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State Estimation with Sensor Failure for Discrete-Time Nonlinear Stochastic Systems

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Abstract— In this paper, state estimation is considered for discrete-time nonlinear systems with uncertain observations and sensor failure. We focus on the multi-sensor case where sensors may fail independently of each other at different rates. The local unbiased minimum variance estimator is developed for this case. An illustrative example is included to show the performance of the proposed approach.

I. MODEL

The problem of random sensor failure has received a lot of attention over the years. Several solutions have been proposed, e.g. [1-5] to name a few. It was only recently that the results have been extended to the case of the multiple sensors that may fail independently [6]. Due to the importance of nonlinear system models, in this work, we are presenting an extension of [6] to the case of nonlinear systems. Reference [6] also treats stochastic robustness and resilience (for linear models) which are not discussed here.
Consider the dynamical system and the measurement model

$$\begin{cases} x_{k+1} = f(x_k) + v_k \\ y_k^i = \gamma_k^i C_k^i x_k + w_k^i \end{cases}, \quad i = 1 \cdots p \tag{1}$$

where f is a differentiable nonlinear function, $x_k \in R^n$, with x_0 having the mean $E\{x_0\} = \bar{x}_0$ and covariance X_0, v_k is zero mean white noise vector uncorrelated with x_0 with covariance V, y_k^i $i = 1 \cdots p$ are scalar sensor outputs with zero mean white scalar sensor noise w_k^i that are uncorrelated with v_k and x_0, γ_k^i are scalar binary Bernoulli distributed random variables with mean π_i and variance $\pi_i(1-\pi_i)$ whose possible outcomes $\{1,0\}$ are defined as $P(\gamma_k^i = 1) = \pi_i$ and $P(\gamma_k^i = 0) = 1 - \pi_i$.

$$Y_k = diag\left(\gamma_k^1, \cdots, \gamma_k^p\right)\left[\left(C_k^1\right)^T, \cdots, \left(C_k^p\right)^T\right]^T x_k + W_k \tag{2}$$

$$Y_k = \Gamma_k C_k x_k + W_k \tag{3}$$

where W_k is a zero mean white noise vector of covariance W. This is a formulation that involves only hard failures, i.e. either the sensor works or it does not. There is no other alternative considered in this work.
The following nonlinear state estimator will be used:

$$\hat{x}_{k+1} = G_k^1\left(\hat{x}_k\right) + G_k^2 Y_k \tag{4}$$

where $G_k^1(\cdot)$ is a nonlinear time-varying function of $\hat{x}_k$ and G_k^2 is a time–varying coefficient used to weigh the previous measurements Y_k in the update of the state estimate.

II. THE NONLINEAR ESTIMATOR

In this section, we derive the locally unbiased and minimum variance state estimator for the model introduced before. The estimation error dynamics is given by:

$$e_{k+1} = x_{k+1} - \hat{x}_{k+1} = f\left(x_k\right) + v_k - G_k^1\left(\hat{x}_k\right) - G_k^2 Y_k \tag{5}$$

Since $f(\cdot)$ is differentiable, by expanding $f(\cdot)$ in Taylor series around x_k, and by assuming that the effect of the higher order derivatives is negligible, we have:

$$\begin{aligned} e_{k+1} &\cong f\left(\hat{x}_k\right) - G_k^1\left(\hat{x}_k\right) - G_k^2 \Gamma_k C_k \hat{x}_k \\ &+ \left(A_k - G_k^2 \Gamma_k C_k\right) e_k - G_k^2 W_k + v_k \end{aligned} \tag{6}$$

Since v_k, W_k, are of zero mean, if we impose the unbiasedness requirement $E\{e_k\} = 0$ for all $k \geq 0$ which is desirable for estimators, this can be satisfied by taking $\hat{x}_0 = \bar{x}_0$, so that $E\{e_0\} = E\{x_0 - \hat{x}_0\} = 0$ and $G_k^1\left(\hat{x}_k\right) = f\left(\hat{x}_k\right) - G_k^2 \Pi C_k \hat{x}_k$.
where $\Pi = \bar{\Gamma}_k = diag\left(\pi_1, \cdots, \pi_p\right)$. The error equation becomes

$$e_{k+1} \cong -G_k^2 \tilde{\Gamma}_k C_k \hat{x}_k + \left(A_k - G_k^2 \Gamma_k C_k\right) e_k - G_k^2 W_k + v_k \tag{7}$$

where $\tilde{\Gamma}_k = \Gamma_k - \bar{\Gamma}_k$.
We now find the minimum variance estimator. To do that we look at the local error covariance P_k

$$P_k = E\{e_k e_k^T\} \tag{8}$$

which after simplification, evolves as

$$\begin{aligned} P_{k+1} &= E\left\{\left(G_k^2 \tilde{\Gamma}_k C_k \hat{x}_k\right)\left(G_k^2 \tilde{\Gamma}_k C_k \hat{x}_k\right)^T\right\} \\ &- E\left\{\left(G_k^2 \tilde{\Gamma}_k C_k \hat{x}_k\right)\left(\left(A_k - G_k^2 \Gamma_k C_k\right) e_k\right)^T\right\} \\ &- E\left\{\left(A_k - G_k^2 \Gamma_k C_k\right) e_k \left(G_k^2 \tilde{\Gamma}_k C_k \hat{x}_k\right)^T\right\} \\ &+ E\left\{\left(A_k - G_k^2 \Gamma_k C_k\right) e_k \left(\left(A_k - G_k^2 \Gamma_k C_k\right) e_k\right)^T\right\} \\ &+ G_k^2 W \left(G_k^2\right)^T + V \end{aligned} \tag{9}$$

Using Lemma 1 in [6], we get

$$Z_{1k} \equiv E\left\{\left(\tilde{\Gamma}_k C_k \hat{x}_k\right)\left(\tilde{\Gamma}_k C_k \hat{x}_k\right)^T\right\} = \Upsilon \otimes \left(C_k \hat{x}_k \hat{x}_k^T C_k^T\right) \quad (10)$$

where $\otimes$ denotes the Hadamard product [7], and $\Upsilon = diag\left(\pi_1(1-\pi_1), \cdots, \pi_p(1-\pi_p)\right)$

Let us focus on the computation of

$$E\left\{\left(A_k - G_k^2 \Gamma_k C_k\right) e_k e_k^T \left(A_k - G_k^2 \Gamma_k C_k\right)^T\right\}$$
$$= A_k P_k A_k^T - G_k^2 \Pi C_k P_k A_k^T$$
$$- A_k P_k C_k^T \Pi G_k^{2T} + G_k^2 E\left\{\Gamma_k C_k e_k e_k^T C_k^T \Gamma_k^T\right\} G_k^{2T} \quad (11)$$

Since $\Gamma_k = \bar{\Gamma}_k + \tilde{\Gamma}_k = \Pi + \tilde{\Gamma}_k$ therefore

$$E\left\{\Gamma_k C_k e_k e_k^T C_k^T \Gamma_k^T\right\} = E\left\{\Pi C_k e_k e_k^T C_k^T \Pi\right\} + E\left\{\tilde{\Gamma}_k C_k e_k e_k^T C_k^T \tilde{\Gamma}_k^T\right\}$$

Using the smoothing property of the expectation,

$$E\left\{\Gamma_k C_k e_k e_k^T C_k^T \Gamma_k^T\right\} = \Pi C_k P_k C_k^T \Pi + E\left\{\tilde{\Gamma}_k C_k P_k C_k^T \tilde{\Gamma}_k^T\right\} \quad (12)$$

Applying Lemma 1 [6] , we get:

$$Z_{2k} \equiv E\left\{\tilde{\Gamma}_k C_k P_k C_k^T \tilde{\Gamma}_k^T\right\} = \Upsilon \otimes \left(C_k P_k C_k^T\right) \quad (13)$$

So we are left with:

$$P_{k+1} = G_k^2 Z_{1k} G_k^{2T} + A_k P_k A_k^T - G_k^2 \Pi C_k P_k A_k^T$$
$$- A_k P_k C_k^T \Pi G_k^{2T} + G_k^2 \left[\Pi C_k P_k C_k^T \Pi + Z_{2k}\right] G_k^{2T} \quad (14)$$
$$+ G_k^2 W G_k^{2T} + V$$

Now we can come back to the minimization of the error covariance

$$P_{k+1} = A_k P_k A_k^T + V - G_k^2 \Pi C_k P_k A_k^T - A_k P_k C_k^T \Pi G_k^{2T}$$
$$+ G_k^2 \left[\Pi C_k P_k C_k^T \Pi + Z_1 + Z_2 + W\right] G_k^{2T} \quad (15)$$

$$P_{k+1} = A_k P_k A_k^T + V - G_k^2 \Lambda_k^T - \Lambda_k G_k^{2T} + G_k^2 \Omega_k G_k^{2T} \quad (16)$$

where:

$$\Omega_k = \Pi C_k P_k C_k^T \Pi + Z_{1k} + Z_{2k} + W \quad (17)$$

$$\Lambda_k = A_k P_k C_k^T \Pi \quad (18)$$

Let us now complete the square in G_k^2

$$P_{k+1} = A_k P_k A_k^T + V + \left(G_k^2 - G_k^{2o}\right) \Omega_k \left(G_k^2 - G_k^{2o}\right)^T$$
$$- G_k^{2o} \Omega_k \left(G_k^{2o}\right)^T \quad (19)$$

For Equation (16) to be equal to equation (19), we must have:

$G_k^{2o} \Omega_k \left(G_k^{2o}\right)^T = \Lambda_k G_k^{2T}$ which yields the optimal gain $G_k^{2o} = \Lambda_k \Omega_k^{-1}$

The resulting matrix difference equation for the minimum error covariance when we let $G_k^2 = G_k^{2o}$ is:

$$P_{k+1} = A_k P_k A_k^T + V - \Lambda_k \Omega_k^{-1} \Lambda_k^T \quad (20)$$

III. SIMULATION EXAMPLE

In this example, the sensor fails with a nonzero probability.

$$\begin{cases} x_{k+1} = \sin\left(x_k\right) + v_k \\ y_k = \gamma_k x_k + w_k \end{cases},$$

where v_k and w_k have variance $v = w = 0.001$, $x_0 = 0.7$, $X_0 = 0.01$, γ_k is Bernoulli distributed with mean $\pi = 0.7$. The performance of our newly designed estimator is compared to that of the regular Extended Kalman Filter (without failure model) in a Monte Carlo simulation and the resulting mean square errors (MSEs) are given in Figure 1. This result shows that the proposed approach is more appropriate for this system with sensor failure as it has a smaller MSE compared to the EKF.

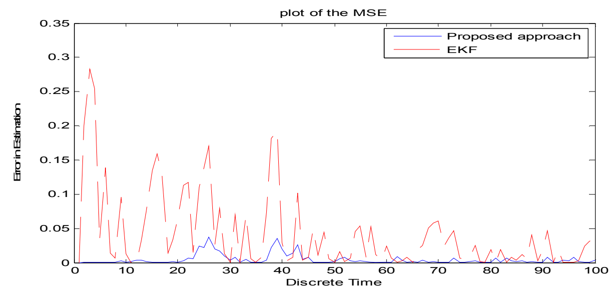

Figure 1: MSEs of the state estimation - The proposed approach vs. EKF

IV. CONCLUSION

In this paper, the state estimation problem for systems with sensor failure is extended to a class of nonlinear systems with uncertain measurements. We derived analytic expressions for the approximate (local) minimum variance unbiased estimator for nonlinear multi-sensor systems featuring failure. A simulation example is also provided for illustration purposes.

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