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Two-stage Unscented Kalman Filter for Nonlinear Systems in the Presence of Unknown Random Bias

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***Abstract*—The two-stage Unscented Kalman Filter (TUKF) is proposed to consider the nonlinear system in the presence of unknown random bias in a number of practical situations. The adaptive fading UKF is designed by using the forgetting factor to compensate the effects of incomplete information. The TUKF to estimate unknown random bias is designed by using the adaptive fading UKF. This filter can be used for nonlinear systems with unknown random bias on the assumption that the stochastic information of a random bias is incomplete. The stability of the TUKF is analyzed and ensured under certain conditions. The performance of the TUKF is verified by using MATLAB simulation on the high-update rate Wheel Mobile Robot (WMR).**

I. INTRODUCTION

THE well-known Unscented Kalman filtering (UKF)[1] has been widely used in many industrial areas as it aims at the nonlinear system directly [2]-[4]. The difference from Extended Kalman Filter (EKF) is that UKF need not the linearization of the system models by Jacobian matrix. This avoids the error produced by the interruption of higher-order terms and the precision can reach second-order even higher (as precise as third-order to the Gauss noise). Unscented Transformation (UT) is introduced into the UKF, so it is free to debug. The resemblances between the UKF and the EKF is that the implementations of the two algorithms all consist of the prediction of the state mean and covariance and the update of the measurement [5], [6].

In order to satisfy the conditions of Kalman filter, the standard UKF requires an accurate system model and exact stochastic information. However, in a number of practical situations, these models contain parameters, which may deviate from their nominal values by unknown constant or unknown random bias. Although, some procedures for estimating the dynamic states of a linear system in the presence of unknown constant bias [7], [8] or a random bias [9]-[13] were suggested as the two-stage Kalman filter (TKF)[14], few of scholars researched on the filter for nonlinear systems in the presence of random bias based on the UKF.

Because the information of unknown random bias is incomplete, the adaptive fading UKF is proposed using the innovation covariance in Section 3. The proposed adaptive fading UKF compensates the effect of inaccuracy information by rescaling of the error covariance and Kalman gain through the forgetting factor. Then the two-stage Unscented Kalman filter (TUKF) is proposed by using the adaptive fading UKF in Section 3. This TUKF can be used for system with the unknown random bias on the assumption that the stochastic information of the random bias is incomplete. In Section 4, some techniques based on an augmented-state TUKF equivalent to the TUKF [15]-[17] are used. We show that the augmented-state UKF is uniformly asymptotically stable and the stability of the augmented-state TUKF means the stability of the TUKF. Finally in Section 5, the performance of the TUKF is verified by MATLAB simulation on the high-update rate Wheel Mobile Robot (WMR) [18] and the results show the effectiveness of the algorithm.

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II. PROBLEM STATEMENT

Consider the following nonlinear discrete-time stochastic system represented by:

$$x_{k+1} = f(x_k) + B_k b_k + w_k^x \tag{1a}$$

$$b_{k+1} = A_k b_k + w_k^b \tag{1b}$$

$$z_k = h(x_k) + D_k b_k + v_k \tag{1c}$$

where x_k is the $n \times 1$ state vector and z_k is the $m \times 1$ measurement vector. The nonlinear function $f(\bullet)$ and $h(\bullet)$ are state transition function and observation function, respectively, which are assumed to continuously differentiable with respect to x_k. b_k is the $p \times 1$ bias vector of unknown magnitude. All matrices have the appropriate dimensions. The noise sequence w_k^x, w_k^b and v_k are zero mean uncorrelated Gaussian random sequences with

$$E\left[\begin{bmatrix} w_k^x \\ w_k^b \\ v_k \end{bmatrix}\begin{bmatrix} w_j^x \\ w_j^b \\ v_j \end{bmatrix}^{\mathrm{T}}\right] = \begin{bmatrix} Q_k^x & 0 & 0 \\ 0 & Q_k^b & 0 \\ 0 & 0 & R_k \end{bmatrix}\delta_{kj} \tag{1d}$$

where $Q_k^x > 0$, $Q_k^b > 0$, $R_k > 0$ and δ_{kj} is the Kronecker delta. The initial states x_0 and b_0 are assumed to be uncorrelated with the white noise processes. Assume that x_0 and b_0 are Gaussian random variables with

$$E[x_0] = x_0^*,\ E[(x_0 - x_0^*)(x_0 - x_0^*)^{\mathrm{T}}] = P_0^x > 0$$

$$E[b_0] = b_0^*,\ E[(b_0 - b_0^*)(b_0 - b_0^*)^{\mathrm{T}}] = P_0^b > 0$$

$$E[(x_0 - x_0^*)(b_0 - b_0^*)^{\mathrm{T}}] = P_0^{xb} > 0$$

The problem is to design a two-stage Unscented Kalman filter (TUKF) to give a solution for nonlinear system with the unknown random bias on the assumption that the stochastic information of the random bias is incomplete.

III. Adaptive Two-stage Unscented Kalman Filter

However, the optimal TKF assumes that A_k and Q_k^b are known. In most cases, these are unknown. If this information is incomplete, the performance of the TKF may be degraded or diverged. To solve this problem, the TUKF in the section has to be adapted to environment of incomplete random bias information. Firstly we propose an adaptive fading UKF using innovation information and secondly propose a TUKF using this adaptive filter.

A. Adaptive Fading UKF Using Innovation Covariance

Consider the following nonlinear discrete-time stochastic system represented by:

$$x_{k+1} = f(x_k) + w_k \tag{2a}$$

$$z_k = h(x_k) + v_k \tag{2b}$$

where, x_k is the $n\times1$ state vector and z_k is the $m\times1$ measurement vector. The nonlinear function $f(\bullet)$ and $h(\bullet)$ are state transition function and observation function, respectively, which are assumed to continuously differentiable with respect to x_k. w_k and v_k denote sequences of uncorrelated Gaussian random vectors with zero means. Each covariance matrix is $E[w_k w_j^{\mathrm{T}}] = Q_k\delta_{kj}$, $E[v_k v_j^{\mathrm{T}}] = R_k\delta_{kj}$ and $E[w_k v_j^{\mathrm{T}}] = 0$ where δ_{kj} is the Kronecker delta function. The initial state x_0 is a random variable with mean $\hat{x}_0$ and covariance matrix P_0 and is independent of w_k and v_k.

Under UKF, the n-dimensional random variable x_k with mean $\tilde{x}_k$ and covariance P_k can be approximated by sigma points $\chi_{i,k}$ selected from the columns of $\tilde{x}_k \pm (a\sqrt{LP_k})_i$, $i = 0,\ldots,2L$. The opposite weight ω_i is $\omega_0 = 1-(1/a^2)$, $\omega_i = 1/2La^2$ ($i = 1,\ 2,\ldots,\ 2L$).

The predicted mean and covariance are computed as

$$\chi_{i,k-1}(+) = f(\chi_{i,k-1}),\quad x_k(-) = \sum_{i=0}^{2n}\omega_i\chi_{i,k-1}(+)$$

$$P_k(-) = \sum_{i=0}^{2n}\omega_i(\chi_{i,k-1}(+) - x_k(-))(\chi_{i,k-1}(+) - x_k(-))^{\mathrm{T}} + Q_k$$

$$\bar{P}_k(-) = \lambda_k P_k(-) = \lambda_k[\sum_{i=0}^{2n}\omega_i(\chi_{i,k-1}(+) - x_k(-))(\chi_{i,k-1}(+) - x_k(-))^{\mathrm{T}} + Q_k]$$

where, λ_k is the forgetting factor introduced into the error covariance equation. The measurement update can be performed with the equations as

$$z_{i,k-1}(+) = h(\chi_{i,k-1}) \quad,\quad z_k(-) = \sum_{i=0}^{2n}\omega_i z_{i,k-1}(+),\quad \varepsilon_k = z_k - z_k(-)$$

$$P_{zz} = E[\varepsilon_k\varepsilon_k^{\mathrm{T}}] = \sum_{i=0}^{2n}\omega_i(z_{i,k-1}(+) - z_k(-))(z_{i,k-1}(+) - z_k(-))^{\mathrm{T}} + R_k$$

$$P_{xz} = \sum_{i=0}^{2n}\omega_i(\chi_{i,k-1}(+) - \hat{x}_k(-))(z_{i,k-1}(+) - z_k(-))^{\mathrm{T}}$$

$$K_k = P_{xz}P_{zz}^{-1},\quad x_k(+) = x_k(-) + K_k(z_k - z_k(-)),\quad P_k(+) = P_k(-) - K_kP_{zz}K_k^{\mathrm{T}}$$

$$\bar{P}_{zz} = \frac{1}{M-1}\sum_{i=k\text{-}M+1}^{k}\varepsilon_i\varepsilon_i^{\mathrm{T}},$$

$$\bar{P}_{zz} = \alpha_k P_{zz} = \alpha_k[\sum_{i=0}^{2n}\omega_i(z_{i,k-1}(+) - z_k(-))(z_{i,k-1}(+) - z_k(-))^{\mathrm{T}} + R_k]$$

The scalar variable α_k can be estimated by

$$\alpha_k = \max\{1, \frac{1}{m}\mathrm{tr}(\bar{P}_{zz}P_{zz}^{-1})\} \text{ or } \alpha_k = \max\{1, \frac{\mathrm{tr}(\bar{P}_{zz})}{\mathrm{tr}(P_{zz})}\}$$

From $\bar{P}_k(-) = \lambda_k P_k(-)$ and $\bar{P}_{zz} = \alpha_k P_{zz}$, we can obtain the new Kalman gain and the forgetting factor λ_k.

$$\bar{K}_k = \frac{\lambda_k}{\alpha_k}K_k = \frac{\lambda_k}{\alpha_k}P_{xz}P_{zz}^{-1}$$

$$\lambda_k = \frac{\mathrm{tr}[\alpha_k\sum_{i=0}^{2n}\omega_i(z_{i,k-1}(+) - z_k(-))(z_{i,k-1}(+) - z_k(-))^{\mathrm{T}} + (\alpha_k - 1)R_k]}{\mathrm{tr}[\sum_{i=0}^{2n}\omega_i(z_{i,k-1}(+) - z_k(-))(z_{i,k-1}(+) - z_k(-))^{\mathrm{T}}]},\quad \lambda_k \ge 1$$

Then it gives

$$\bar{x}_k(+) = x_k(-) + \bar{K}_k(z_k - z_k(-)),\quad \bar{P}_k(+) = \bar{P}_k(-) - \bar{K}_k\bar{P}_{zz}\bar{K}_k^{\mathrm{T}}$$

The proposed adaptive filter has several characteristics. First, the adaptive fading UKF proposed in this section has a unified filter structure for system with incomplete dynamic or measurement equation. Secondly, the forgetting factor using innovation information is adaptively adjusted for system with incomplete information. The method using this forgetting factor requires a low computation time. Also the forgetting factor is calculated simply. Hence the proposed adaptive fading UKF can be used for complex nonlinear stochastic system without a heavy burden.

B. Two-stage UKF in the Presence of Random Bias

The TUKF can be designed by the proposed adaptive fading UKF. This TUKF can be used when the information of A_k and Q_k^b are incomplete. Several equations related to the innovation are arranged as follows.

$$\bar{\varepsilon}_k^b = z_k - z_k(-) - N_k\bar{b}_k(-) \tag{3}$$

$$P_{zz}^b = E[\bar{\varepsilon}_k^b\bar{\varepsilon}_k^{b\mathrm{T}}] = \sum_{i=0}^{2n}\omega_i(z_{i,k-1}(+) - z_k(-))(z_{i,k-1}(+) - z_k(-))^{\mathrm{T}} + R_k + N_k\bar{P}_k^b(-)N_k^{\mathrm{T}} \tag{4}$$

$$\bar{P}_{zz}^b = \frac{1}{M-1}\sum_{i=k\text{-}M+1}^{k}\bar{\varepsilon}_k^b\bar{\varepsilon}_k^{b\mathrm{T}} \tag{5}$$

To compensate the effects of incomplete information in the bias filter of the TUKF, the calculated innovation covariance and the estimated innovation covariance are defined by (4) and (5). We use the adaptive fading UKF with rescaling $\bar{P}_k(-)$ because the dynamic equation of the bias filter is incomplete. Then α_k^b is equal to the forgetting factor λ_k^b where $\bar{P}_{zz}^b = \alpha_k^b P_{zz}^b$. By the forgetting factor calculated from (4) and (5), the error covariance equation is changed into $P_k^{b*}(-) = \lambda_k\bar{P}_k^b(-)$.

Next, we consider the modified bias free filter of the TUKF which has u_k and $\bar{Q}_k^x$. For convenience, u_k and $\bar{Q}_k^x$ of TUKF are rewritten as

$$\begin{aligned} u_k &= (\bar{U}_{k+1} - U_{k+1})A_k\bar{b}_k(+) \\ &= \left(\bar{U}_{k+1} - \bar{U}_{k+1}\left[I - Q_k^b[\bar{P}_k^b(-)]^{-1}\right]\right)A_k\bar{b}_k(+) = \bar{U}_{k+1}Q_k^b[\bar{P}_k^b(-)]^{-1}A_k\bar{b}_k(+) \end{aligned} \tag{6}$$

$$\bar{Q}_k^x = Q_k^x + U_{k+1}Q_k^b\bar{U}_{k+1} \tag{7}$$

In (6), u_k is related to the incomplete A_k and Q_k^b. Also in (7), $\bar{Q}_k^x$ is related to the incomplete Q_k^b. These mean that the dynamic equation of the modified bias free filter is incomplete. Therefore we use the adaptive fading UKF with rescaling $\bar{P}_k(-)$. Several equations related to the innovation are arranged as follows:

$$\bar{\varepsilon}_k^x = z_k - z_k(-) = z_k - \sum_{i=0}^{2n}\omega_i h(\chi_{i,k-1}) \tag{8}$$

$$P_{zz}^x = E[\bar{\varepsilon}_k^x\bar{\varepsilon}_k^{x\mathrm{T}}] = \sum_{i=0}^{2n}\omega_i(z_{i,k-1}(+) - z_k(-))(z_{i,k-1}(+) - z_k(-))^{\mathrm{T}} + R_k \tag{9}$$

$$\bar{P}_{zz}^x = \frac{1}{M-1}\sum_{i=k\text{-}M+1}^{k}\bar{\varepsilon}_k^x\bar{\varepsilon}_k^{x\mathrm{T}} \tag{10}$$

To compensate the effects of incomplete information in the modified bias free filter of the TUKF, each innovation covariance is defined by (9) and (10). Here, α_k^x is equal to the forgetting factor λ_k^x where $\bar{P}_{zz}^x = \alpha_k^x P_{zz}^x$. By the forgetting factor calculated from (9) and (10), the error covariance equation is changed into $\bar{P}_k^{x*} = \lambda_k^x \bar{P}_k^x(-)$. From equations above, the TUKF of Definition 1 is proposed.

Definition 1. A discrete-time two-stage Unscented Kalman filter (TUKF) is given by the following coupled difference equations when the information of nonlinear stochastic system given by (1) is incomplete:

$$\hat{x}_k(-) = \bar{x}_k(-) + U_k \bar{b}_k(-),\ \hat{x}_k(+) = \bar{x}_k(+) + V_k \bar{b}_k(+) \tag{11a}$$

$$\hat{P}_k^x(-) = \bar{P}_k^{x*}(-) + U_k \bar{P}_k^{b*}(-) U_k^{\mathrm{T}},\ \hat{P}_k^x(+) = \bar{P}_k^{x*}(+) + V_k \bar{P}_k^{b*}(+) V_k^{\mathrm{T}} \tag{11b}$$

where A_k and Q_k^b are partially known. Here, $\hat{x}_k$, $\bar{x}_k$ and $\bar{b}_k$ are the state vectors of the TUKF, the modified bias-free filter and the bias filter, respectively.

The modified bias free filter is

$$\chi_{i,k-1}(+) = f(\chi_{i,k-1}),\ \bar{x}_k(-) = \sum_{i=0}^{2n} \omega_i \chi_{i,k-1}(+) + u_{k-1}$$

$$\bar{P}_k^{x*}(-) = \lambda_k^x [\sum_{i=0}^{2n} \omega_i (\chi_{i,k-1}(+) - \bar{x}_k(-))(\chi_{i,k-1}(+) - \bar{x}_k(-))^{\mathrm{T}} + \bar{Q}_{k-1}^x]$$

$$z_{i,k-1}(+) = h(\chi_{i,k-1}),\ \hat{z}_k(-) = \sum_{i=0}^{2n} \omega_i z_{i,k-1}(+)$$

$$\bar{\varepsilon}_k^x = z_k - z_k(-) = z_k - \sum_{i=0}^{2n} \omega_i h(\chi_{i,k-1})$$

$$P_{zz}^x = E[\bar{\varepsilon}_k^x \bar{\varepsilon}_k^{x\mathrm{T}}] = \sum_{i=0}^{2n} \omega_i (z_{i,k-1}(+) - z_k(-))(z_{i,k-1}(+) - z_k(-))^{\mathrm{T}} + R_k$$

$$\bar{P}_{zz}^x = \lambda_k^x P_{zz}^x,\ \lambda_k^x \ge 1,\ \bar{P}_{zz}^x = \frac{1}{M-1} \sum_{i=k-M+1}^{k} \bar{\varepsilon}_k^x \bar{\varepsilon}_k^{x\mathrm{T}}$$

$$\lambda_k^x = \max\{1, \frac{1}{m} \mathrm{tr}[\bar{P}_{zz}^x (P_{zz}^x)^{-1}]\} \text{ or } \lambda_k^x = \max\{1, \frac{\mathrm{tr}(\bar{P}_{zz}^x)}{\mathrm{tr}(P_{zz}^x)}\}$$

$$\bar{P}_{xz}^{x*} = \sum_{i=0}^{2n} \omega_i (\chi_{i,k-1}(+) - \bar{x}_k(-))(z_{i,k-1}(+) - z_k(-))^{\mathrm{T}}$$

$$\bar{K}_k^{x*} = \bar{P}_{xz}^{x*} (\bar{P}_{zz}^x)^{-1}$$

$$\bar{x}_k(+) = \bar{x}_k(-) + \bar{K}_k^{x*} \bar{\varepsilon}_k^x,\ \bar{P}_k^{x*}(+) = \bar{P}_k^{x*}(-) - \bar{K}_k^{x*} \bar{P}_{zz}^x (\bar{K}_k^{x*})^{\mathrm{T}}$$

and the bias filter is

$$\bar{b}_k(-) = A_{k-1} \bar{b}_{k-1}(+),\ \bar{P}_k^{b*}(-) = \lambda_k^b [A_{k-1} \bar{P}_{k-1}^{b\ *}(+) A_{k-1}^{\mathrm{T}} + \bar{Q}_{k-1}^b]$$

$$\bar{K}_k^{b*} = \bar{P}_k^{b*}(-) N_k^{\mathrm{T}} [\sum_{i=0}^{2n} \omega_i (z_{i,k-1}(+) - z_k(-))(z_{i,k-1}(+) - z_k(-))^{\mathrm{T}} + R_k + N_k \bar{P}_k^{b*}(-) N_k^{\mathrm{T}}]^{-1}$$

$$\bar{P}_k^{b*}(+) = [I - \bar{K}_k^{b*} N_k] \bar{P}_k^{b*}(-),\ \bar{b}_k(+) = \bar{b}_k(-) + \bar{K}_k^{b*} \bar{\varepsilon}_k^b$$

$$\bar{\varepsilon}_k^b = z_k - z_k(-) - N_k \bar{b}_k(-) = \bar{\varepsilon}_k^x - N_k \bar{b}_k(-)$$

$$P_{zz}^b = \sum_{i=0}^{2n} \omega_i (z_{i,k-1}(+) - z_k(-))(z_{i,k-1}(+) - z_k(-))^{\mathrm{T}} + R_k + N_k \bar{P}_k^{b*}(-) N_k^{\mathrm{T}}$$

$$\bar{P}_{zz}^b = \lambda_k^b P_{zz}^b,\ \lambda_k^b \ge 1,\ \bar{P}_{zz}^b = \frac{1}{M-1} \sum_{i=k-M+1}^{k} \bar{\varepsilon}_k^b \bar{\varepsilon}_k^{b\mathrm{T}}$$

$$\lambda_k^b = \max\{1, \frac{1}{m} \mathrm{tr}[\bar{P}_{zz}^b (P_{zz}^b)^{-1}]\} \text{ or } \lambda_k^b = \max\{1, \frac{\mathrm{tr}(\bar{P}_{zz}^b)}{\mathrm{tr}(P_{zz}^b)}\}$$

with the coupling equations

$$N_k = \gamma_k H_k U_k + D_k,\ U_k = \bar{U}_{k+1} \left[I - \lambda_k^b Q_{k-1}^b [\bar{P}_k^{b*}(-)]^{-1} \right],$$

$$V_k = U_k - \bar{K}_k^{b*} N_k,\ \bar{U}_{k+1} = (\beta_k \Phi_{k-1} V_{k-1} + B_{k-1}) A_{k-1}^{-1},$$

$$u_k = (\bar{U}_{k+1} - U_{k+1}) A_k \bar{b}_k(+),\ \bar{Q}_k^x = Q_k^x + U_{k+1} Q_k^b \bar{U}_{k+1}$$

where, $\Phi_k = \left(\frac{\partial f(x)}{\partial x} \Big|_{x=\hat{x}_k} \right)$ and $H_k = \left(\frac{\partial h(x)}{\partial x} \Big|_{x=\hat{x}_k} \right)$.

And the unknown instrumental diagonal matrices $\beta_k = \mathrm{diag}(\beta_{1,k}, \beta_{2,k}, \cdots, \beta_{N,k})$ and $\gamma_k = \mathrm{diag}(\gamma_{1,k}, \gamma_{2,k}, \cdots, \gamma_{M,k})$ are introduced in order to take these residuals into account and obtain a more exact equality.

Also, the initial conditions are

$$\bar{x}_0(+) = x_0^* - V_0 b_0^*,\ \bar{b}_0(+) = b_0^*,\ V_0 = P_0^{xb} (P_0^b)^{-1},$$

$$\bar{P}_0^x = P_0^x - V_0 P_0^b V_0^{\mathrm{T}},\ \bar{P}_0^b(+) = P_0^b$$

Remark 1. To compensate the effects of incomplete information in the modified bias free filter of the TUKF, the forgetting factor λ_k is introduced into the predicted covariance $P_k(-)$. The error covariance equation is changed into $\bar{P}_k^{x*} = \lambda_k^x \bar{P}_k^x(-)$. This enlarges the predicted covariance $P_k(-)$ and make more error, which is not established in the model, be included. Then the algorithm is simpler and more reliable.

IV. Stability Analysis

In this section, the stability of the TUKF of Definition 1 is analyzed. Firstly, instrumental time-varying matrices are introduced to give a formulation for the UT technique. Then an augmented-state TUKF can be obtained as a simple structure of the TUKF. Secondly, we show that the augmented-state TUKF is uniformly asymptotically stable by Theorem 1 in order to discuss the stability of the TUKF further more.

A. Instrumental diagonal matrix and equivalence system

Expanding $f(\bullet)$ and $h(\bullet)$ in (1) by a Taylor series about $\hat{x}_k$ yields an approximate equality

$$x_{k+1} \approx \beta_k \Phi_k x_k + B_k b_k + w_k^x$$

$$b_{k+1} = A_k b_k + w_k^b$$

$$z_k \approx \gamma_k H_k x_k + D_k b_k + v_k$$

where, $\Phi_k = \left(\frac{\partial f(x)}{\partial x} \Big|_{x=\hat{x}_k} \right)$ and $H_k = \left(\frac{\partial h(x)}{\partial x} \Big|_{x=\hat{x}_k} \right)$.

It is obvious that there always exist residuals of state prediction. In order to take these residuals into account and obtain a more exact equality, the unknown instrumental diagonal matrices [15] $\beta_k = \mathrm{diag}(\beta_{1,k}, \beta_{2,k}, \cdots, \beta_{N,k})$ and $\gamma_k = \mathrm{diag}(\gamma_{1,k}, \gamma_{2,k}, \cdots, \gamma_{M,k})$ are introduced, so that the nonlinear system can be transformed into the equivalence linear system as follow.

$$x_{k+1} = \beta_k \Phi_k x_k + B_k b_k + w_k^x \tag{12a}$$

$$b_{k+1} = A_k b_k + w_k^b \tag{12b}$$

$$z_k = \gamma_k H_k x_k + D_k b_k + v_k \tag{12c}$$

Here, if $x_k(\bullet)$ and $b_k(\bullet)$ are augmented as the system state, we can sample the system as follow.

$$x_{k+1}^a = \Phi_k^a x_k^a + w_k^a \tag{13a}$$

$$z_k^a = H_k^a x_k^a + v_k \tag{13b}$$

where $\bar{x}_k(\bullet)$ represents the estimate of the modified bias-free filter of the TUKF of Definition 1, $\bar{b}_k(\bullet)$ represents the estimate of the bias filter of the TUKF of Definition 1.

$$\bar{x}_k^a = \begin{bmatrix} \bar{x}_k \\ \bar{b}_k \end{bmatrix},\ \Phi_k^a = \begin{bmatrix} \beta_k \Phi_k & B_k \\ 0 & A_k \end{bmatrix},\ H_k^a = [\gamma_k H_k \quad D_k],\ w_k^a = \begin{bmatrix} w_k^x \\ w_k^b \end{bmatrix},\ Q_k^a = \begin{bmatrix} Q_k^x & 0 \\ 0 & Q_k^b \end{bmatrix}$$

And the augmented-state TUKF can be given by the following coupled equations when the information of the nonlinear stochastic system given by (1) is partially known.

$$\bar{x}_k^a(\bullet)=\begin{bmatrix}\bar{x}_k(\bullet)\\ \bar{b}_k(\bullet)\end{bmatrix},\ \ \bar{K}_k^{a*}=\begin{bmatrix}\bar{K}_k^{x*}\\ \bar{K}_k^{b*}\end{bmatrix} \tag{14a}$$

$$\bar{P}_k^{a*}(\bullet)=\begin{bmatrix}\bar{P}_k^{x*}(\bullet) & P_k^{xb*}(\bullet)\\ (P_k^{xb*}(\bullet))^{\mathrm{T}} & \bar{P}_k^{b*}(\bullet)\end{bmatrix}=\Lambda_k^a\begin{bmatrix}P_k^{x*}(\bullet) & P_k^{xb*}(\bullet)\\ (P_k^{xb*}(\bullet))^{\mathrm{T}} & P_k^{b*}(\bullet)\end{bmatrix} \tag{14b}$$

$$\Lambda_k^a=\begin{bmatrix}\lambda_k^x I_n & -(\lambda_k^x-\lambda_k^b)U_k\\ 0 & \lambda_k^b I_p\end{bmatrix},\ \ \lambda_k^x\ge 1,\ \ \lambda_k^b\ge 1 \tag{14c}$$

$$U_k\equiv P_k^{xb*}(-)[\bar{P}_k^{b*}(-)]^{-1} \tag{14d}$$

We use the following two-stage U-V transformation. Two-stage U-V transformation [13] is

$$\hat{x}_k^a(-)=T(U_k)\bar{x}_k^a(-),\ \ \hat{x}_k^a(+)=T(U_k)\bar{x}_k^a(+) \tag{15a}$$

$$\hat{P}_k^a(-)=T(U_k)\bar{P}_k^{a*}(-)T^{\mathrm{T}}(U_k),\ \ \hat{P}_k^a(+)=T(V_k)\bar{P}_k^{a*}(+)T^{\mathrm{T}}(V_k) \tag{15b}$$

$$\hat{K}_k^a=T(V_k)\bar{K}_k^{a*} \tag{15c}$$

where , $T(M)=\begin{bmatrix}I & M\\ 0 & I\end{bmatrix}$, $V_k\equiv P_k^{xb*}(+)[\bar{P}_k^{b*}(+)]^{-1}$.

Two-stage U-V transformation has a good advantage as $T^{-1}(M)=T(-M)$.

B. Stability Analysis

For stability analysis of the TUKF, some standard results [16]-[17] about should be recalled.

Lemma 1. If the system given by (2) with complete information is stochastically controllable and stochastically observable, the system $y_k=P_k^a(+)[P_k^a(-)]^{-1}\Phi(k,k-1)y_{k-1}$ is uniformly asymptotically stable.

Lemma 2. The augmented-state TUKF (14) is equivalent to the TUKF of Definition 1 with

$$\hat{x}_k^a(+)=\begin{bmatrix}\bar{x}_k(+)\\ \bar{b}_k(+)\end{bmatrix}=\hat{P}_k^a(+)[\hat{P}_k^a(-)]^{-1}\Phi(k,k-1)\hat{x}_{k-1}^a(+)+\hat{K}_k^a z_k \tag{16}$$

To show that the augmented-state TUKF (14) is uniformly asymptotically stable, Theorem 1 is proposed below. The system given by (12) is said to be stochastically controllable if there exist positive numbers μ_1 and μ_2, $0<\mu_1<\mu_2<\infty$, and a positive integer N such that, for all $k\ge N$,

$$\mu_1 I\le\sum_{i=k-N}^{k-1}\Phi(k,i+1)Q_i^a\Phi^{\mathrm{T}}(k,i+1)\le\mu_2 I \tag{17}$$

and the system given by (12) is said to be stochastically observable if there exist positive numbers η_1 and η_2, $0<\eta_1<\eta_2<\infty$, and a positive integer N such that, for all $k\ge N$,

$$\eta_1 I\le\sum_{i=k-N}^{k}\Phi^{\mathrm{T}}(i,k)H_i^{a\mathrm{T}}R_i^{-1}H_i^a\Phi(i,k)\le\eta_2 I \tag{18}$$

where the transition matrix $\Phi(k+1,k)$ has the following characteristics. $\Phi(k+1,k)=\Phi_k^a$, $\Phi(k,i)=\Phi(k,k-1)\Phi(k-1,k-2)\cdots\Phi(i+1,i)$, $\Phi(i,k)=\Phi^{-1}(k,i)$ and $\Phi(k,k)=I$. Here, $M_1\ge M_2$ means $(M_1-M_2)\ge 0$, i.e. (M_1-M_2) is positive semidefinite.

The system y_k is assumed as $y_k=P_k^a(+)[P_k^a(-)]^{-1}\Phi(k,k-1)y_{k-1}$. If there exist real scalar functions $V(y_k,k)$, $\xi_1(\|y_k\|)$, $\xi_2(\|y_k\|)$ and $\xi_3(\|y_k\|)$ such that for some finite $N\ge 0$

$$0<\xi_1(\|y_k\|)\le V(y_k,k)\le\xi_2(\|y_k\|),\ V(y_k,k)=y_k^{\mathrm{T}}P_k^{a-1}(+)y_k,\ \ y_k\ne 0 \tag{19}$$

$$\xi_1(0)=\xi_2(0)=0,\ \lim_{\rho\to\infty}\xi_1(\rho)=\infty \tag{20}$$

$$V(y_k,k)-V(y_{k-N},k-N)\le\xi_3(\|y_k\|)<0,\ k\ge N,\ y_k\ne 0 \tag{21}$$

then the system $y_k=P_k^a(+)[P_k^a(-)]^{-1}\Phi(k,k-1)y_{k-1}$ is uniformly asymptotically stable. These equations (19)-(21) are the requirement for $V(y_k,k)$ to be a Lyapunov function.

From Lemma 1, if the system given by (12) with complete information is stochastically controllable and stochastically observable, then the system y_k for (14) is uniformly asymptotically stable. Also, the upper bound of $P_k^a(+)$ is

$$P_k^a(+)\le[\sum_{i=k-N}^{k}\Phi^{\mathrm{T}}(i,k)H_i^{a\mathrm{T}}R_i^{-1}H_i^a\Phi(i,k)]^{-1}+\frac{N^2\mu_2\eta_2}{\mu_1\eta_1}\sum_{i=k-N}^{k-1}\Phi(k,i+1)Q_i^a\Phi^{\mathrm{T}}(k,i+1)$$
$$\le(\frac{1}{\eta_1}+\frac{N^2\mu_2\eta_2}{\mu_1\eta_1})I \tag{21}$$

and a lower bound on $V(y_k,k)$ is

$$V(y_k,k)=y_k^{\mathrm{T}}[P_k^a(-)]^{-1}y_k\ge(\frac{1}{\eta_1}+\frac{N^2\mu_2\eta_2}{\mu_1\eta_1})^{-1}\|y_k\|^2=\xi_1(\|y_k\|) \tag{22}$$

Also the upper bound of $[P_k^a(+)]^{-1}$ is

$$[P_k^a(+)]^{-1}\le[\sum_{i=k-N}^{k-1}\Phi(k,i+1)Q_i^a\Phi^{\mathrm{T}}(k,i+1)]^{-1}+\frac{N^2\mu_2\eta_2}{\mu_1\eta_1}\sum_{i=k-N}^{k}\Phi^{\mathrm{T}}(i,k)H_i^{a\mathrm{T}}R_i^{-1}H_i^a\Phi(i,k)$$
$$\le(\frac{1}{\mu_1}+\frac{N^2\mu_2\eta_2}{\mu_1\eta_1})I \tag{23}$$

and an upper bound on $V(y_k,k)$ is

$$V(y_k,k)=y_k^{\mathrm{T}}[P_k^a(-)]^{-1}y_k\le(\frac{1}{\mu_1}+\frac{N^2\mu_2\eta_2}{\mu_1\eta_1})^{-1}\|y_k\|^2=\xi_2(\|y_k\|) \tag{24}$$

Finally, we can obtain

$$V(y_k,k)-V(y_{k-N},k-N)\le-\sum_{i=k-N+1}^{k}[y_i^{\mathrm{T}}H_i^{a\mathrm{T}}R_i^{-1}H_i^a y_i+u_i^{\mathrm{T}}[P_k^a(-)]^{-1}u_i]$$
$$\le -J_m\le-\vartheta_1\|y_k\|^2\le\xi_3(\|y_k\|)<0 \tag{25}$$

The bound conditions of (21)-(25) are used for Theorem 1. Additionally, the following equations are needed to obtain the upper bounds of $\Lambda_k^a P_k^{a*}(-)$ and $\Lambda_k^a P_k^{a*}(+)$:

$$P_k^{a*}(-)=\Lambda_k^a P_k^{a*}(-)\le\left\|\Lambda_k^a\right\|_{\mathrm{F}}P_k^a(-)=(\sqrt{\textstyle\sum_{i=1}^{n+p}\sigma_{i,k}^2})P_k^a(-)=\lambda_k P_k^a(-) \tag{26}$$

$$P_k^{a*}(+)\le\Lambda_k^a P_k^{a*}(+)\le\left\|\Lambda_k^a\right\|_{\mathrm{F}}P_k^a(+)=(\sqrt{\textstyle\sum_{i=1}^{n+p}\sigma_{i,k}^2})P_k^a(+)=\lambda_k P_k^a(+) \tag{27}$$

where $\|\bullet\|_{\mathrm{F}}$ is Frobenius norm and $\sigma_{i,k}$ is a singular value of Λ_k^a.

Theorem 1. Assume that the system given by (12) with incomplete information is stochastically controllable and observable. Then, the augmented-state TUKF (14) is uniformly asymptotically stable.

Proof. From Lemma 2, A posteriori estimate of the augmented-state TUKF is derived as (16).

$$\hat{x}_k^a(+)=\begin{bmatrix}\bar{x}_k(+)\\ \bar{b}_k(+)\end{bmatrix}=\hat{P}_k^a(+)[\hat{P}_k^a(-)]^{-1}\Phi(k,k-1)\hat{x}_{k-1}^a(+)+\hat{K}_k^a z_k$$

The homogeneous part of (16) is defined as $y_k=\hat{P}_k^a(+)[\hat{P}_k^a(-)]^{-1}\Phi(k,k-1)y_{k-1}$. From (14), the error covariance has a relation such as $\hat{P}_k^a(+)\ge P_k^a(+)$ and $\hat{P}_k^a(+)\le\Lambda_k^a P_k^a(+)$. From (27), there exists λ_k where $\hat{P}_k^a(+)\ge P_k^a(+)$ and $\hat{P}_k^a(+)\le\Lambda_k^a P_k^a(+)\le\lambda_k P_k^a(+)$. As the forgetting factor Λ_k^a is inserted into the error covariance equation, (22) and (24) are changed as

$$\bar{V}_p(y_k,k)=y_k^{\mathrm{T}}[P_k^{a*}(+)]^{-1}y_k\ge\frac{1}{\lambda_k}(\frac{1}{\eta_1}+\frac{N^2\mu_2\eta_2}{\mu_1\eta_1})^{-1}\|y_k\|^2=\bar{\xi}_1(\|y_k\|) \tag{28}$$

$$\bar{V}_p(y_k,k)=y_k^{\mathrm{T}}[P_k^{a*}(+)]^{-1}y_k\le(\frac{1}{\mu_1}+\frac{N^2\mu_2\eta_2}{\mu_1\eta_1})^{-1}\|y_k\|^2=\bar{\xi}_2(\|y_k\|) \tag{29}$$

Therefore, conditions (19) and (20) are satisfied by (28) and (29). From (15), (25) and (26),

$$\bar{V}_p(y_k,k)-\bar{V}_p(y_{k-N},k-N)\le-\sum_{i=k-N+1}^{k}[y_i^{\mathrm{T}}H_i^{a\mathrm{T}}R_i^{-1}H_i^a y_i+u_i^{\mathrm{T}}[P_k^{a*}(-)]^{-1}u_i]$$

$$\begin{aligned} &\le -\sum_{i=k-N+1}^{k}[y_i^{\mathrm{T}} H_i^{a\mathrm{T}} R_i^{-1} H_i^a y_i + \frac{u_i^{\mathrm{T}}[P_k^a(-)]^{-1}u_i}{\lambda_{\max i}}] \\ &\le -\frac{J_m}{\lambda_{\max i}} \le -\frac{\vartheta_1}{\lambda_{\max i}}\| y_k \|^2 \le \bar{\xi}_3(\| y_k \|) < 0 \end{aligned} \quad (30)$$

where $\lambda_{\max i} = \max(\lambda_i)$, $k-N+1 \le i \le k$. By (28), (29) and (30), the augmented-state TUKF (14) is uniformly asymptotically stable when the system given by (12) is stochastically controllable and stochastically observable. □

Remark 2. β_k and γ_k are unknown instrumental diagonal matrices introduced to evaluate the residuals introduced by the UT. And the stability of the augmented-state TUKF (14) does not depend on the magnitude of β_k and γ_k . According to (28), (29) and (30), although different β_k and γ_k may change the value of Φ_k^a , H_k^a in (30), $\bar{V}_p(y_k,k)-\bar{V}_p(y_{k-N},k-N)$ will remain negative and the relationship shown in (30) will not be changed.

Remark 3. from lemma 2, the augmented-state TUKF (14) equivalent to the TUKF of Definition 1. Therefore, the stability of the augmented-state TUKF (14) means the stability of the TUKF when the system given by (12) is stochastically controllable and stochastically observable. Because the stability of the augmented-state TUKF (14) does not depend on the magnitude of β_k and γ_k , the stability of the augmented-state TUKF (14) also means the stability of the TUKF when the system given by (1) is stochastically controllable and stochastically observable.

V. Simulation Results

The results in the preceding two sections clarify the TUKF for Nonlinear Systems in the Presence of Unknown Random Bias and the stability analysis of the TUKF, respectively.

In order to show the efficiency of the TUKF, it is applied to the high-update rate Wheel Mobile Robot (WMR) posture, velocities, and perturbation estimation using Real-time Kinematic Global Positioning System (RTK-GPS) and inertial sensors for WMR control in the presence of wheel skidding and slipping [18] in comparison with the TKF.

The discretized equations of the WMR are

$$x_{k+1} = f(x_k) + B_k b_k + w_k^x \, , \ b_{k+1} = b_k + w_k^b$$
$$z_k = h(x_k) + v_k$$

where

$$f(x_k) = \begin{bmatrix} X_k + \Delta t V_{l,k}\cos(\theta_k) - \Delta t V_{y,k}\sin(\theta_k) \\ Y_k + \Delta t V_{l,k}\sin(\theta_k) - \Delta t V_{y,k}\cos(\theta_k) \\ V_{l,k} + \Delta t V_{y,k} r_k + \Delta a_{X,k} \\ V_y - \Delta t V_l r_k + \Delta a_{Y,k} \\ \theta_k + \Delta t r_{m,k} \end{bmatrix}, \ h(x_k) = \begin{bmatrix} X_k \\ Y_k \\ \cos\theta_k V_{l,k} - \sin\theta_k V_{y,k} \\ \sin\theta_k V_{l,k} + \cos\theta_k V_{y,k} \\ \theta_k \end{bmatrix}$$

The state vector $x_k = [X_k \ Y_k \ V_{l,k} \ V_{y,k} \ \theta_k]^{\mathrm{T}}$ uncorrelated with the bias b_k and $x_0 \sim \mathrm{N}(2, 0.05^2)$. The observation vector $z_k = [z_{X,k} \ z_{Y,k} \ z_{X',k} \ z_{Y',k} \ z_{\theta,k}]^{\mathrm{T}}$ consists of absolute position, velocity and orientation readings. The process noise vector $w_k^x = [0 \ \ 0 \ \ \Delta t\omega_{a,X} \ \ \Delta t\omega_{a,Y} \ \ \Delta t\omega_r]^{\mathrm{T}}$ and $w_k^x \sim \mathrm{N}(0, 0.05^2)$. The observation noise $v_k^x = [\upsilon_{X,k} \ \ \upsilon_{Y,k} \ \ \upsilon_{X',k} \ \ \upsilon_{Y',k} \ \ \upsilon_{\theta,k}]^{\mathrm{T}}$ and $v_k \sim \mathrm{N}(0, 0.05^2)$.

The time-varying parameters $\{a_{X,k} \ \ a_{Y,k} \ \ r_{m,k}\}$ at time k are provided by the accelerometer and gyroscope. Δt denotes the sample time of the discrete system. We assume the instantaneous yaw rate r_k is measurable by a low-noise gyroscope; hence, let $r_k = r_m$.

To estimate the innovation covariance, a window size is selected as $M = 20$. To verify the performance of the TUKF, we assume that the information of a random bias is incomplete. The TKF and the TUKF use $b_{k+1} = 0.02 b_k + w_k^b$ and $w_k^b \sim \mathrm{N}(0, 0.5^2)$.

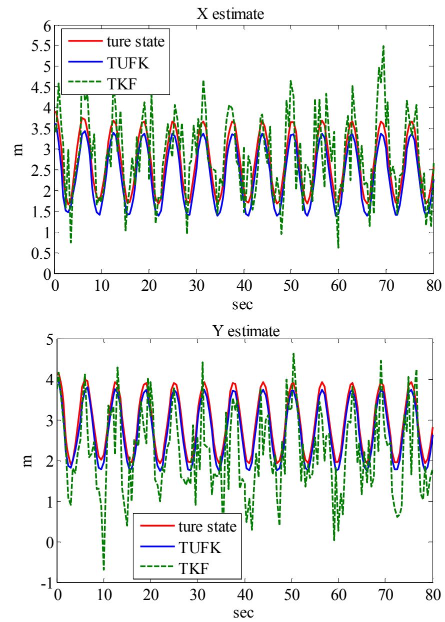

Figure 1. The comparison of true and estimated states

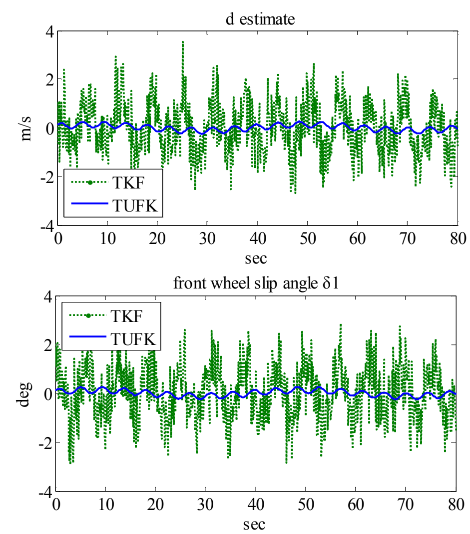

Figure 2. The comparison of the kinematic perturbations estimation

Figure 1 shows the true state, a position posteriori estimate of the TKF and a position posteriori estimate of the TUKF. Figure 2 depicts the kinematic perturbations estimates. Figure 3 shows the posteriori estimation error of the TKF and that of the TUKF. Totally, the TUKF well tracks the true state. The estimation error of the TUKF is smaller than that of the TKF. As a result, the tracking and the estimation performance of the TUKF are better than those of the TKF for the nonlinear systems that the information of a random bias is incomplete.

The simulations on the high-update rate Wheel Mobile Robot (WMR) estimation using Real-time Kinematic Global

Positioning System (RTK-GPS) and inertial sensors in the presence of wheel skidding and slipping in this section verify the proposed TUKF and its performance from the view of experimentation. It is shown that the proposed algorithm has practicability to a certain extent.

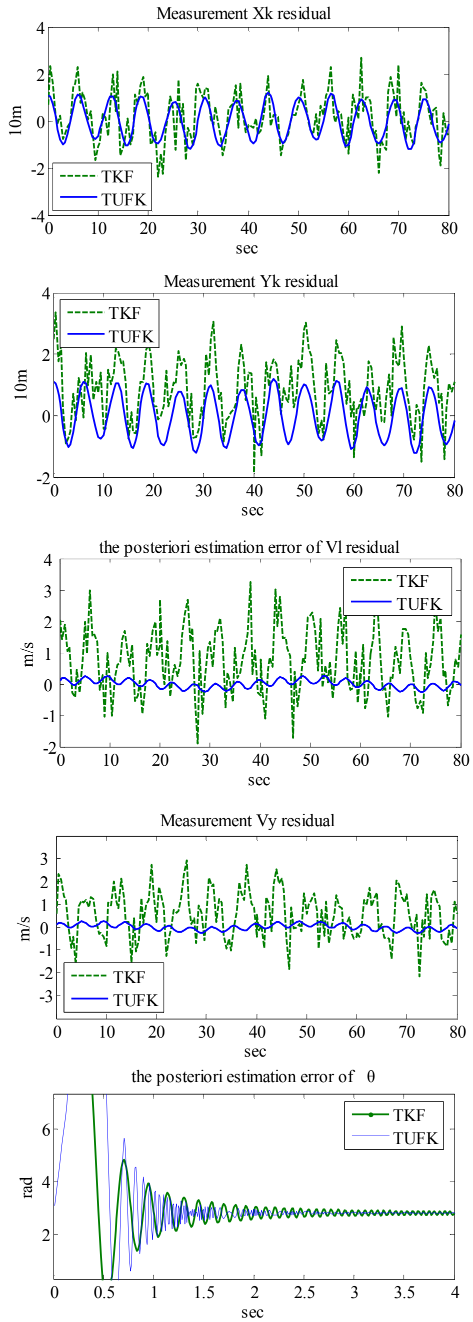

Figure 3. The comparison of the posteriori estimation error

VI. Conclusion

This paper proposes the two-stage Unscented Kalman filter (TUKF) for nonlinear system with unknown random bias with incomplete bias information. Adaptive fading UKF is presented using the ratio between the calculated innovation covariance and the estimated innovation covariance. And it proposes the TUKF that is designed by using the adaptive fading UKF. The stability of the two-stage Unscented Kalman filter (TUKF) is analyzed. According to some standard results, it is pointed out that, the stability of the TUKF may be ensured when the system given by (1) is stochastically controllable and stochastically observable and do not depend on the magnitude of β_k and γ_k which are unknown instrumental diagonal matrix introduced to evaluate the residuals introduced by the UT. Moreover, the high-update rate Wheel Mobile Robot (WMR) estimation using Real-time Kinematic Global Positioning System (RTK-GPS) and inertial sensors in the presence of wheel skidding and slipping are introduced to show the high performances of the UTKF.

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