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# Sliding Mode Estimation Schemes for Unstable Systems Subject to Incipient Sensor Faults

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*Abstract*— **This paper proposes a new method for the analysis and design of sliding mode observers for fault reconstruction which is applicable for unstable systems. The proposed design addresses one of the restrictions in the existing literature (in which the open–loop system needs to be stable). Simulation results from an open–loop unstable system representing a fighter jet model show good fault estimation, even when simulated on the full nonlinear model.**

## I. INTRODUCTION

In active fault tolerant control (FTC), one of the important components is the fault detection and isolation (FDI) scheme [15]. The FDI scheme detects and isolates the faults that exist in the system and initiates controller reconfiguration to allow the faults/failures to be 'tolerated' and to enable safe degraded performance [23]. Most model based FDI schemes are residual based and an analytical redundancy approach is adopted to compare the system measurements with a mathematical model of the system, and the difference provides residual signals from which the faults/failures can be detected and isolated. Work on residual based FDI is discussed extensively in the literature: see for example [4]. Some active fault tolerant control schemes however require more information regarding the faults: see for example [24], where the estimate of the actuator efficiency is required to allow the FTC scheme to accommodate the faults/failures. This information can be provided by schemes such as those proposed in [21], [22], [24] which use the so–called modified two stage Kalman filter. In terms of sensor fault tolerant control, if the sensor fault can be estimated/reconstructed, this information can be used directly to correct the corrupted sensor measurements before they are used by the controller. This avoids reconfiguring or restructuring the controller [1].

Recent sliding mode based fault reconstruction ideas can be found in [10]. Here, the novel idea of using the 'equivalent output error injection signal' to reconstruct faults was introduced. This method was later improved for robust actuator and sensor fault estimation by Tan & Edwards [19] using a Linear Matrix Inequality (LMI) formulation. The methods for sensor fault estimation proposed in [19], [18] require one (testable) assumption, to guarantee the existence of the observer design. A sufficient condition in [19], [18] is that the system needs to be open–loop stable in order to robustly estimate the sensor faults. Open loop stability is not a necessary condition, but for open loop unstable systems with certain classes of faults, examples can be constructed such that the methods in [19], [18] are not applicable. Classical linear unknown input observers (UIO) also cannot be employed in this situation [11], [3], [5], [6], [17].

This paper proposes a new observer design for sensor fault reconstruction which addresses this restriction. In particular the proposed observer designs are applicable for open–loop stable and unstable systems. The structure of the paper is as follows: Firstly, the paper considers systems without uncertainty to convey the basic idea. Later, analysis which includes uncertainty is made to ensure a robust design. Two aircraft model examples – a passenger transport aircraft (which is open–loop stable) and a fighter jet aircraft (which is open–loop unstable) are presented to illustrate the proposed methods.

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## II. PRELIMINARIES

This section develops the preliminaries necessary for the work presented in this paper. Consider a dynamical system affected by sensor faults described by

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$

$$y(t) = Cx(t) + Ff_o(t) \quad (2)$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}$ and $F \in \mathbb{R}^{p\times q}$, with $n \geq p > q$. Assume that the matrices $C$ and $F$ have full row and column rank respectively. Without loss of generality, it can be assumed that the outputs of the system have been reordered (and scaled if necessary) so that the matrix $F$ has a structure

$$F = \begin{bmatrix} 0 \\ I_q \end{bmatrix} \quad (3)$$

The function $f_o : \mathbb{R}_+ \to \mathbb{R}^q$ is unknown but smooth and bounded so that

$$\|f_o(t)\| \leq \alpha(t) \quad (4)$$

where $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a known function. The signal $f_o(t)$ represents (additive) sensor faults and $F$ represents a distribution matrix, which indicates which of the sensors providing measurements are prone to possible faults.

*Remark*: The assumption that only certain sensors are fault prone is a limitation. However in practical situations, some sensors may be more vulnerable to damage or may be more sensitive or delicate in terms of construction than others, and so such a situation is not unrealistic. Also certain key sensors may have back-ups (hardware redundancy) and so

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essentially a fault free signal can be assumed from a certain subset of the sensors.

The objective is to design a *sliding mode* observer [20], [7], [9] in order to *reconstruct* the faults $f_o(t)$ using only measurements of $y(t)$ and $u(t)$. Suppose the signal $f_o$ is smooth and so assume

$$\xi(t) := \dot{f}_o(t) \tag{5}$$

In this paper it is assumed that the sensor faults are incipient and so $\|\xi(t)\|$ is small in magnitude, but over time the effects of the fault increment, and become significant. Equations (1) and (5) can be combined to give a system of order $n+q$ with states $x_a := \mathrm{col}(x, f_o)$ in the form

$$\begin{bmatrix} \dot{x}(t) \\ \dot{f}_o(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}}_{A_a} \begin{bmatrix} x(t) \\ f_o(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_a} u(t) + \underbrace{\begin{bmatrix} 0 \\ I_q \end{bmatrix}}_{F_a} \xi(t) \tag{6}$$

$$y(t) = \underbrace{\begin{bmatrix} C & F \end{bmatrix}}_{C_a} \begin{bmatrix} x(t) \\ f_o(t) \end{bmatrix} \tag{7}$$

and $A_a \in \mathbb{R}^{(n+q)\times(n+q)}$, $B_a \in \mathbb{R}^{(n+q)\times m}$, $C_a \in \mathbb{R}^{p\times(n+q)}$ and $F_a \in \mathbb{R}^{(n+q)\times q}$. Equations (6) and (7) represent an unknown input problem for the triple $(A_a, F_a, C_a)$ driven by the unmeasurable signal $\xi(t)$.
From (7), and based on the structure of $F$ in (3),

$$C_a = \begin{bmatrix} C & F \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ C_2 & I_q \end{bmatrix} \tag{8}$$

where $C_1 \in \mathbb{R}^{p-q\times n}$ and $C_2 \in \mathbb{R}^{q\times n}$. Notice that the triple $(A_a, F_a, C_a)$ is inherently relative degree one since $C_a F_a = F$ and $\mathrm{rank}(F) = q$ by assumption.
*Lemma 1:* The triple $(A_a, F_a, C_a)$ is minimum phase if and only if $(A, C_1)$ is detectable.
**Proof**: Consider the Rosenbrock system matrix [16] associated with $(A_a, F_a, C_a)$:

$$R(s) = \begin{bmatrix} sI-A & 0 & 0 \\ 0 & sI & -I_q \\ C_1 & 0 & 0 \\ C_2 & I_q & 0 \end{bmatrix} \tag{9}$$

The invariant zeros of $(A_a, F_a, C_a)$ are given by the values of $s \in \mathbb{C}$ where $R(s)$ loses normal rank. It is clear from (9) that

$$rank\ R(s) = rank \begin{bmatrix} sI-A & 0 \\ C_1 & 0 \\ C_2 & I_q \end{bmatrix} + q$$

and so $R(s)$ loses rank if and only if

$$rank \begin{bmatrix} sI-A \\ C_1 \end{bmatrix} < n$$

It follows from the (Popov-Belevitch-Hautus) PBH rank test that the invariant zeros of the triple $(A_a, F_a, C_a)$ are the unobservable modes of $(A, C_1)$. Consequently $(A_a, F_a, C_a)$ is minimum phase if and only if $(A, C_1)$ is detectable. ∎
*Lemma 2:* The pair $(A_a, C_a)$ is observable if $(A, C_1)$ does not have an unobservable mode at zero.

**Proof**: From the PBH test and the definition of $A_a$ and $C_a$ in (6) and (7), the pair $(A_a, C_a)$ is observable if and only if

$$rank \left[\begin{array}{c|c} sI-A & 0 \\ 0 & sI_q \\ \hline C_1 & 0 \\ C_2 & I_q \end{array}\right] = n+q, \quad \text{for all } s \in \mathbb{C} \tag{10}$$

For $s \neq 0$

$$\begin{bmatrix} sI-A & 0 \\ 0 & sI_q \\ C_1 & 0 \\ C_2 & I_q \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = 0 \Rightarrow \eta_2 = 0 \Rightarrow \begin{bmatrix} sI-A \\ C_1 \\ C_2 \end{bmatrix} \eta_1 = 0 \Rightarrow \eta_1 = 0 \tag{11}$$

since $(A, C)$ is observable, and so for $s \neq 0$, the rank of the PBH matrix in (10) is $n+q$. When $s = 0$,

$$rank \begin{bmatrix} sI-A & 0 \\ 0 & sI_q \\ C_1 & 0 \\ C_2 & I_q \end{bmatrix}_{s=0} = rank \begin{bmatrix} -A & 0 \\ C_1 & 0 \\ C_2 & I_q \end{bmatrix} = rank \begin{bmatrix} -A \\ C_1 \end{bmatrix} + q \tag{12}$$

Consequently (10) holds if and only if

$$rank \begin{bmatrix} -A \\ C_1 \end{bmatrix} = n$$

A sufficient condition for this is that $(A, C_1)$ does not have an unobservable mode at $s = 0$. ∎

*Corollary 1:* If the open loop system in (1) is stable the pair $(A_a, C_a)$ is observable.

∎

Assume without loss of generality that $C$ from (2) is given as

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & I_{p-q} \\ 0 & I_q & 0 \end{bmatrix} \tag{13}$$

For any system with $C$ of full row rank, this canonical form can be achieved by a change of coordinates in (1)–(2). Change coordinates in the augmented system in (6) and (7) according to

$$T = \begin{bmatrix} I_n & 0 \\ C_2 & I_q \end{bmatrix} \tag{14}$$

The coordinate change

$$x_a \mapsto T x_a \tag{15}$$

gives a system triple in the new coordinates as $(TA_aT^{-1}, TF_a, C_aT^{-1})$ where

$$TA_aT^{-1} = \begin{bmatrix} I_n & 0 \\ C_2 & I_q \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -C_2 & I_q \end{bmatrix} = \begin{bmatrix} A & 0 \\ C_2A & 0 \end{bmatrix} \tag{16}$$

and

$$C_aT^{-1} = \begin{bmatrix} C_1 & 0 \\ C_2 & I_q \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -C_2 & I_q \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & I_q \end{bmatrix} = [0 \quad I_p] \tag{17}$$

from the definition of $C_1$ in (13). It is also easy to check that

$$TF_a = F_a = \begin{bmatrix} 0 \\ I_q \end{bmatrix} \tag{18}$$

where $F_a$ is defined in (6).

In the $x_a$ coordinates, the states corresponding to $f_o$ are given by the last $q$ components i.e.

$$f_o(t) = C_f x_a(t) \tag{19}$$

where

$$C_f := \begin{bmatrix} 0_{q\times n} & I_q \end{bmatrix} \tag{20}$$

After the change of coordinates $x_a \mapsto T x_a$ the new matrix relating the states to the fault signals $f_o$ is

$$C_f T^{-1} = \begin{bmatrix} 0 & I_q \end{bmatrix} \begin{bmatrix} I & 0 \\ -C_2 & I_q \end{bmatrix} = \begin{bmatrix} 0_{q\times(n-p)} & -I_q & 0_{q\times(p-q)} & I_q \end{bmatrix} \tag{21}$$

using $C_2$ as defined in (13).

## III. Main Results

This section will consider a system, arising from the augmented sensor fault system (6)-(7), of the form

$$\dot{x}_a(t) = A_a x_a(t) + B_a u(t) + F_a \xi(t) \tag{22}$$

$$y(t) = C_a x_a(t) \tag{23}$$

where the faults $f_o(t) = C_f x_a(t)$. Without loss of generality, (following the series of transformations described above) the matrices $A_a$, $F_a$, $C_a$ and $C_f$ have the forms given in (16), (17), (18) and (21) respectively. Write

$$A_a = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{211} & \\ A_{212} & A_{22} \end{array}\right] \tag{24}$$

where $A_{11} \in \mathbb{R}^{(n+q-p)\times(n+q-p)}$. Define $A_{211}$ as the top $p-q$ rows of $A_{21}$. By construction, the unobservable modes of $(A_{11}, A_{211})$ are the invariant zeros of $(A_a, F_a, C_a)$ [10]. Also define $F_2 \in \mathbb{R}^{p\times q}$ as the bottom $p$ rows of $F_a$ so from (18)

$$F_2 = \begin{bmatrix} 0_{(p-q)\times q} \\ I_q \end{bmatrix} \tag{25}$$

**Assumption 1**: Assume that the triple $(A, B, C)$ is such that the new pair $(A, C_1)$ resulting from the reordering and partitioning of the outputs as shown in (6)-(8), does not have any unobservable modes at the origin.

**Remark 1:** It follows from Assumption 1 and Lemma 1, that the pair $(A_a, C_a)$ is observable. Using the results of Lemma 1, Assumption 1 is equivalent to the assumption that $(A_a, C_a)$ is observable. It is then straightforward to show using the PBH test that the pair $(A_{11}, A_{21})$ from the partition in (24) is also observable.

### A. Observer analysis

For the system in (6) - (7) a sliding mode observer of the form

$$\dot{z}(t) = A_a z(t) + B_a u(t) - G_l e_y(t) + G_n \nu \tag{26}$$

will be considered. In (26) the discontinuous output error injection term

$$\nu = -\rho(t, y, u) \frac{P_o e_y}{\|P_o e_y\|} \qquad \text{if } e_y \neq 0 \tag{27}$$

where $e_y(t) := C_a z(t) - y(t)$ is the output estimation error and $P_o$ is a symmetric positive definite (s.p.d.) matrix. The matrix $G_l$ is a traditional Luenberger observer gain used to make $(A_a - G_l C_a)$ stable. The scalar function $\rho(\cdot)$ must be an upper bound on the uncertainty and the faults; for details see [19].

An appropriate gain $G_n$ for the nonlinear injection term $\nu$ in (26) has the structure

$$G_n = \begin{bmatrix} -L \\ I_p \end{bmatrix} \quad \text{where} \quad L = \begin{bmatrix} L_1 & L_2 \end{bmatrix} \tag{28}$$

and $L_1 \in \mathbb{R}^{(n+q-p)\times(p-q)}$ and $L_2 \in \mathbb{R}^{(n+q-p)\times q}$ represent design freedom [8], [20]. In particular the gain $L$ must be chosen so that $A_{11} + LA_{21}$ is stable. If $e := z - x_a$ is the state estimation error then from (22) and (26)

$$\dot{e}(t) = (A_a - G_l C_a) e(t) - F_a \xi + G_n \nu \tag{29}$$

where $\xi$ is defined in (5), and represents the derivative of the sensor fault signal. For an appropriate choice of $\rho(t, y, u)$ in (27), it can be shown using arguments similar to those used in [19], that an ideal sliding motion takes place on

$$\mathscr{S} = \{e : C_a e = 0\}$$

in finite time: for details see [19]. During the ideal sliding motion [20], [9], $e_y = \dot{e}_y = 0$ and the discontinuous signal $\nu$ must take on average a value to compensate for $\xi$ to maintain sliding. The average quantity, denoted by $\nu_{eq}$, is referred to as the *equivalent output error injection term* (the natural analogue of the concept of equivalent control [20]). It follows from (29) that during the sliding motion,

$$\nu_{eq} = -(C_a G_n)^{-1} (C_a A_a e - C_a F_a \xi) \tag{30}$$

and so the sliding motion is governed by

$$\dot{e} = (A_a - G_n (C_a G_n)^{-1} C_a A_a) e - (F_a - G_n (C_a G_n)^{-1} C_a F_a) \xi \tag{31}$$

Ideally the effect of the unknown disturbance $\xi$ on the state estimation, particularly on the states which correspond to estimates of $f_o$, need to be minimized.

The effect of $\xi$ on the estimate of $f_o$ is given by $C_f e(t)$, where $e(t)$ evolves according to (31). Therefore, the impact of $\xi$ on the estimate of $f_o$ can be expressed as $G(s)\xi$ where

$$G(s) := \left[\begin{array}{c|c} (A_a - G_n (C_a G_n)^{-1} C_a A_a) & (F_a - G_n (C_a G_n)^{-1} C_a F_a) \\ \hline C_f & 0 \end{array}\right] \tag{32}$$

For accurate estimation of the faults $f_o$, the transfer function matrix $G(s)$ must be 'small'. Here, the $\mathscr{H}_\infty$ norm of $G(s)$ will be minimized by choice of $G_n$.

Partition the state error vector $e$ from (29), conformably with the canonical form in (24), as $\mathrm{col}(e_1, e_y)$. One way to identify the reduced order sliding motion is to perform a further change of coordinates according to the nonsingular matrix

$$T_L = \begin{bmatrix} I_{n+q-p} & L \\ 0 & I_p \end{bmatrix} \tag{33}$$

so that

$$e = (e_1, e_y) \rightarrow (e_1 + L e_y, e_y) \equiv (\tilde{e}_1, e_y) =: \tilde{e} \tag{34}$$

It can be easily verified that in the coordinate system in (34), during the sliding motion, the error system i.e. (the reduced order sliding motion) can be written as

$$\dot{\tilde{e}}_1(t) = \big(A_{11}+L_1A_{211}+L_2A_{212}\big)\tilde{e}_1(t)+L_2\xi \quad (35)$$

$$\dot{e}_y(t) = e_y(t)=0 \quad (36)$$

The gain matrices $L_1$ and $L_2$ needed to be chosen to ensure $A_{11}+LA_{211}+L_2A_{212}$ is stable for the sliding motion to be stable. Therefore the effect of $\xi$ on the estimation $\hat{f}_o$ is given by $C_f e = \tilde{C}_f\tilde{e}$ where $\tilde{C}_f = C_f T_L^{-1}$ and $C_f$ is given in (20). It can be verified

$$\tilde{C}_f = \left[\begin{array}{ccc} 0_{n-p\times q} & I_q & * \end{array}\right] \quad (37)$$

where $*$ represents a matrix which plays no part in the subsequent analysis. During the sliding motion,

$$\tilde{C}_f\tilde{e} = \left[\begin{array}{cc|c} 0_{n-p\times q} & I_q & * \end{array}\right]\left[\begin{array}{c} \tilde{e}_1 \\ e_y \end{array}\right] = \underbrace{\left[\begin{array}{cc} 0_{n-p\times q} & I_q \end{array}\right]}_{C_e}\tilde{e}_1 \quad (38)$$

since $e_y=0$ during sliding. Consequently,

$$G(s)\xi = \tilde{G}(s)\xi \quad (39)$$

where

$$\tilde{G}(s) := \left[\begin{array}{c|c} A_{11}+L_1A_{211}+L_2A_{212} & L_2 \\ \hline C_e & 0 \end{array}\right] \quad (40)$$

and $C_e$ is defined in (38). As argued in Remark 1, the pair $(A_{11}, A_{211})$ is observable, and so from the partition of $A_{21}$ in (24) to obtain $A_{211}$ and $A_{212}$, it follows that there exist $L_1$ and $L_2$ so that $A_{11}+L_1A_{211}+L_2A_{212}$ is stable.

*Proposition 1:* If $(A_a, F_a, C_a)$ from (22)-(23) is minimum phase, then a sliding mode observer of the form in (26) exists such that $\hat{f}_o = C_f x_a \to f_o$ as $t\to\infty$.

Proof: If $(A_a, F_a, C_a)$ from (22)-(23) is minimum phase, then the pair $(A_{11}, A_{211})$ is detectable [10], and so there exists an $L_o$ such that $(A_{11}+L_oA_{211})$ is stable. Consequently the selection $L_1 = L_o$ and $L_2 = 0$ is a feasible choice which makes $A_{11}+L_1A_{211}+L_2A_{212} = A_{11}+L_oA_{211}$ stable. Furthermore for this choice of $L_1$ and $L_2$ it follows that $\|\tilde{G}(s)\|_\infty = 0$ and $\xi$ has no impact on the estimation error and so asymptotic tracking of the states takes place. It follows $\hat{f}_o(t) - f(t) = C_f e(t) \to 0$ since $e(t)\to 0$ and the fault is estimated asymptotically. ■

*Proposition 2:* If the plant system matrix $A$ from (1) is stable, $\hat{f}_o = C_f z_a \to f_o$ as $t\to\infty$.

Proof: If the plant system matrix $A$ from (1) is stable, then $(A, C_1)$ is automatically detectable and from Lemma 1, $(A_a, F_a, C_a)$ is minimum phase. Therefore from Proposition 1, $\hat{f}_o = C_f z_a \to f_o$ since $e(t)\to 0$. ■

**Remark 2:** If $A$ from (1) is unstable then for certain fault conditions, $(A, C_1)$ may be unobservable and perfect reconstruction is not possible. Furthermore if $(A, C_1)$ is undetectable making $(A_a, F_a, C_a)$ nonminimum phase, then as argued in [11] classical unknown input observers UIOs also cannot be employed to reject the unknown input $\xi(t)$: see for example [17], [6], [5], [3].

The next subsection considers the ramifications of this.

### B. Observer Desgin

As in [19], define a Lyapunov matrix for the error system in (29) to have the form

$$P = \left[\begin{array}{cc} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22} \end{array}\right] \quad (41)$$

where $P_{11} \in \mathbb{R}^{(n+q-p)\times(n+q-p)}$ is s.p.d. Let $G_l \in \mathbb{R}^{(n+q)\times p}$ be any matrix which satisfies

$$P(A_a - G_lC_a) + (A_a - G_lC_a)^{\mathrm{T}}P < 0 \quad (42)$$

Here, the design of the linear gain $G_l$ for the sliding mode observer from (26) will be chosen to satisfy

$$\left[\begin{array}{ccc} P(A_a-G_lC_a)+(A_a-G_lC_a)^{\mathrm{T}}P & P(G_lD-B_d) & E^{\mathrm{T}} \\ (G_lD-B_d)^{\mathrm{T}}P & -\gamma_0 I_{p+q} & 0 \\ E & 0 & -\gamma_0 I_q \end{array}\right] < 0 \quad (43)$$

The matrices $B_d \in \mathbb{R}^{(n+q)\times(p+q)}$, $D \in \mathbb{R}^{p\times(p+q)}$ in (43) are defined as

$$B_d := \left[\begin{array}{cc} 0 & F_a \end{array}\right] \quad (44)$$

$$D := \left[\begin{array}{cc} D_1 & 0 \end{array}\right] \quad (45)$$

where $D_1 \in \mathbb{R}^{p\times p}$, $F_a$ is defined in (18), and

$$E = \left[\begin{array}{cc} C_e & F_2^{\mathrm{T}} \end{array}\right] \quad (46)$$

where $C_e$ is defined in (38). From (43), it can be seen that $D_1$ is the only visible design freedom. As argued in [19], inequality (43) is feasible if and only if

$$\left[\begin{array}{ccc} PA_a + A_a^{\mathrm{T}}P - \gamma_0 C_a^{\mathrm{T}}(DD^{\mathrm{T}})^{-1}C_a & -PB_d & E^{\mathrm{T}} \\ -B_d^{\mathrm{T}}P & -\gamma_0 I_{(p+q)} & 0 \\ E & 0 & -\gamma_0 I_q \end{array}\right] < 0 \quad (47)$$

in which case

$$G_l = \gamma_0 P^{-1}C_a^{\mathrm{T}}(DD^{\mathrm{T}})^{-1}C_a \quad (48)$$

is a choice of Luenberger gain. Let

$$PA_a + A_a^{\mathrm{T}}P := \left[\begin{array}{cc} X_{11} & X_{12} \\ X_{12}^{\mathrm{T}} & X_{22} \end{array}\right] \quad (49)$$

where $P$ is defined in (41) and $X_{11} \in \mathbb{R}^{(n+q-p)\times(n+q-p)}$ is defined as

$$X_{11} = P_{11}A_{11} + P_{12}A_{21} + (P_{11}A_{11}+P_{12}A_{21})^{\mathrm{T}} \quad (50)$$

From (44)

$$PB_d = P\left[\begin{array}{cc} 0 & F_a \end{array}\right] = \left[\begin{array}{cc} 0 & P_{122} \\ 0 & P_{222} \end{array}\right] \quad (51)$$

where $P_{122}$ and $P_{222}$ are the last $q$ columns of $P_{12}$ and $P_{22}$ respectively. Using (49) and (51), equation (47) can be written as

$$\left[\begin{array}{ccccc} X_{11} & X_{12} & 0 & -P_{122} & C_e^{\mathrm{T}} \\ X_{12}^{\mathrm{T}} & X_{22} - \gamma_0^{\mathrm{T}}(DD^{\mathrm{T}})^{-1} & 0 & -P_{222} & F_2 \\ 0 & 0 & -\gamma_o I_p & 0 & 0 \\ -P_{122}^{\mathrm{T}} & -P_{222}^{\mathrm{T}} & 0 & -\gamma_o I_q & 0 \\ C_e & F_2^T & 0 & 0 & -\gamma_o I_q \end{array}\right] < 0 \quad (52)$$

A necessary condition for the inequality above to hold is that

$$\begin{bmatrix} X_{11} & -P_{122} & C_e^{\mathrm{T}} \\ -P_{122}^{\mathrm{T}} & -\gamma_0 I_q & 0 \\ C_e & 0 & -\gamma_0 I_q \end{bmatrix} < 0 \tag{53}$$

If $L := P_{11}^{-1}P_{12}$ then (53) can be re-written as

$$\begin{bmatrix} P_{11}(A_{11}+LA_{21})+(A_{11}+LA_{21})^T P_{11} & -P_{11}L_2 & C_e^{\mathrm{T}} \\ * & -\gamma_0 I_q & 0 \\ * & * & -\gamma_0 I_q \end{bmatrix} < 0 \tag{54}$$

which is the Bounded Real Lemma [2] associated with the transfer function $\tilde{G}(s) = C_e(sI-(A_{11}+LA_{21}))^{-1}L_2$ and implies $\|\tilde{G}(s)\|_\infty < \gamma_0$.

Formally the design problem is: for a given matrix $D_1$ and scalar $\gamma_0$, minimize $\gamma$ with respect to $P$, subject to

$$\begin{bmatrix} X_{11} & -P_{122} & C_e \\ -P_{122}^{\mathrm{T}} & -\gamma I_q & 0 \\ C_e & 0 & -\gamma I_q \end{bmatrix} < 0 \tag{55}$$

$$P > 0 \tag{56}$$

and (47). This is a convex optimization problem. Standard LMI software such as [13] can be used to synthesize numerically $\gamma$ and $P$. Once $P$ has been determined, $L$ can be determined as $L = P_{11}^{-1}P_{12}$. The observer gain $G_l$ can be determined from (48) and $G_n$ is determined from (28). As argued in [18] a possible choice of the s.p.d matrix $P_0$ associated with the unit-vector term (27) is $P_0 = P_{22} - P_{21}P_{11}^{-1}P_{12}$.

**Remark 3:** If (52) holds, then (55) holds for $\gamma = \gamma_0$ and so the minimum value of $\gamma$ represented by $\hat{\gamma}$ satisfies $\hat{\gamma} \le \gamma_0$.

### *C. System Uncertainty*

Suppose the system in (1) is subject to uncertainty so that

$$\dot{x}(t) = Ax(t) + Bu(t) + M\psi(t,x) \tag{57}$$

where $\psi(\cdot)$ represents a bounded unknown disturbance. Therefore the augmented system in (6) - (7) becomes

$$\dot{x}_a(t) = A_a x_a(t) + B_a u(t) + M_a\psi(t,x) + F_a\xi(t) \tag{58}$$

$$y(t) = C_a x_a(t) \tag{59}$$

where the term $M_a\psi(t,x)$ represents the effect of additive bounded uncertainty. Again the fault to be reconstructed is given by $f_o = C_f x_a$. The idea now is to represent (58) as

$$\dot{x}_a(t) = A_a x_a(t) + B_a u(t) + \begin{bmatrix} M_a & F_a \end{bmatrix} \begin{bmatrix} \psi(t,x) \\ \xi(t) \end{bmatrix} \tag{60}$$

and to minimize the effect of $(\psi,\xi)$ on the reconstruction of $f_o$. As a consequence, the disturbance matrix $B_d$ from (44) must be augmented and becomes

$$\bar{B}_d = \begin{bmatrix} 0 & F_a & M_a \end{bmatrix} \tag{61}$$

and the matrix $D$ from (45) becomes

$$\bar{D} = \begin{bmatrix} D_1 & 0 & 0 \end{bmatrix} \tag{62}$$

The new optimization problem becomes:

For a given matrix $D_1$ and $\gamma_0$, minimize with respect to $\gamma$ and $P$, inequalities (55), (56) and

$$\begin{bmatrix} PA_a + A_a^{\mathrm{T}}P - \gamma_0 C_a^{\mathrm{T}}(\bar{D}\bar{D}^{\mathrm{T}})^{-1}C_a & -P\bar{B}_d & E^{\mathrm{T}} \\ -\bar{B}_d^{\mathrm{T}}P & -\gamma_0 I & 0 \\ E & 0 & -\gamma_0 I \end{bmatrix} < 0 \tag{63}$$

**Remark 4:** Note $M_a$ needs to be pre–scaled appropriately so that $\psi_a$ and $\xi$ are of the same order, or suitably weighted to reflect the importance of rejection of uncertainty compared to the effect of the fault derivative.

## IV. Simulation Results

The ADMIRE model represents a rigid small fighter aircraft with a delta-canard configuration based on a real fighter aircraft. Details of the model can be found in [12]. The linear model used here has been obtained at a low speed flight condition of Mach 0.22 at an altitude of 3000m and is similar to the one in [14]. The states are $x = [\alpha\ \beta\ p\ q\ r]^{\mathrm{T}}$ with controlled outputs $\alpha,\beta,p$; where $\alpha$ is angle of attack (AoA) (rad), $\beta$ is sideslip angle (rad), $p$ is roll rate (rad/sec), $q$ is pitch rate (rad/sec) and $r$ is yaw rate (rad/sec). The control surfaces are $\delta = [\delta_c\ \delta_{re}\ \delta_{le}\ \delta_r]^{\mathrm{T}}$, which represent the deflections (rad) of the canard, right elevon, left elevon and rudder respectively. A linearized model [14] is:

$$A = \begin{bmatrix} -0.5432 & 0.0137 & 0 & 0.9778 & 0 \\ 0 & -0.1179 & 0.2215 & 0 & -0.9661 \\ 0 & -10.5128 & -0.9967 & 0 & 0.6176 \\ 2.6221 & -0.0030 & 0 & -0.5057 & 0 \\ 0 & 0.7075 & -0.0939 & 0 & -0.2127 \end{bmatrix} \tag{64}$$

$$B = \begin{bmatrix} 0.0069 & -0.0866 & -0.0866 & 0.0004 \\ 0 & 0.0119 & -0.0119 & 0.0287 \\ 0 & -4.2423 & 4.2423 & 1.4871 \\ 1.6532 & -1.2735 & -1.2735 & 0.0024 \\ 0 & -0.2805 & 0.2805 & -0.8823 \end{bmatrix} \tag{65}$$

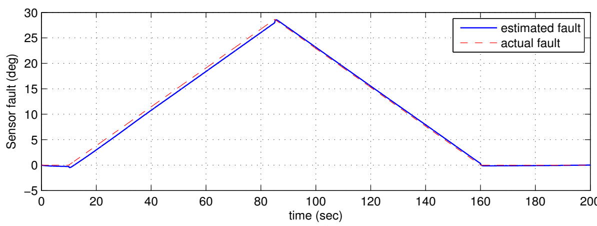
$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{66}$$

The linear model given above is open–loop unstable, which is a typical characteristic of fighter aircraft to allow high manoeuvrability. It is assumed that the sensor for the pitch rate ($q$) is prone to faults. This system is an example where the fault estimation scheme in [18], [19] will not work because it can be shown that if

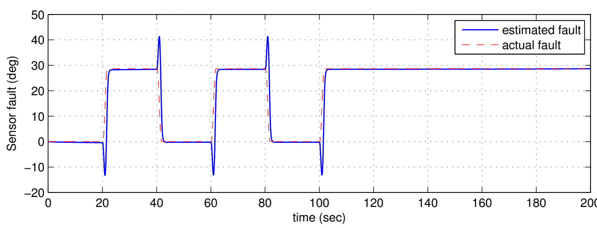
$$F = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$$

in (2), then the associated augmented system $(A_a, F_a, C_a)$ is non–minimum phase with a zero at $\{1.0769\}$. Note that the $C$ matrix has been reordered to comply with the requirements in (3) where the sensors that are prone to faults are in the lower part of the $C$ matrix. However, the approach proposed in this paper is applicable for this particular system. The design parameters for the observer were chosen as, $\gamma_0 = 10$ from (43) and $D_1 = I_3$ from (45) to yield $\|\tilde{G}(s)\|_\infty = 1.2212$. Based on this choice and the associated observer gains, the closed–loop reduced order eigenvalues for the observer are given by $\{-3.8496, -2.1258, -0.6089\}$. The nonlinear gain in (27) has been chosen as $\rho = 1$. During simulation the signum function from (27) has been approximated by the smooth function $\frac{P_o e_y}{\|P_o e_y\|+\delta}$ where $\delta = 0.001$.

The simulation in Figure 1 has been obtained from the full nonlinear ADMIRE model with the aircraft undergoing a banking manoeuvre and change in altitude. Figure 1 shows the results of the fault reconstruction using different sensor fault shapes, to show the effectiveness of the method. Figure 1(a) shows a slow incipient ramp fault where the fault drifts to a maximum value and then returns to a nominal condition. In Figure 1(b), a sensor fault is considered in which the fault fluctuates between a nominal and a maximum value before finally maintaining a constant fault level. In both conditions, the proposed scheme provides satisfactory fault reconstructions on the $q$ sensor when tested on the full nonlinear model. As expected, in this situation, perfect fault estimation cannot be achieved.




(a) ramp fault




(b) smooth step fault

Fig. 1. Sensor fault reconstruction on the pitch rate ($q$) sensor on ADMIRE full nonlinear model

## V. Conclusion

This paper has addressed one of the design restrictions in the literature for sensor fault reconstruction based on sliding mode observers as proposed in [19], [18]. The existing literature guarantees that a sensor fault reconstruction observer exists for open loop stable systems. In this paper, a sliding mode observer for fault reconstruction which is applicable for both open–loop stable and unstable systems has been proposed. Simulation results from an open–loop unstable system of a fighter jet called ADMIRE (for which the schemes from [19], [18] and classical linear unknown input observers cannot be designed), shows good fault estimation properties even when simulated on the full nonlinear model.

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