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Fast-switching pulse synchronization of chaotic oscillators

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***Abstract*—We study pulse synchronization of chaotic systems in master-slave configuration. The slave system is unidirectionally coupled to the master system through an intermittent linear error feedback coupling, whose gain matrix periodically switches among a finite set of constant matrices. Using Lyapunov-stability theory, fast-switching techniques, and the concept of matrix measure, we derive sufficient conditions for global synchronization. The derived conditions are specialized to the case of Chua's circuits. An inductorless realization of coupled Chua's circuits is developed to illustrate the effectiveness of the proposed approach.**

***Index Terms*—Global synchronization, fast-switching, Chua's circuits, master-slave synchronization**

I. INTRODUCTION

Synchronization of chaotic oscillators has incurred great interest in recent years, see for example the excellent reviews [1], [2], [3]. Chaos synchronization has been observed in a wide variety of phenomena ranging from biological systems, that include animal groups [4], fireflies [5], animal gaits [6], heart stimulation [7], and neural activity [8], to secure communications [9], [10], [11], [12], [13], [14], chemistry [15], meteorology [16], and nonlinear optics [17].

Synchronization schemes for two identical chaotic oscillators can be classified into bidirectional [18], [19], [20] or unidirectional [21], [22], [23], [24], [25], [26], [27], [28], [29] depending on the coupling between the oscillators [30]. Synchronization of unidirectionally coupled oscillators is generally referred to as master-slave synchronization. In this case, one system acts as a "master" by driving the other system, that consequently behaves as a "slave".

Most of the research efforts on master-slave synchronization focuses on time-invariant coupling, see for example [21], [22], [24], [25], [26], [27], [28], [29]. Nevertheless, experimental results on Chua's circuits [23] and modified Chua's circuits [31] indicate that master-slave synchronization can also be achieved under time-varying intermittent coupling.

In this paper, we establish sufficient conditions for global synchronization of master-slave coupled chaotic systems with time-varying coupling. We consider the case of pulse synchronization experimentally investigated in [23], [31]. That is, we assume that the systems are unidirectionally coupled via a linear error feedback, whose gain matrix periodically switches over time among a finite set of constant gain matrices. We introduce a partially averaged system, see for example [32], that describes master-slave synchronization under a time-invariant coupling, whose constant gain matrix equals the time-average of the switching gain. Using recent results on partial averaging tecniques [33], [34], well-established global synchronization criteria based on Lyapunov-stability theory [25], [29], and the concept of matrix measure [35], we establish sufficient conditions for pulse synchronization. In particular, we provide easily manageable conditions on the time-average gain matrix, and on the switching period for global synchronization.

In order to illustrate the proposed approach, we specialize our results to the synchronization of Chua's circuits. Theoretical results are validated through experiments conducted on coupled Chua's circuits. Each circuit is developed using the inductorless synthesis proposed in [36], that combines the Chua's diode realization of [37] with the Antoniou RC active realization of a grounded inductance [38]. An RC active circuit implementation of a full-state linear error feedback, involving both voltage and current states, is presented.

The rest of the paper is organized as follows. In Section II, we present our results on fast-switching global synchronization of chaotic oscillators. In Section III, we propose a hardware demonstration based on Chua's circuits. Section IV is left for conclusions.

Our notation throughout is standard. $\|\cdot\|$ refers to a norm in $\mathbb{R}^n$ and the corresponding induced norm in $\mathbb{R}^{n\times n}$, where n is a positive integer. $\mathbb{Z}^+$ refers to the set of nonnegative integers, also I_n is the $n \times n$ identity matrix. Matrix transposition is indicated with superscript T. The symmetric part of a matrix $A \in \mathbb{R}^{n\times n}$ with $n \in \mathbb{Z}^+$ is indicated with $\mathrm{sym}A$, that is $\mathrm{sym}A = \frac{1}{2}(A + A^{\mathrm{T}})$. The one sided directional derivative of $\|\cdot\|$ at the identity matrix I_n in the direction $A \in \mathbb{R}^{n\times n}$ is called the matrix measure of A and denoted by $\mu(A)$. That is,

$$\mu(A) = \lim_{h\to 0^+} \frac{\|I + hA\| - 1}{h} \tag{1}$$

Basic properties of the matrix measure for $A, B \in \mathbb{R}^{n\times n}$ are [35]

$$\mu(A + B) \le \mu(A) + \mu(B) \tag{2a}$$
$$\mu(A) \le \|A\| \tag{2b}$$

In the case of the Euclidean norm $\|\cdot\|_2$, the matrix measure is computed by

$$\mu_2(A) = \max_{i=1,\ldots,n} \lambda_i(\mathrm{sym}A) \tag{3}$$

where $\lambda_i(A)$ indicates the i^{th} eigenvalue of A.

II. Pulse Synchronization of Chaotic Systems

A. Problem Statement

We consider the master system

$$\dot{x}(t) = Ax(t) + g(x(t)) + u(t) \tag{4}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^n$ is the input vector, $A \in \mathbb{R}^{n\times n}$ is a constant matrix, g is a non linear function, and $t \in \mathbb{R}^+$ the time variable. We construct a slave system for (4)

$$\dot{\widetilde{x}}(t) = A\widetilde{x}(t) + g(\widetilde{x}(t)) + u(t) + K(t)(x(t) - \widetilde{x}(t)) \tag{5}$$

Slave system (5) is unidirectionally coupled to the master system (4) through the feedback matrix function $K : \mathbb{R}^+ \rightarrow \mathbb{R}^{n\times n}$. We consider the case where $K(t)$ is a piecewise constant bounded matrix function. At any point in time, $K(t)$ takes values from the set $\Theta = \{K_1, K_2, \ldots, K_N\}$ where $N \in \mathbb{Z}^+$. Over the interval $[0, T)$,

$$K(t) = K_r \text{ for } t \in \left[T\sum_{i=1}^{r-1}\delta_i, T\sum_{i=1}^{r}\delta_i\right) \tag{6}$$

where δ_i is the duty cycle of the subsystem corresponding to the matrix K_i, so that $0 < \delta_i < 1$ and $\sum_{i=1}^{N}\delta_i = 1$.

Following [25], we assume that

$$g(x) - g(\widetilde{x}) = M_{x,\widetilde{x}}(x - \widetilde{x}) \tag{7}$$

for some bounded matrix $M_{x,\widetilde{x}}$, whose elements depend on x and $\widetilde{x}$. As discussed in [25], this condition applies to a large variety of chaotic systems.

We express the system of equations (4) and (5) in terms of the error function $e = x - \widetilde{x}$, that is

$$\dot{e}(t) = Ae(t) + g(x(t)) - g(\widetilde{x}(t)) - K(t)e(t) \tag{8}$$

Using (7), equation (8) can be compactly rewritten as

$$\dot{e}(t) = f(e(t), t, t/T) \tag{9a}$$

$$f(e(t), t, t/T) = (A + M_{x(t),x(t)-e(t)})e(t) - K(t)e(t) \tag{9b}$$

Equation (9a) shows that two different time scales are involved in the problem: a fast time scale t/T representing the switching process and a slow time scale t describing the chaotic dynamics. We say that the oscillators globally synchronize if the nonautonomous nonlinear system described by (8) is globally asymptotically stable, see for example [39].

We note that, since $M_{x,\widetilde{x}}$ is bounded and Θ is finite, $f(e(t), t, t/T)$ is globally Lipschitz in $\mathbb{R}^+$, that is

$$\|f(x, t, t/T) - f(y, t, t/T)\| \leq L\|x - y\| \tag{10}$$

for any $x, y \in \mathbb{R}^n$ and any $t \in \mathbb{R}^+$, and with Lipshitz constant $L > 0$. Along with the Lipshitz constant L we define the following bound for the matrix measure of $A + M_{x(t),\widetilde{x}(t)} - K(t)$

$$\mu(A + M_{x,\widetilde{x}} - K) \leq \mathfrak{L} \tag{11}$$

that holds for any $x, \widetilde{x} \in \mathbb{R}^n$ and $K \in \Theta$. Unlike the Lipshitz constant L, the constant $\mathfrak{L}$ can be positive or negative. Following the same line of argument of the proof of Theorem 1 of [29], we can use the constant $\mathfrak{L}$ to quantify the rate of growth of the error $e(t)$ as

$$\|e(t)\| \leq \exp(\mathfrak{L}(t - t_0))\|e(t_0)\| \tag{12}$$

for any $t_0 \in \mathbb{R}^+$ and $t > t_0$. We also note that using the triangle inequality and the properties of the matrix measure recalled in (2), we can choose $L = L_*$ and $\mathfrak{L} = \mathfrak{L}_* \leq L_*$ given by

$$L_* = m + \max_{1\leq i\leq N}\{\|A - K_i\|\} \tag{13a}$$

$$\mathfrak{L}_* = m + \max_{1\leq i\leq N}\{\mu(A - K_i)\} \tag{13b}$$

where $m > 0$ is such that $\|M_{x,\widetilde{x}}\| \leq m$.

Using (12), a simple criterion for global synchronization can be stated.

Theorem 1: Consider the nonlinear nonautonomous switched system (9a). If the feedback gain matrix $K(t)$ is chosen such that

$$\mathfrak{L} \leq -\kappa \tag{14}$$

for any $t \in \mathbb{R}^+$ and some $\kappa > 0$, then (9a) is globally exponentially stable, implying that the oscillators globally synchronize.

Proof: If condition (14) holds, then equation (12) yields directly

$$\|e(t)\| \leq \exp(-\kappa(t - t_0))\|e(t_0)\| \tag{15}$$

which implies the claim since $\kappa > 0$. ■

B. Fast-Switching Global Synchronization

We associate to the nonlinear nonautonomous system (9a) the partially averaged system

$$\dot{e}(t) = \overline{f}(e(t), t, t/T) \tag{16a}$$

$$\overline{f}(e(t), t, t/T) = (A + M_{x(t),x(t)-e(t)})e(t) - \overline{K}e(t) \tag{16b}$$

where $\overline{K} = \sum_{i=1}^{N}\delta_i K_i$ represents the time-average gain matrix.

In what follows, we show that pulse synchronization of coupled oscillators can be assessed through the analysis of the partially averaged system (16a). In particular, we show that if the time-average gain matrix is chosen such that the partially averaged system (16a) is globally exponentially stable with a monotonically decaying quadratic Lyapunov function, then the oscillators globally synchronize under fast-switching intermittent coupling. This means that if the switching period T is sufficiently small, compared to the individual oscillator's time dynamics, then global pulse synchronization is achieved.

The proof of our claim combines results from global synchronization of coupled oscillators, based on Lyapunov-stability theory [25], [29], with stability results from partial averaging techniques [33], [34]. In particular, we build on the use of a quadratic Lyapunov function for chaotic systems coupled via a time-invariant coupling [25], [29] and on the stability results established in [33], [34] that are valid for more general Lyapunov functions.

Theorem 2: Consider the nonautonomous nonlinear switched system (9a) and the corresponding partially averaged system (16a). If the feedback gain matrix function $K(t)$ is chosen such that $\forall \xi \in \mathbb{R}^n$

$$\bar{l}_i(\xi, t) \leq -w < 0, \quad i = 1, 2, ...n \tag{17}$$

where $w > 0$, $\bar{l}_i(\xi, t)$' s are the eigenvalues of the matrix

$$Q(\xi, t) = (A - \overline{K} + M_{x(t),x(t)-\xi})^{\mathrm{T}} P + P(A - \overline{K} + M_{x(t),x(t)-\xi}) \tag{18}$$

$x(t)$ is a solution of (4), and P is a positive definite symmetric constant matrix, then the partially averaged system (16a) is globally exponentially stable. Moreover there exists a $T^* > 0$ such that such that $\forall T < T^*$ (9a) is also globally exponentially stable, implying that the oscillators globally synchronize.

Proof: Define the quadratic function

$$V(\xi) = \xi^{\mathrm{T}} P \xi \tag{19}$$

The derivative of V along the trajectory of the partially averaged system (16a) is

$$\dot{V}(e(t)) = 2e(t)^{\mathrm{T}} P \bar{f}(e(t), t) \tag{20}$$

From equations (16b) and (18) we have that $\forall \xi \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$

$$2\xi^{\mathrm{T}} P \bar{f}(\xi, t) = \xi^{\mathrm{T}} Q(\xi, t) \xi \leq -w\|\xi\|^2 \tag{21}$$

We note that from equations (17) and (21), V is a Lyapunov function for the partially averaged system whose derivative along the flow of (16a) is negative definite. Thus, the partially averaged system (16a) is globally exponentially stable [39], and the two oscillators globally synchronize under the time constant coupling matrix $\overline{K}$.

In general, the function V is not a Lyapunov function for the switched system (9a) even under fast-switching conditions. Nevertheless, global exponential stability of (9a) can be enforced by using the weaker stability conditions presented in [33]. Indeed Theorem 1 of [33], applied to the case at hand, states that if there exists $\nu > 0$ such that for any $k \in \mathbb{Z}^+$, $V(e((k+1)T)) - V(e(kT)) \leq -\nu\|e(kT)\|^2$, where $e((k+1)T)$ is the solution of (9a) with initial condition $\mathrm{e}(kT)$ at $t = kT$, then (9a) is globally exponentially stable. In what follows, we show that there exists a finite switching period T^* such that conditions of Theorem 1 of [33] apply for any $T < T^*$. To this aim, we define for every $k \in \mathbb{Z}^+$

$$\Delta_k V = V(e((k+1)T)) - V(e(kT)) \tag{22}$$

The derivative of V along the trajectory of the switched system is

$$\dot{V}(e(t)) = \frac{dV}{de}(e(t)) f(e(t), t, t/T) = 2e^{\mathrm{T}}(t) P f(e(t), t, t/T) \tag{23}$$

From (9a), (23), and (22), we have

$$\begin{aligned}\Delta_k V = \int_{kT}^{(k+1)T} \dot{V}(e(t)) dt = \\ 2\int_{kT}^{(k+1)T} e^{\mathrm{T}}(t) P f(e(t), t, t/T) dt = \\ 2\Big[\int_{kT}^{(k+1)T} e^{\mathrm{T}}(t) P f(e(t), t, t/T) - \\ e^{\mathrm{T}}(kT) P f(e(kT), t, t/T) dt\Big] + \\ 2e^{\mathrm{T}}(kT) P \Big[\int_{kT}^{(k+1)T} f(e(kT), t, t/T) dt\Big]\end{aligned} \tag{24}$$

We seek an upper bound for the absolute values of the two terms in the right hand side of (24). We start our analysis by considering the first term, that we rewrite for convenience as

$$\int_{kT}^{(k+1)T} 2e^{\mathrm{T}}(t) P (f(e(t), t, t/T) - f(e(kT), t, t/T)) + 2(e^{\mathrm{T}}(t) - e^{\mathrm{T}}(kT)) P f(e(kT), t, t/T) dt \tag{25}$$

Using (12) we have

$$\|e(t)\| \leq \|e(kT)\| \exp(\mathfrak{L}T) \tag{26}$$

for any $t \in [kT, (k+1)T)$. In addition, we note that for any $t \in [kT, (k+1)T)$

$$e(t) = e(kT) + \int_{kT}^{t} f(e(s), s, s/T) ds \tag{27}$$

Using the Lipshitz condition (10) in (27), we obtain

$$\|e(t) - e(kT)\| \leq \int_{kT}^{t} L\|e(s)\| ds \tag{28}$$

Accounting for (26) in (28), we find

$$\|e(t) - e(kT)\| \leq LT\|e(kT)\| \exp(\mathfrak{L}T) \tag{29}$$

Using (10), (26), and (29) in (25), we find

$$\Big| \int_{kT}^{(k+1)T} 2e^{\mathrm{T}}(t) P f(e(t), t, t/T) - 2e^{\mathrm{T}}(kT) P f(e(kT), t, t/T) dt \Big| \leq g(T)\|e(kT)\|^2 \tag{30}$$

where we added and subtracted in equation (30) the function $g(T)$ is given by

$$g(T) = 2T^2 L^2 \|P\| \exp(\mathfrak{L}T)(1 + \exp(\mathfrak{L}T)) \tag{31}$$

Now, we consider the second term in the right side of (24). First, we note that from the definition of the partially averaged system and the fact that $e(kT)$ is not a function of time, see equations (9b) and (16b), we have

$$\int_{kT}^{(k+1)T} f(e(kT), t, t/T) dt = \int_{kT}^{(k+1)T} \overline{f}(e(kT), t) dt \tag{32}$$

Using (21), the second term in (24) can be bounded by

$$2e^{\mathrm{T}}(kT)P\int_{kT}^{(k+1)T} f(e(kT),t,t/T)dt \le -wT\|e(kT)\|^2 \tag{33}$$

Using (30) and (33) in (24), we obtain

$$\Delta_k V \le (g(T) - wT)\|e(kT)\|^2 \tag{34}$$

Noticing that $g(0) = 0$ and $g'(0) = 0$, we find that there exists $T^* > 0$ such that

$$\Delta_k V \le -\nu\|e(kT)\|^2 \tag{35}$$

where $\nu = [wT - g(T)] > 0$, for every $T < T^*$. This implies that (9a) is globally exponentially stable according to Theorem 1 of [33]. ■

Choosing $P = I_n$ and making use of the constants defined in (13), we obtain the following following specialization of Theorem 2.

Corollary 1: Consider the nonautonomous nonlinear switched system (9a) and the corresponding partially averaged system (16a). If the feedback gain matrix $K(t)$ is chosen such that its time average is a diagonal matrix $\overline{K} = \mathrm{diag}[\overline{k}_1, \cdots \overline{k}_n]$ and

$$\overline{k}_i \ge \frac{1}{2}\left(b_{ii} + \sum_{j\ne i}|b_{ij}| + w\right), \quad i = 1, 2, ...n \tag{36}$$

where $B = [b_{ij}]$ is defined by

$$B = 2\mathrm{sym}(A + M_{x(t),x(t)-e(t)}) \tag{37}$$

with $w > 0$. Then, (9a) is globally exponentially stable for any $T < T^*$, where T^* is the smallest nonzero solution of

$$2TL_*^2\exp(\mathfrak{L}_*T)(1+\exp(\mathfrak{L}_*T)) - w = 0 \tag{38}$$

and L_* and $\mathfrak{L}_*$ are given in (13).

Proof: By applying Gerschgorin's theorem [40], it can be shown that conditions (17) follow from (36). By applying Theorem 2, and by noticing that $\|P\|_2 = 1$ in (31), the claim follows. ■

We note that the matrix B in (37) is a function of $x(t)$ and $e(t)$.

III. Case Study: Synchronization of Two Chaotic Chua's Circuit

A. Governing Equations

As an example, we specialize our results to synchronization of Chua's circuits, see for example [41]. A Chua's circuit is described by

$$\begin{cases} \dot{x}_1 &= a(x_2 - x_1 - h(x_1)) \\ \dot{x}_2 &= x_1 - x_2 + x_3 \\ \dot{x}_3 &= -bx_2 \end{cases} \tag{39}$$

where $a > 0$, $b > 0$ and the nonlinear function h has the form

$$h(x_1) = m_1x_1 + \frac{1}{2}(m_0 - m_1)(|x_1+1| - |x_1-1|) \tag{40}$$

with $m_0 < 0$ and $m_1 < 0$. We define

$$h(x_1) - h(\widetilde{x}_1) = w_{x_1,\widetilde{x}_1}(x_1 - \widetilde{x}_1) \tag{41}$$

where $w_{x_1,\widetilde{x}_1}$ depends on x_1 and $\widetilde{x}_1$ and is bounded by $m_0 \le w_{x_1,\widetilde{x}_1} \le m_1$, see for example [25]. We consider the case where the feedback matrices in Θ are all diagonal. Following (5), the slave system of (39) below is constructed

$$\begin{cases} \dot{\widetilde{x}}_1 &= a(\widetilde{x}_2 - \widetilde{x}_1 - h(\widetilde{x}_1)) + k_1(t)(x_1 - \widetilde{x}_1) \\ \dot{\widetilde{x}}_2 &= \widetilde{x}_1 - \widetilde{x}_2 + \widetilde{x}_3 + k_2(t)(x_2 - \widetilde{x}_2) \\ \dot{\widetilde{x}}_3 &= -b\widetilde{x}_2 + k_3(t)(x_3 - \widetilde{x}_3) \end{cases} \tag{42}$$

Combining (39) and (42), we obtain the error dynamics (8), with

$$\begin{aligned} A &= \begin{bmatrix} -a & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix}, \quad g(x) = \begin{bmatrix} -ah(x_1) \\ 0 \\ 0 \end{bmatrix} \\ K(t) &= \begin{bmatrix} k_1(t) & 0 & 0 \\ 0 & k_2(t) & 0 \\ 0 & 0 & k_3(t) \end{bmatrix} \end{aligned} \tag{43}$$

We observe that $g(x) - g(\widetilde{x}) = M_{x,\widetilde{x}}e$, with

$$M_{x,\widetilde{x}} = \begin{bmatrix} -aw_{x_1,\widetilde{x}_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{44}$$

Using the Euclidean norm, we have $\|M\|_2 \le a|m_0|$, see for example [25]. The constants L_* and $\mathfrak{L}_*$ are computed specializing (13) to the Euclidean norm

$$L_* = a|m_0| + \max_{1\le i\le N}\{\|A - K_i\|_2\} \tag{45a}$$

$$\mathfrak{L}_* = a|m_0| + \max_{1\le i\le N}\{\mu_2(A - K_i)\} \tag{45b}$$

where the matrix measure is defined in (3).

Global pulse synchronization is achieved by enforcing the conditions of Corollary 1 for some $w > 0$. In particular, conditions (36) specified to the case at hand, yield

$$\begin{aligned} \overline{k}_1 &\ge \frac{1}{2}(1 - a - 2am_0 + w) \\ \overline{k}_2 &\ge \frac{1}{2}(a - 1 + |1-b| + w) \end{aligned} \tag{46}$$

$$\overline{k}_3 \ge \frac{1}{2}(|1-b| + w) \tag{47}$$

Equations (46) guarantee that the partially averaged system is globally exponentially stable. Solving equation (38), we determine the slowest switching rate that guarantees global pulse synchronization.

B. Experimental Results

The experimental test-bed is constituted of two Chua's circuits. The simplest implementation of a Chua's circuit comprises four passive components, including two capacitors, one resistor, and one inductor, and a single nonlinear component called Chua's diode. Following [36], we synthesize the Chua's diode using the circuit realization proposed in [37], and we synthesize the inductor using the Antoniou's circuit [38]. This inductorless synthesis permits a fine tuning of the nominal parameters of the circuit, and provides a robust circuit realization. For ease of implementation, we consider a feedback gain matrix $K(t)$ that switches between two constant matrices K_1 and K_2. We further specialize our experiments to the case where the feedback gain for the x_3 variable is constant in time and is equal to k_3. Moreover, we assume that the feedback gains for the x_1 and x_2 variables switch between 0 and positive values, k_1 and k_2 respectively. Switching is driven by a square wave whose duty cycle is kept constant to $\delta_1 = \delta_2 = 0.5$.

The realization of the experimental platform includes passive and active circuit elements. Employed elements are carbon film resistors (5% of tolerance respect to the nominal value), ceramic disk capacitors (20% of tolerance respect to the nominal value), $TL082$ operational amplifiers (biased at $\pm 9V$), and $DG419$ analogue switches. Switches are driven by a square wave generated by the $CF0250$ function generator by Tektronix. The choice of the hardware components implies the following values for the parameters describing the Chua's circuits in (39): $a = 9.80$, $b = 13.44$, $m_0 = -1.217$, and $m_1 = -0.648$. The values of the feedback gains are $k_1 = 40$, $k_2 = 55$, and $k_3 = 20$. Data are collected with National Instruments $PCI - 6229$ acquisition board with a sample period $T_s = 0.1006$. All the quantities reported in the following figures are nondimensional. We note that the dimensional time in the actual experiment is equal to $t\bar{t}_0$, where $\bar{t}_0 = 0.1656ms$, and t is the dimensionless time appearing in equation (39). Fig. 1 shows the chaotic dynamics of one Chua's circuit.

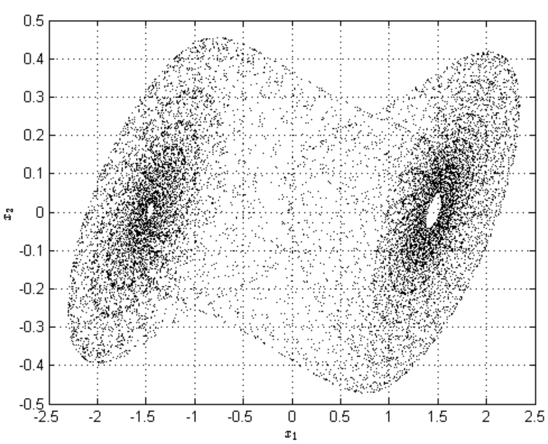

Fig. 1. Trajectory of the system in the $x_1 - x_2$ plane.

The time-average coupling matrix $\overline{K} = \mathrm{diag}[20, 27.5, 20]$ satisfies condition (36) with $w = 25$. Therefore, Corollary 1 guarantees global pulse synchronization under fast-switching. Experimentally, we note that the coupled circuits synchronize for switching periods lower than $T = 0.6039$. As the switching period increases synchronization becomes weaker and weaker, and for the switching period $T = 6.0386$ the oscillators lose synchronization. Fig. 2 shows how the coupled circuits synchronize for a switching period $T = 0.0403$, and Fig. 3 instead illustrates how oscillators lose synchronization at a switching period $T = 6.0386$.

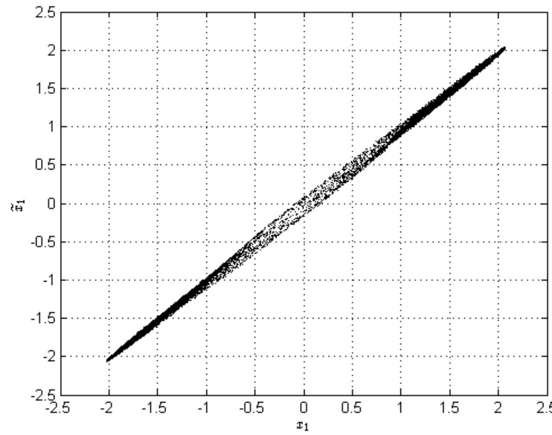

Fig. 2. Experimental synchronization graph in the $x_1 - \widetilde{x}_1$ plane for a time-varying coupling of period $T = 0.0403$.

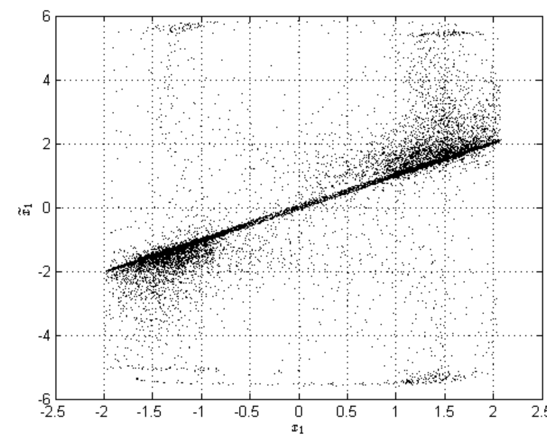

Fig. 3. Experimental synchronization graph in the $x_1 - \widetilde{x}_1$ plane for a time-varying coupling of period $T = 6.0386$.

For brevity, the trajectories in the $x_2 - \widetilde{x}_2$ and $x_3 - \widetilde{x}_3$ planes are not reported. Nevertheless the graphs confirm synchronization achievement shown in Fig. 2. It is important to notice that the existence of a residual synchronization error is due to a constructive difference between parameters of the two Chua's circuits. Solving equation (38) for the present case, we find a relatively conservative estimate of the slowest switching period T^* for global pulse synchronization. From (45), we find $\mathfrak{L}^* = 14.89$ and $L^* = 84.32$. Numerically solving equation (38), we find that the maximum switching period that guarantees global synchronization is $T^* = 0.000857$.

IV. Conclusions

In this paper, we study global pulse synchronization of two coupled chaotic systems. The systems are unidirectionally coupled through a linear error feedback that periodically switches over time. The global pulse synchronization problem is transformed into a global exponential stability problem for a nonlinear nonautonomous switched system. We associate to the switched system a partially averaged one that describes synchronization under the time-average coupling feedback gain matrix. We derive sufficient conditions for global pulse synchronization in terms of the time-average feedback gain matrix, and the switching rate. We show that synchronizability under time-average coupling can be inherited in the case of switched coupling if the switching rate is sufficiently fast.

Proposed criteria are easily applicable to a wide class of chaotic systems. We illustrate our general findings through the analysis of coupled Chua's circuits. Theoretical findings are validated through experimental results. An inductorless realization of two Chua's circuit coupled through a full-state linear diagonal switched feedback is developed. Experimental results show that synchronization is possible if the time-average feedback gains are properly chosen, and the switching rate is sufficiently large.

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