$$

Finally, using (21) to bound the right side of (20), we obtain

$$\mathsf{E}[\|x(t)\|^2] \leq \alpha\|x(t_0)\|^2 e^{-\beta(t-t_0)} \quad (22)$$

where $\alpha = \gamma^4(1-\nu)^{-2}\lambda_{max}/\lambda_{min}$ and $\beta = -ln(1-\nu)/\varepsilon$. Therefore, according to Definition 1, the system (5) is globally mean square exponentially stable. ■

We associate to (5) the partially averaged system

$$\dot{x}(t) = \overline{f}(x(t),t) = \mathsf{E}[f(x(t),t,\Omega)] \quad (23)$$

Equation (23) represents a deterministic, time-varying, non linear system. We notice that $\overline{f}(0,t) = 0$, $\forall t \in \mathbb{R}^+$. If (23) is globally exponentially stable, by the Converse Theorem of Lyapunov (see [29], Theorem 3.12) we know that there exists a Lyapunov function $V(x,t)$, whose time derivative is negative definite along its trajectories. In the following theorem, we show that if $V(x,t)$ satisfies further regularity conditions and the switching period is sufficiently small, the original system (5) is globally mean square exponentially stable.

Theorem 2: Consider the system (5) and the associated partially averaged system (23) and suppose that there exists a Lyapunov function $V(x,t)$ which satisfies the following conditions:

1. $V(0,t) = 0$, $\forall t \in \mathbb{R}^+$ and there exist positive numbers $\lambda_{min}, \lambda_{max}$ such that for every $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$,

$$\lambda_{min}\|x\|^2 \leq V(x,t) \leq \lambda_{max}\|x\|^2 \quad (24)$$

2. there exists a positive number w such that for every $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$

$$\frac{\partial V}{\partial t}(x,t) + \frac{\partial V}{\partial x}(x,t)\overline{f}(x,t) \leq -w\|x\|^2 \quad (25)$$

3. $\forall t \in \mathbb{R}^+$, $\frac{\partial V}{\partial x}(0,t) = 0$ and $\frac{\partial V}{\partial x}(x,t)$ is globally Lipschitz, with Lipschitz constant C_v. Moreover, $\forall t \in \mathbb{R}^+$, $\frac{\partial^2 V}{\partial x \partial t}(0,t) = 0$, and $\frac{\partial^2 V}{\partial x \partial t}(x,t)$ is globally Lipschitz, with Lipschitz constant C_{vt}.

Then, there exists an $\varepsilon^* > 0$ such that $\forall \varepsilon < \varepsilon^*$ system (5) is globally mean square exponentially stable.

Proof: Consider the Lyapunov function $V(x,t)$. Its derivative along the solution of (5) is

$$\dot{V}(x(t),t) = \frac{\partial V}{\partial t}(x(t),t) + \frac{\partial V}{\partial x}(x(t),t)f(x(t),t,\Omega) \quad (26)$$

For every nonnegative integer k, we define

$$\begin{aligned}\Delta V(\sigma_{k+1},\sigma_k) = \mathsf{E}[V(x(\sigma_{k+1}),\sigma_{k+1})|x(\sigma_k)] \\ - V(x(\sigma_k),\sigma_k) \quad (27)\end{aligned}$$

From (5), (26) and (27) we have

$$\Delta V(\sigma_{k+1},\sigma_k) = \mathsf{E}\Big[\int_{\sigma_k}^{\sigma_{k+1}} \dot{V}(x(t),t)dt\Big] = \\ \mathsf{E}\Big[\int_{\sigma_k}^{\sigma_{k+1}} \frac{\partial V}{\partial x}(x(t),t)f(x(t),t,\Omega) \\ - \frac{\partial V}{\partial x}(x(\sigma_k),t)f(x(\sigma_k),t,\Omega)dt\Big] \\ + \mathsf{E}\Big[\int_{\sigma_k}^{\sigma_{k+1}} \frac{\partial V}{\partial t}(x(t),t) - \frac{\partial V}{\partial t}(x(\sigma_k),t)dt\Big] \\ + \mathsf{E}\Big[\int_{\sigma_k}^{\sigma_{k+1}} \frac{\partial V}{\partial t}(x(\sigma_k),t) + \frac{\partial V}{\partial x}(x(\sigma_k),t)f(x(\sigma_k),t,\Omega)dt\Big] \tag{28}$$

We seek an upper bound for the absolute values of the three terms in the summation above. We start our analysis by considering the first and the second terms. Using the Lipschitz conditions on f_ω and on the first and second derivatives of V, considering that for each realization ω of Ω $\sum_{i=1}^{N} P\{\Omega = \omega_i\} = 1$ and $L_{\omega,\varepsilon} < L$ for each ω), and following the arguments of [28] (see parts II and III of the proof of Theorem 2), we find that the absolute value of the first term of the summation (28) is less than or equal to $2L^2 C_v e^{2\varepsilon L}\varepsilon^2 \|x(\sigma_k)\|^2$. Similarly, we find that absolute value of second term is less than or equal to $LC_{vt}e^{2\varepsilon L}\varepsilon^2\|x(\sigma_k)\|^2$.

As a result, we have that (25) provides a bound for the third term in the right side of (28).

Similarly to all the above boundaries we obtain

$$\Delta V(\sigma_{k+1},\sigma_k) \le [g(\varepsilon) - w\epsilon]\|x(\sigma_k)\|^2 \tag{29}$$

where the function $g(\varepsilon)$ is defined by

$$g(\varepsilon) = (2L^2 K_v e^{2\varepsilon L} + LK_{vt}e^{2\varepsilon L})\varepsilon^2 \tag{30}$$

Noticing that $g(0) = 0$ and $g'(0) = 0$, we have that there exists $\varepsilon^* > 0$ such that

$$\Delta V(\sigma_{k+1},\sigma_k) \le -\overline{w}\|x(\sigma_k)\|^2 \tag{31}$$

where $\overline{w} = [w\epsilon - g(\varepsilon)] > 0$, for every $\varepsilon < \varepsilon^*$.
Therefore, if the switching period ε is sufficiently small, (31) and (27) imply that hypotheses of Theorems 1 are satisfied. Thus the claim follows. ■

IV. Stochastic chaos synchronization

In this section, we combine the general findings of Section 3 on stochastic stability of non linear systems with available results on synchronizability of deterministic systems, to provide sufficient conditions for the global synchronization of system (5). In particular, we make use of the results of [20], where a criterion for global exponential synchronization of (1) and (2) is given in the case of constant feedback gain. The error system equation (3), when $K(t)$ equals the constant K^*, becomes

$$\dot{e}(t) = (A - K^*)e(t) + M_{x(t),x(t)-e(t)}e(t) \tag{32}$$

For clarity we restate here the main theorem of [20].

Theorem 3: The system (32) is globally exponentially stable, if the feedback gain matrix K^* is chosen such that

$$l_i(\xi,t) \le -w < 0, \quad i = 1,2,...n \tag{33}$$

for every $\xi \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$, where $l_i(\xi,t)$' s are the eigenvalues of the matrix

$$Q(\xi,t) = (A - K^* + M_{x(t),x(t)-\xi})^T P \\ + P(A - K^* + M_{x(t),x(t)-\xi}) \tag{34}$$

and P is a positive definite symmetric constant matrix. A Lyapunov function for (32) can be constructed as

$$V(e(t)) = e(t)^T Pe(t) \tag{35}$$

with

$$\dot{V}(e(t)) = e(t)^T Q(\xi,t)e(t) \le -w\|e(t)\|^2 < 0 \tag{36}$$

We note that, for $i = 1,...,N$, the function $f(e(t),t,K_i) = (A + M_{x,x-e} - K_i)e(t)$ is globally Lipschitz in $\mathbb{R}^+$ with Lipschitz constant $L_i = \|A\| + m + \|K_i\|$, where $\|M\| \le m$. The Lipschitz constants L_i are bounded by $L = \|A\| + m + \max_{1\le i\le N}\{\|K_i\|\}$. We further notice that $f(0,t,K_i) = 0$, $\forall t \in \mathbb{R}^+$. We associate to the system (3) the partially averaged system

$$\dot{e}(t) = (A + M_{x(t),x(t)-\xi})e(t) - \overline{K}e(t) \tag{37}$$

where $\overline{K} = \mathsf{E}[K(t)] = \sum_{i=1}^{N} p_i K_i$. Here, p_i indicates the probability of $K(t)$ assuming value K_i, that is $p_i = P\{K(t) = K_i\}$.

Since $\overline{K}$ is constant, we can apply Theorem 3 to (37). The Lyapunov function (35) constructed for the partially averaged system, can be used to assess the stability of the stochastic system. In fact $V(0,t) = 0$ and (24) holds for $\lambda_{min} = \min\{\lambda(P)\}$ and $\lambda_{max} = \max\{\lambda(P)\}$, since P is a constant matrix (here, $\lambda(\bullet)$ indicates the spectrum of the matrix). Furthermore, (36) is equivalent to (25) and Condition 3 of Theorem 2 is satisfied with $C_v = 2\|P\|$ and $C_{vt} = 0$. Thus equation (31), specified for the case at hand, reads

$$2L^2 C_v e^{2L\varepsilon}\varepsilon - w = 0 \tag{38}$$

and it yields the sought value of ε^*. By applying Theorem 2, we claim that the system (3) is globally mean square exponentially stable $\forall\varepsilon < \varepsilon^*$. We summarize the above arguments in the following Corollary.

Corollary 1: Consider the system (3) and the corresponding partially averaged system (37). If the feedback gain matrix $K(t)$ is chosen such that

$$\bar{l}_i(\xi,t) \le -w < 0, \quad i = 1,2,...n \tag{39}$$

for every $\xi \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$, where $\bar{l}_i(\xi,t)$' s are the eigenvalues of the matrix

$$Q(\xi,t) = (A - \overline{K} + M_{x(t),x(t)-\xi})^T P + P(A - \overline{K} + M_{x(t),x(t)-\xi}) \quad (40)$$

and P is a positive definite symmetric constant matrix, then there exists an $\varepsilon^* > 0$ such that such that $\forall \varepsilon < \varepsilon^*$ system (3) is mean square globally exponentially stable.

V. Case study: synchronization of two chaotic Chua's circuit

As an example, we apply our results to synchronization of Chua's circuits, see for example [42]. A Chua's circuit is described in terms of vector state $x(t) = [x_1(t)\ x_2(t)\ x_3(t)]^T$ by

$$\dot{x}(t) = Ax(t) + g(x(t)) \quad (41)$$

with

$$A = \begin{bmatrix} -a & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix}, \quad g(x) = \begin{bmatrix} -ah(x_1) \\ 0 \\ 0 \end{bmatrix} \quad (42)$$

$a > 0$, and $b > 0$, and the nonlinear function h equal to $h(x_1) = m_1 x_1 + \frac{1}{2}(m_0 - m_1)\{|x_1 + 1| - |x_1 - 1|\}$, with $m_0 < 0$ and $m_1 < 0$.

We define $h(x_1) - h(\widetilde{x}_1) = w_{x_1,\widetilde{x}_1}(x_1 - \widetilde{x}_1)$, where $w_{x_1,\widetilde{x}_1}$ depends on x_1, and $\widetilde{x}_1$ and is bounded by $m_0 \leq w_{x_1,\widetilde{x}_1} \leq m_1$, see for example [20]. Following (2), the slave system of (41) is constructed

$$\dot{\widetilde{x}}(t) = A\widetilde{x}(t) + g(x(t)) - g(\widetilde{x}(t)) + K(t)(x(t) - \widetilde{x}(t)) \quad (43)$$

We consider the case where $K(t) = diag[k_1(t)\ k_2(t)\ k_3(t)]$ is a diagonal matrix. We observe that $g(x) - g(\widetilde{x}) = M_{x,x-e}e$, where $M_{x,x-e} = \begin{bmatrix} -aw_{x_1,\widetilde{x}_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\|M\| \leq a|m_0|$.

We associate to the system (3) the partially averaged system

$$\dot{e} = Ae + M_{x,x-e}e - \overline{K}e \quad (44)$$

where $\overline{K} = diag[\overline{k}_1\ \overline{k}_2\ \overline{k}_3]$.
By choosing $P = I$, and by setting

$$\begin{aligned} \overline{k}_1 &\geq \frac{1}{2}(1 - a - 2am_0 + w) \\ \overline{k}_2 &\geq \frac{1}{2}(a - 1 + |1 - b| + w) \\ \overline{k}_3 &\geq \frac{1}{2}(|1 - b| + w) \end{aligned} \quad (45)$$

the hypothesis of corollary 1 are satisfied, and there exists ε^* such that (44) is globally mean exponentially stable for any $\varepsilon < \varepsilon^*$. The critical value ε^* can be determined by equation (38) with $C_v = 2$ and $L = \|A\| + a|m_0| + \max_{1 \leq i \leq N}\{\|K_i\|\}$.

Equation (38) gives the value of ε^* that assures the global mean square exponential stability of the stochastic system $\forall \varepsilon < \varepsilon^*$.

Here, we present a few numerical results that illustrate how two stochastically coupled Chua's circuits synchronize for a sufficiently fast switching rate. We select $a = 9.78$, $b = 14.97$, $m_0 = -1.31$, and $m_1 = -0.75$ in order to generate chaotic behavior [20]. We let $K(t)$ switching randomly between the two constant matrices K_1 and K_2, where K_1 is the zero matrix and $K_2 = diag[20\ 27.5\ 20]$. For these parameters we have $\|A\| = 18.8859$, $\max_{1 \leq i \leq 2}\{\|K_i\|\} = \|K_2\| = 27.5$, and $L = 59.1977$. Selecting $p_1 = 0.6$ and $p_2 = 0.4$, w can be chosen from (45) to be equal to 0.5. From equation (38) we have that for $\varepsilon \leq \varepsilon^* = 3.5629 \cdot 10^{-5}$ the system synchronizes globally mean square exponentially. Fig. 1 and Fig. 2 depicts the trajectories of the master and slave systems on the x_1x_2 and x_1x_3 planes for a switching period $\varepsilon = 10^{-5}$.

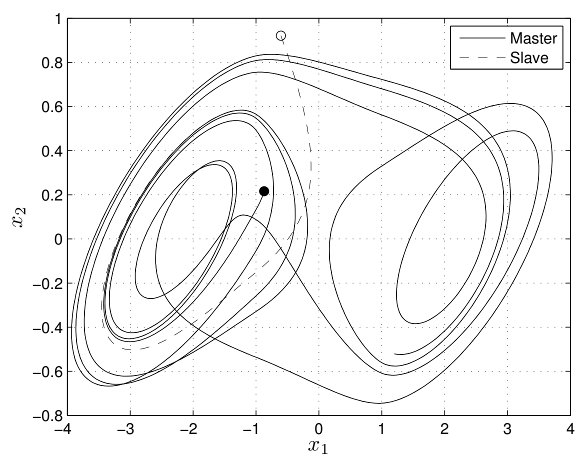

Fig. 1. Trajectories of the master and slave systems in the x_1x_2 plane

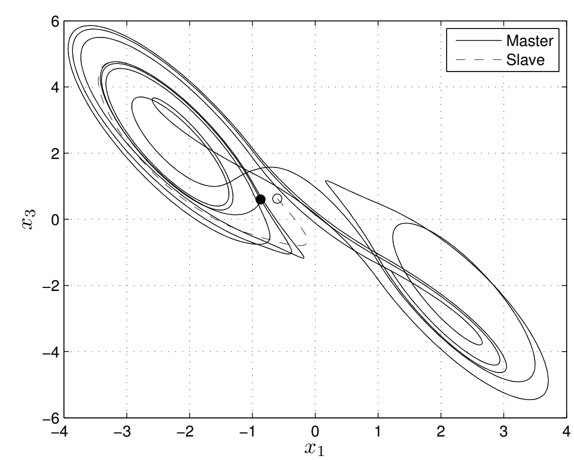

Fig. 2. Trajectories of the master and slave systems in the x_1x_3 plane

VI. Conclusions

In this paper we presented a general criterion for global synchronization of two chaotic oscillators in a master-slave configuration. The two systems are coupled through a stochastic unidirectional feedback, that switches randomly among a finite set of constant values. Using tools based on Lyapunov stability theory and partially averaging techniques we showed that, under suitable regularity conditions, the synchronization characteristics of the partially averaged system are inherited by the stochastic system. Our findings are illustrated through numerical simulations on Chua's circuits.

REFERENCES

[1] S. Camazine, W. Ristine, and M. E. Didion, *Self-Organization in Biological Systems*. Princeton University Press, 2003.

[2] J. Buck and E. Buck, "Synchronous fireflies," *Scientific American*, vol. 234, no. 5, pp. 75–85, 1976.

[3] J. J. Collins and I. Stewart, "Coupled nonlinear oscillators and the symmetries of animal gaits," *Journal of Nonlinear Science*, vol. 3, no. 1, pp. 349–392, 1993.

[4] A. Goldbeter, *Biochemical Oscillations and Cellular Rhythms: The Molecular Bases of Periodic and Chaotic Behaviour*. Cambridge University Press, 1996.

[5] A. Garfinkel, M. Spano, , W. Ditto, and J. Weiss, "Controlling cardiac chaos," *Science*, vol. 257, pp. 1230–1235, 1992.

[6] P. Rapp, T. R. Bashore, J. M. Martinerie, A. M. Albano, I. D. Zimmerman, and M. A. I., "Dynamics of brain electrical activity," *Brain Topography*, vol. 2, no. 1-2, pp. 99–118, 1989.

[7] G. S. Duane, P. J. Webster, and J. B. Weiss, "Go-occurrence of northern and southern hemisphere blocks as partially synchronized chaos," *Journal of Atmospheric sciences*, vol. 56, no. 24, pp. 4183–4205, 1999.

[8] G. D. Van Wiggeren and R. Roy, "Communication with chaotic lasers," *Science*, vol. 279, pp. 1198–1200, 1998.

[9] B. R. Andrievskii and A. L. Fradkov, "Control of chaos: Methods and applications," *Automation and Remote Control*, vol. 64(5), pp. 673–713, 2003.

[10] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. S. Zhou, "The synchronization of chaotic systems," *Physics Reports*, vol. 366, no. 1, pp. 1–101, 2002.

[11] G. Chen and X. Yu, Eds., *Chaos Control Theory and Applications*, ser. Lecture Notes in Control and Information Sciences. Berlin: Springer, 2003, vol. 292.

[12] E. Mosekilde, Y. Maistrenko, and D. Postnov, *Chaotic Synchronization: Applications to Living systems*. New Jersey: World Scientific, 2002.

[13] L. M. Pecora, T. L. Carroll, G. A. Johnson, and D. J. Mar, "Fundamentals of synchronization in chaotic systems, concepts, and applications," *Chaos*, vol. 7, no. 4, pp. 520–543, 1997.

[14] Z. Ge and Y. Chen, "Synchronization of mutual coupled chaotic systems via partial stability theory," *Chaos, Solitons and Fractals*, vol. 34, no. 3, pp. 787–794, 2007.

[15] M. Porfiri, D. J. Stilwell, E. M. Bollt, and J. D. Skufca, "Random talk: Random walk and synchronizability in a moving neighborhood network," *Physica D*, vol. 224, pp. 102–113, 2006.

[16] D. J. Stilwell, E. M. Bollt, and D. G. Roberson, "Sufficient conditions for fast switching synchronization in time varying network topologies," *SIAM Journal on Applied Dynamical Systems*, Accepted for publication.

[17] S. Bowong and F. M. M. Kakmeni, "Synchronization of uncertain chaotic system via backstepping approach," *Chaos Solitons and Fractals*, vol. 21, pp. 999–1011, 2004.

[18] R. Zhang, Z. Xu, S. X. Yang, and X. He, "Generalized synchronization via impulsive control," *Chaos, solitons and Fractals*, vol. Article in Press, 2006.

[19] T. L. Carroll and L. M. Pecora, "Synchronizing chaotic circuits," *IEEE Transaction on Circuits and Systems*, vol. 38, no. 4, pp. 453–456, 1991.

[20] G. P. Jiang, W. K. S. Tang, and G. Chen, "A simple global synchronization criterion for coupled chaotic system," *Chaos, Solitons and Fractals*, vol. 15, no. 5, pp. 925–935, 2003.

[21] A. I. Lerescu, N. Contandache, S. Oancea, and I. Grosu, "Collection of master-slave synchronized chaotic systems," *Chaos, Solitons and Fractals*, vol. 22, no. 3, pp. 599–604, 2004.

[22] L. M. Pecora and T. L. Carroll, "Master stability functions for synchronized coupled systems," *Physical Review Letters*, vol. 80, no. 10, pp. 2109–2112, March 1998.

[23] L. Fortuna, M. Frasca, and A. Rizzo, "Experimental pulse synchronization of two chaotic circuits," *Chaos, Solitons and Fractals*, vol. 17, no. 2-3, pp. 335–361, 2003.

[24] M. Itoh, T. Yang, and L. O. Chua, "Conditions for impulsive synchronization of chaotic and hyperchaotic systems," *International Journal of Bifurcation and Chaos*, vol. 11, pp. 551–560, 2001.

[25] M. Zochowski, "Intermittent dynamical control," *Physica D*, vol. 145, pp. 181–190, 2000.

[26] G. Grammel and I. Maizurna, "Exponential stability and partial averaging," *Journal of Mathematical Analysis and Applications*, vol. 283, pp. 276–286, 2003.

[27] J. K. Hale, *Ordinary Differential Equations*. New York: John Wiley & Sons, 1969.

[28] J. Peuteman and D. Aeyels, "Exponential stability of nonlinear time-varying differential equations and partial averaging," *Mathematics of Control, Signals and System*, vol. 15, no. 1, pp. 42–70, 2002.

[29] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ: Prentice Hall, 2002.

[30] H. Kushner, *Introduction to Stochastic Control*. New York, NY: Holt, Rinehart and Winston, Inc., 1971.

[31] H. J. Kushner, *Stochastic Stability and Control*. New York: Academic Press, 1967.

[32] Y. Hatano and M. Mesbahi, "Agreement over random networks," *IEEE Transactions on Automatic Control*, vol. 50, no. 11, pp. 1867 – 1872, 2005.

[33] M. Porfiri and D. J. Stilwell, "Consensus seeking over random weighted directed graphs," *IEEE Transactions on Automatic Control*, accepted for pubblication.

[34] K. M. Cuomo, V. A. Oppenheim, and S. H. Strogatz, "Synchronization of lorentz-based chaotic circuits with application to communications," *IEEE Transactions on Circuits and Systems II*, vol. 40, no. 10, pp. 626–633, 1993.

[35] B. Jovic, C. P. Unsworth, G. S. Sandhu, and S. M. Berber, "A robust sequence synchronization unit for multi-user ds-cdma chaos-based communication systems," *Signal Processing*, vol. 87, no. 7, pp. 1692–1708, 2007.

[36] I. V. Belykh, V. N. Belykh, and M. Hasler, "Blinking model and synchronization in small-world networks with a time-varying coupling," *Physica D: Nonlinear Phenomena*, vol. 195, no. 1-2, pp. 188–206, 2004.

[37] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, *Discrete-Time Markov Jump Linear systems*. London, UK: Springer, 2005.

[38] X. Feng, K. A. Loparo, Y. Ji, and H. J. Chizeck, "Stochastic stability properties of jump linear systems," *IEEE Transactions on Automatic Control*, vol. 37(1), pp. 38–53, 1992.

[39] D. Aeyels and J. Peuteman, "On exponential stability of nonlinear time-varying differential equations," *Automatica*, vol. 35, no. 6, pp. 1091–1100, 1999.

[40] A. Gut, *Probability: A Graduate Course*. New York, NY: Springer, 2005.

[41] P. Bremaud, *Markov Chains, Gibbs Fields, Monte Carlo Simulation, and Queues*, Springer-Verlag, Ed., 1999.

[42] L. P. Shil'nikov, "Chua's circuit: rigorous results and future problems," *International Journal of Bifurcation and Chaos*, vol. 4, no. 10, pp. 489–519, 1993.